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Abstract

The mobility of a Kähler metric is the dimension of the space of metrics with which it is c-projectively equivalent. The mobility is at least two if and only if the Kähler metric admits a nontrivial hamiltonian 2-form. After summarizing this relationship, we present necessary conditions for a Kähler metric to have mobility at least three: its curvature must have nontrivial nullity at every point. Using the local classification of Kähler metrics with hamiltonian 2-forms, we describe explicitly the Kähler metrics with mobility at least three and hence show that the nullity condition on the curvature is also sufficient, up to some degenerate exceptions. In an appendix, we explain how the classification may be related, generically, to the holonomy of a complex cone metric.

Introduction

This paper weaves together two threads in Kähler geometry which have been running in parallel for 40–60 years with remarkably little interaction, given their common themes.

The first thread concerns a notion of projective equivalence between Kähler metrics. The classical notion is too strong when applied to Kähler metrics: if two metrics that are hermitian with respect to the same almost complex structure have the same geodesics, they have the same Levi-Civita connection. In 1954, Otsuki and Tashiro [OT54] introduced a complex, but nonholomorphic, version of projective equivalence, which acquired the unfortunate name of 'holomorphically projective' or 'h-projective' equivalence in the literature. We prefer the term 'c-projective', which is intended to suggest 'complex projective', without implying that the geometry is holomorphic.

Definition 1. Let (M, J) be a complex manifold of real dimension $2m \geqslant 4$. Then two J-hermitian Kähler metrics $g, \widetilde{g}$ on M, with Levi-Civita connections $\nabla, \widetilde{\nabla}$, are called *c-projectively equivalent* if there is a 1-form Φ such that

$$\widetilde{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX \tag{1}$$

for all vector fields X, Y.

This notion has been extensively studied by Russian and Japanese schools (see [Mik98] for a list of references up to 1998). One common theme has been the relationship between special curvature properties of a Kähler metric and the existence of metrics c-projectively equivalent to it (e.g. [IT61]).

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The second thread concerns the explicit construction of 'optimal' Kähler metrics on complex manifolds, generalizing the constant curvature metrics used in the uniformization of Riemann surfaces. The idea of seeking such metrics goes back to Calabi's famous conjectures in the 1950s (e.g., [Cal57]), but the problem was attacked primarily using analytical methods until the late 1970s. Then Calabi provided fresh impetus by introducing the notion of an extremal Kähler metric and constructing explicit examples on total spaces of complex projective line bundles [Cal79, Cal82]. Calabi's construction has been refined and extended considerably by many authors (e.g., [Abr98, HS02]), providing a rich supply of Kähler metrics with special curvature properties (such as extremal Kähler metrics). These generalizations have in common that they introduce first order structure to simplify the second (and higher) order partial differential equations (PDEs) that describe curvature. A single source for this structure was identified in [ACG06], where it was observed that Calabi's construction and its generalizations reflect the presence of a nontrivial solution to an overdetermined linear differential equation, called a hamiltonian 2-form.

Definition 2. Let (M, g, J, ω) be a Kähler manifold of real dimension $2m \geqslant 4$. Then a (real) J-invariant 2-form ϕ on M is *hamiltonian* if

$$\nabla_X \phi = \tfrac{1}{2}(d\,\mathrm{tr}_\omega\,\phi \wedge JX^\flat - J\,d\,\mathrm{tr}_\omega\,\phi \wedge X^\flat) \tag{2}$$

for all vector fields X, where $X^\flat = g(X, \cdot)$, $JX^\flat = -X^\flat \circ J = (JX)^\flat$, and $\mathrm{tr}_\omega\,\phi = g(\omega, \phi)$ is the trace of ϕ with respect to the Kähler form ω.

Kähler manifolds with hamiltonian 2-forms are classified locally in [ACG06] and globally in [ACGT04], with applications to extremal Kähler metrics in [ACGT08].

The origins of the present paper are somewhat serendipitous. In April 2011, the first author was asked to referee the paper [MR12] by the second and third authors, which proves that the only compact c-projective manifold with a one-parameter subgroup of 'essential' symmetries is complex projective space. This drew the first author's attention to the 'main equation' of c-projective equivalence ((4) below), which is manifestly equivalent to the equation for hamiltonian 2-forms (see Remark 1).

As noted in the published version of [MR12], this equivalence has two main ramifications. First, the organizing principle observed in [ACG06] to underpin explicit constructions of Kähler metrics coincides with the notion of a c-projectively equivalent metric, a topic studied independently for many years previously. Secondly, the classification results in [ACG06, ACGT04] solve open problems in the theory of c-projective equivalence, as well as providing new examples.

Our interest here is in a third ramification: although the methodologies employed in the theories of c-projective equivalence and hamiltonian 2-forms have a large overlap (e.g., as both depend upon the theory of overdetermined PDEs of finite type), they have quite different flavours which might be combined with profit to prove new results. This paper is a first attempt to exploit both theories in this way.

We focus on the *mobility* $D(g, J)$ of a Kähler metric g on (M, J), which is the dimension of the space $Sol(g, J)$ of solutions of (4), or equivalently (2). Since the identity map Id (corresponding to the Kähler form ω) is always a solution, $D(g, J) \geqslant 1$, and the presence of an independent solution (or a nontrivial hamiltonian 2-form) means equivalently that $D(g, J) \geqslant 2$.

Our plan is to study the case $D(g, J) \geqslant 3$, using [FKMR12, Theorem 5], quoted as Theorem 1 below, which states that any such Kähler metric g is $C_{\mathbb{C}}(B)$ (for some $B \in \mathbb{R}$) in the sense of Definition 4 (unless all solutions of (4) are parallel). The converse is not true: it is straightforward to construct $C_{\mathbb{C}}(B)$ metrics with mobility two (e.g., using the cone construction described in the appendix; see § A.4). In Theorems 2 and 3 we establish necessary and sufficient conditions for a Kähler metric to be $C_{\mathbb{C}}(B)$, and then, in Theorem 6, describe the additional conditions such that a $C_{\mathbb{C}}(B)$ metric g has mobility $D(g, J) \geqslant 3$.

Whereas Theorem 2 draws upon curvature conditions from the theory of c-projective equivalence, Theorem 3 uses hamiltonian 2-form methods. It follows, in Corollary 2, that an extremal Kähler metric with mobility at least three must have constant scalar curvature.

Our results are closely related to the cone construction of [MR15] (cf. [Arm08a, Arm08b, Mik98]) discussed in the appendix. More precisely, for $C_{\mathbb{C}}(B)$ metrics with $B < 0$ (and we may assume $B = -1$ by rescaling), this construction gives an explicit isomorphism between $Sol(g, J)$ and the space of parallel hermitian endomorphisms on a complex cone $(\hat{M}, \hat{g}, \hat{J})$ over (M, g, J), which we summarize in § A.1. The cone is a Kähler manifold of dimension $\dim_{\mathbb{C}} M + 1$, and (M, g, J) may be recovered from it by taking a Kähler quotient. It is known, at least since Eisenhart [Eis23], that the existence of a parallel hermitian endomorphism $\hat{A}$ on $\hat{M}$ is (locally) equivalent to a decomposition of $\hat{M}$ into a direct product of Kähler manifolds.

In § A.2, we derive a formula for the Kähler quotient metric g in terms of radial and angular coordinates on $\hat{M}$ coming from the decomposition of $\hat{M}$ induced by $\hat{A}$. In § A.3, we (partially) rederive the local classification formula (9) for g relative to $A \in Sol(g, J)$ corresponding to $\hat{A}$; this yields another proof of (one direction of) Theorem 3 by a direct calculation; see Proposition A.1. In § A.4 we use the cone construction to give an alternative proof of Theorem 6 for a $C_{\mathbb{C}}(-1)$ metric.

1. C-projective equivalence and hamiltonian 2-forms

1.1 C-projective equivalence and $C_{\mathbb{C}}(B)$ metrics

Let (M, J) be a complex manifold of real dimension $2m \geqslant 4$. For J-hermitian metrics $g, \widetilde{g}$ on M, we introduce the nondegenerate (g, J)-hermitian (i.e., g-symmetric, J-complex-linear) endomorphism

$$A(g, \widetilde{g}) := \left(\frac{\det \widetilde{g}}{\det g}\right)^{1/2(m+1)} \widetilde{g}^{-1} g, \tag{3}$$

where we view $g, \widetilde{g} \colon TM \to T^*M$ as bundle isomorphisms. A fundamental observation by Mikeš and Domashev [MD78] is that g and $\widetilde{g}$ are c-projectively equivalent if and only if there is a vector field Λ such that $A = A(g, \widetilde{g})$ satisfies the 'main equation'

$$\nabla_X A = X^\flat \otimes \Lambda + \Lambda^\flat \otimes X + JX^\flat \otimes J\Lambda + J\Lambda^\flat \otimes JX. \tag{4}$$

Conversely, a nondegenerate solution A of (4) determines a Kähler metric

$$\widetilde{g} = (\det A)^{-1/2} g A^{-1} \tag{5}$$

(obtained by solving (3) with respect to $\widetilde{g}$) c-projectively equivalent to g. Since Id is always a solution of (4), we can add a multiple of Id to any solution A to obtain (at least locally) a solution which is nondegenerate. In this sense, the solutions A of (4) are (locally, generically) in bijection with Kähler metrics $\widetilde{g}$ that are c-projectively equivalent to g.

Definition 3. The space of hermitian endomorphisms A satisfying (4) will be denoted by $Sol(g, J)$. The *mobility*[1] $D(g, J)$ of (M, g, J) is the dimension of $Sol(g, J)$.

[1] In the classical c-projective literature, this is known as the 'degree of mobility'.

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Remark 1. Obviously, two metrics $g, \widetilde{g}$ are affinely equivalent ($\widetilde{\nabla} = \nabla$) if and only if the endomorphism $A = A(g, \widetilde{g})$ is parallel. By (4), if the metrics are c-projectively equivalent, they are affinely equivalent if and only if the vector field Λ is identically zero.

Taking the trace on both sides of (4) shows that

$$\Lambda = \tfrac{1}{4} \operatorname{grad}_g \operatorname{tr} A, \tag{6}$$

hence (4) is a linear PDE system on A, which is equivalent to (2) for a hamiltonian 2-form ϕ by writing $g(AX, Y) = \phi(X, JY)$.

In [ACG06, MR12], the nonconstant eigenvalues $\xi_1, \dots, \xi_\ell$ of A, considered as functions on M, are shown to be continuous, and smooth on a dense open subset M^0. Moreover, their (complex) multiplicity on this subset is one. Thus we can express Λ on M^0 as

$$\Lambda = \frac{1}{2} \sum_{i=1}^{\ell} \operatorname{grad}_g \xi_i. \tag{7}$$

For each nonconstant eigenvalue ξ_i of A, $\operatorname{grad}_g \xi_i$ lies in the corresponding eigenspace (see [ACG06, MR12]). Hence the vanishing of Λ is equivalent to all eigenvalues of the endomorphism A (considered as functions on the manifold) being constant.

An important standard result in c-projective geometry is the fact that $J\Lambda$ is Killing.

Lemma 1. *Let (M, g, J) be a Kähler manifold of real dimension $2m \geqslant 4$. Then for any $A \in Sol(g, J)$, the corresponding vector field Λ is holomorphic, and $J\Lambda$ is a Killing vector field; equivalently $\nabla\Lambda$ is (g, J)-hermitian.*

Proof. This is well known: see [MD78, (13)], [ACG06, Proposition 3] and [FKMR12, Corollary 3]. □

As the introduction explains, our study builds on the following theorem.

Theorem 1 [FKMR12]. *Let (M, g, J) be a connected Kähler manifold of real dimension $2m \geqslant 4$ and mobility $D(g, J) \geqslant 3$. Then there is a unique $B \in \mathbb{R}$ such that for every $A \in Sol(g, J)$, with corresponding vector field Λ, there is a function μ such that the system*

$$\begin{aligned} \nabla_X A &= X^\flat \otimes \Lambda + \Lambda^\flat \otimes X + JX^\flat \otimes J\Lambda + J\Lambda^\flat \otimes JX, \\ \nabla\Lambda &= \mu \operatorname{Id} + BA, \\ \nabla\mu &= 2B\Lambda^\flat \end{aligned} \tag{8}$$

holds at every point of M.

Remark 2. If for $A \in Sol(g, J)$, $A \neq \text{const} \cdot \operatorname{Id}$, with corresponding vector field Λ, there exists a function μ such that (A, Λ, μ) solves (8) for a certain constant B, then this holds for any other element $\tilde{A} \in Sol(g, J)$. This is clear if $\tilde{A}$ is a linear combination of Id and A and follows from Theorem 1 if Id, $A, \tilde{A}$ are linearly independent.

Definition 4. Let B be a real number. A Kähler metric (g, J) is called[2] $C_{\mathbb{C}}(B)$ if it admits a solution (A, Λ, μ) to the system (8) with Λ not identically zero.

[2] Here '$C_{\mathbb{C}}$' suggests constant/curvature/cone and complex/c-projective, and replaces the term '$K_n(B)$', often used in the classical c-projective literature, in which K_n denotes a Kähler n-manifold.

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Remark 3. In Definition 4 we require B to be a constant. If B is initially assumed to be a function, it turns out that this function must be (locally) constant provided there exists at almost every point a nonzero vector contained in the B-nullity of the curvature; see Definition 5 and Theorem 2 below.

Remark 4. Neither (2) nor (4) provides the most natural formulation of c-projective equivalence and mobility because they treat the metrics g and $\widetilde{g}$ asymmetrically. This can be remedied by observing that the definition (1) for c-projective equivalence is really an equivalence relation between complex affine connections (connections ∇ on TM with $\nabla J = 0$). A *c-projective structure* on a complex manifold (M, J) is a c-projective equivalence class of such complex affine connections. Equation (4) can be rewritten without reference to a background metric g by replacing A with the metric h on T^*M defined by $h(\alpha, \beta) = g(\alpha \circ A, \beta)$. Then (4) becomes

$$\nabla_X h = X \otimes \Lambda + \Lambda \otimes X + JX \otimes J\Lambda + J\Lambda \otimes JX$$

(for all vector fields X) and this equation for h depends only on the c-projective class of ∇ provided that h is viewed as a section of $\mathcal{L}^* \otimes S^2TM$, where $\mathcal{L}^{\otimes(m+1)} = \wedge^{2m}TM$.

This viewpoint is developed in detail in a forthcoming survey [CEMN] on c-projective geometry; see also [Yos78]. For the present paper, we shall always have in mind a background metric, and so we do not pursue this reformulation any further.

1.2 The classification of hamiltonian 2-forms

According to [ACG06], a Kähler metric (g, J, ω) admitting a hamiltonian 2-form (or equivalently an $A \in \mathit{Sol}(g, J)$) is locally a bundle over a product of Kähler $2m_\eta$-manifolds indexed by the constant eigenvalues η of A (m_η being the multiplicity of η), whose 'orthotoric' fibres are totally geodesic with the nonconstant eigenvalues $\xi_1, \ldots, \xi_\ell$ of A as coordinates. On a dense open set, we may write

$$g = \underbrace{\sum_\eta p_{\mathrm{nc}}(\eta) g_\eta}_{\text{base metric}} + \underbrace{\sum_{i=1}^{\ell} \frac{\Delta_j}{\Theta_j(\xi_j)} d\xi_j^2 + \sum_{j=1}^{\ell} \frac{\Theta_j(\xi_j)}{\Delta_j} \Big(\sum_{r=1}^{\ell} \sigma_{r-1}(\hat{\xi}_j) \theta_r \Big)^2}_{\text{fibre metric}}, \tag{9}$$

$$\omega = \sum_\eta p_{\mathrm{nc}}(\eta) \omega_\eta + \sum_{r=1}^{\ell} d\sigma_r \wedge \theta_r, \quad \text{with } d\theta_r = \sum_\eta (-1)^r \eta^{\ell - r} \omega_\eta, \tag{10}$$

where $p_{\mathrm{nc}}(t) = \prod_{i=1}^{\ell} (t - \xi_i)$, σ_r is the rth elementary symmetric function of $\{\xi_1, \ldots, \xi_\ell\}$, $\sigma_{r-1}(\hat{\xi}_j)$ is the $(r-1)$th such function of $\{\xi_k : k \neq j\}$, $\Delta_j = \prod_{k \neq j} (\xi_j - \xi_k)$, and

$$J\, d\xi_j = \frac{\Theta_j(\xi_j)}{\Delta_j} \sum_{r=1}^{\ell} \sigma_{r-1}(\hat{\xi}_j)\, \theta_r, \quad J\theta_r = (-1)^r \sum_{j=1}^{\ell} \frac{\Delta_j}{\Theta_j(\xi_j)} \xi_j^{\ell - r}\, d\xi_j. \tag{11}$$

For any metric of this form,

$$\begin{aligned} \phi &:= \sum_\eta \eta\, p_{\mathrm{nc}}(\eta) \omega_\eta + \sum_{j=1}^{\ell} \xi_j\, d\xi_j \wedge \Big(\sum_{r=1}^{\ell} \sigma_{r-1}(\hat{\xi}_j) \theta_r \Big) \\ &= \sum_\eta \eta\, p_{\mathrm{nc}}(\eta) \omega_\eta + \sum_{r=1}^{\ell} (\sigma_r\, d\sigma_1 - d\sigma_{r+1}) \wedge \theta_r \end{aligned}$$

is a hamiltonian 2-form. The extension of this local classification to pseudo-riemannian metrics is the subject of the forthcoming paper [BMR15].

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Curvature properties of the metric g in (9) are also computed in [ACG06], to which we refer for details and explanations. Let $p_c(t) = \prod_\eta (t-\eta)^{m_\eta}$ be the (monic) polynomial whose roots are the constant eigenvalues η of ϕ, counted with multiplicity.

(i) g is Bochner flat if and only if the functions $\Theta_j(t)$ are equal, given by a polynomial $\Theta(t)$ of degree at most $\ell+2$, with $\Theta(\eta) = 0$ for all constant eigenvalues η, and the base metrics g_η have constant holomorphic sectional curvature (CHSC), given by $-\Theta'(\eta)$. The metric g is itself CHSC if and only if in addition $\deg \Theta(t) \leqslant \ell+1$.

(ii) g is weakly Bochner flat if and only if the functions $(p_c\Theta_j)'(t)/p_c(t)$ are equal, given by a polynomial $\Psi(t)$ of degree at most $\ell+1$, and the base metrics g_η are Kähler–Einstein, with $(1/m_\eta)Scal_{g_\eta} = -\Psi(\eta)$. The metric g is Kähler–Einstein if and only if in addition $\deg \Psi(t) \leqslant \ell$.

In particular (applying (1) fibrewise, using the case that there are no constant eigenvalues), the orthotoric fibres have CHSC if and only if the functions $\Theta_j(t)$ are equal to a common polynomial of degree at most $\ell+1$.

It will also be useful to recall from [ACG06] that there is a 'Gray–O'Neill' formula [Gra67, O'Ne66] for the Levi-Civita connection of g in terms of the fibre and base metrics, where the Gray–O'Neill tensor of the horizontal distribution is given by

$$2C(X,Y) = \sum_{r=1}^{\ell} \big(\Omega_r(X,Y)J\Lambda_r - \Omega_r(JX,Y)\Lambda_r\big) \tag{12}$$

for $\Omega_r = \sum_\eta (-1)^r \eta^{\ell-r}\omega_\eta$ and $\Lambda_r = \operatorname{grad}_g \sigma_r$.

2. Curvature nullity and the extended system

Let $R \in \Omega^2(M, \mathfrak{gl}(TM))$ denote the curvature of the Kähler manifold (M,g,J),

$$R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z,$$

and let

$$K(X,Y) = \tfrac{1}{4}(Y^\flat \otimes X - X^\flat \otimes Y + JY^\flat \otimes JX - JX^\flat \otimes JY + 2g(X,JY)J) \tag{13}$$

be the algebraic curvature tensor of constant holomorphic sectional curvature.

Lemma 2. *Let (M,g,J) be a Kähler manifold of real dimension $2m \geqslant 4$. Then every $A \in Sol(g,J)$ satisfies the identity*

$$[R(X,Y),A] = -4[K(X,Y),\nabla\Lambda] \tag{14}$$

at every point for all tangent vectors X,Y.

Proof. Equation (14) is well known in the theory of c-projectively equivalent metrics; see, for example, [MD78, Mik98]. To prove it, consider the identity

$$[R(X,Y),A] = \nabla_X(\nabla A)_Y - \nabla_Y(\nabla A)_X \tag{15}$$

which holds for any endomorphism $A \in \Gamma(\mathfrak{gl}(TM))$. Assuming that $A \in Sol(g,J)$, we can replace the covariant derivatives of A in (15) with (4), to derive an integrability condition for (4). A straightforward calculation yields the desired (14). We note that we have to use the fact that $\nabla\Lambda$ commutes with J (see Lemma 1). $\square$

Definition 5. For $p \in M$ and $B \in \mathbb{R}$, the B-nullity space of the curvature R at p is the linear space

$$N(B)_p = \{Z \in T_pM : \mathcal{N}_B(X,Y)Z = 0 \ \forall X, Y \in T_pM\}, \tag{16}$$

where $\mathcal{N}_B(X,Y) = R(X,Y) + 4BK(X,Y)$.

Remark 5. Since $g(\mathcal{N}_B(\cdot,\cdot)\cdot,\cdot)$ is a section of $S^2(\wedge^2 T^*M)$, $N(B)_p$ is the set of $Z \in T_pM$ whose contraction into any entry of $g(\mathcal{N}_B(\cdot,\cdot)\cdot,\cdot)$ is zero. Note also that $N(B)_p$ is J-invariant, i.e., a complex linear subspace of T_pM.

Remark 6. The real number B in the definition of the nullity is unique: if $Z \in N(B)_p$ and $Z' \in N(B')_p$ are nonzero vectors, then $B = B'$. To see this, we replace X by Z' in the nullity condition for Z, and apply the nullity condition for Z' to obtain $(B - B')K(Z, Z') = 0$. Hence, $B = B'$ or $K(Z, Z') = 0$. The last equation implies Z' is a multiple of Z. Thus $(B - B')K(X,Y)Z = 0$ for all vectors X, Y, which, for Z nonzero, forces $B = B'$.

However, B may depend on the point p, and (of course) the metric g.

Proposition 1. *Let (M, g, J) be a Kähler manifold of real dimension $2m \geqslant 4$, and let $A \in$ Sol(g, J) with corresponding vector field Λ. Then for any functions B, μ, we have*

$$[K(X,Y), \nabla\Lambda - BA - \mu\,\mathrm{Id}] + \tfrac{1}{4}[\mathcal{N}_B(X,Y), A] = 0 \tag{17}$$

and, if B and μ are smooth,

$$\nabla_X(\nabla\Lambda - BA - \mu\,\mathrm{Id}) + J\mathcal{N}_B(X, J\Lambda) + (\nabla_X\mu - 2Bg(\Lambda, X))\,\mathrm{Id} + dB(X)A = 0. \tag{18}$$

Proof. Equation (17) is immediate from Lemma 2 (identity (14)). Recall from Lemma 1 that $J\Lambda$ is a Killing vector field, and hence $\nabla_X\nabla\Lambda = -J\nabla_X\nabla J\Lambda = -JR(X, J\Lambda)$ (by the standard formula $\nabla_X\nabla K = R(X,K)$, $X \in TM$, which holds for any Killing vector field K; see [Kos55]). Equation (18) follows from this by expanding $\nabla_X(\nabla\Lambda - BA - \mu\,\mathrm{Id})$ and substituting for $\nabla_X A$ from (4). □

Lemma 3. *Let Q be a hermitian endomorphism and Z a nonzero tangent vector at $p \in M$ such that $[K(X,Z), Q] = 0$ for all $X \in T_pM$. Then Q is a multiple of the identity.*

Proof. We may assume Q is trace-free and prove it vanishes. By definition (13) of K,

$$[Z^\flat \otimes X - X^\flat \otimes Z + JZ^\flat \otimes JX - JX^\flat \otimes JZ, Q] = 0. \tag{19}$$

Let $e_1, \ldots, e_{2m}$ be an orthonormal frame of T_pM. We take a trace by applying (19) to e_i with $X = e_i$ and summing over i. Since Q and $Q \circ J = J \circ Q$ are trace-free, and Q is hermitian, we obtain (with summation understood)

$$0 = g(Z, Qe_i)e_i - g(Z, e_i)Qe_i + g(e_i, e_i)QZ + g(JZ, Qe_i)Je_i - g(JZ, e_i)QJe_i = 2mQZ.$$

Thus $QZ = 0$, which we substitute into (19) to obtain

$$Z^\flat \otimes QX + (QX)^\flat \otimes Z + JZ^\flat \otimes Q(JX) + Q(JX)^\flat \otimes JZ = 0.$$

For any $Y \in \mathrm{span}\{Z, JZ\}^\perp$ this yields (using the fact that Q is hermitian)

$$0 = g(QX, Y)Z + g(Q(JX), Y)JZ = g(X, QY)Z + g(JX, QY)JZ.$$

Since $Z \neq 0$, Q vanishes on $\mathrm{span}\{Z, JZ\}^\perp$. But Q vanishes on $\mathrm{span}\{Z, JZ\}$, so $Q = 0$. □

Theorem 2. *Let (M, g, J) be a connected Kähler manifold of real dimension $2m \geqslant 4$. Then for any $A \in Sol(g, J)$ with corresponding vector field Λ such that A is not parallel (equivalently, $\Lambda \neq 0$) the following statements are equivalent.*

- (i) *There is a constant B such that $[\mathcal{N}_B(X, Y), A] = 0$ for all vector fields X, Y.*
- (ii) *There is a constant B and a smooth function μ such that $\nabla\Lambda = BA + \mu\,\mathrm{Id}$.*
- (iii) *There is a constant B and a smooth function μ such that A satisfies the extended system (8).*
- (iv) *There is a constant B such that Λ is in the B-nullity space $N(B)_p$ at every $p \in M$; equivalently $\mathcal{N}_B(X, J\Lambda) = 0$ for all $X \in TM$.*
- (v) *At every point p of a dense subset, there is a real number $B = B(p)$ such that the B-nullity space $N(B)_p$ is nonzero.*
- (vi) *There is a constant B such that, for any open subset U of M and any eigenvalue ξ of A smoothly defined on U, $\mathrm{grad}_g\,\xi$ is in the B-nullity of the curvature on U.*

If for given B, these conditions hold for some nonparallel $A \in Sol(g, J)$, then they hold for all $A \in Sol(g, J)$ (with the same constant B). In particular, the metric g is $C_{\mathbb{C}}(B)$.

Proof. (1) $\Leftrightarrow$ (2) by (17): if $\nabla\Lambda - BA$ commutes with $K(X, Y)$ for all $X, Y \in T_pM$, then it commutes with all skew-hermitian endomorphisms of T_pM and is hence a multiple of the identity at p.

(2) $\Leftrightarrow$ (3) by (18), which reduces to

$$g(\mathcal{N}_B(X, J\Lambda)Y, Z) = (\nabla_X\mu - 2Bg(\Lambda, X))g(JY, Z)$$

for all X, Y, Z: the left-hand side satisfies the Bianchi identity in X, Y, Z while the right-hand side does not (for $n > 1$), so they must vanish independently.

(3) $\Rightarrow$ (4) by (18) again: the extended system (8) implies $\mathcal{N}_B(X, J\Lambda) = 0$.

(4) $\Rightarrow$ (5) is immediate: if Λ is not identically zero, it is nonzero on an open dense subset, because $J\Lambda$ is a Killing vector field by Lemma 1.

(5) $\Rightarrow$ (3). Given a nonzero $Z \in N(B)_p$, substitute $Y = Z$ and $\mu = 0$ in (17) to obtain $[K(X, Z), \nabla\Lambda - BA] = 0$. Hence by Lemma 3 there is a scalar $\mu = \mu(p)$ such that $\nabla\Lambda - BA = \mu\,\mathrm{Id}$ at p. This holds at every point of a dense subset for functions μ, B defined on this subset. Moreover, A is not proportional to the identity at every point of a dense open set (this is straightforward to show using (4): for a proof, see [FKMR12, Lemma 4]). Then on a neighbourhood U of any point in this dense open set, B and μ are smooth functions (being solutions of an inhomogeneous linear system of maximal rank with smooth coefficients). We need to show that B is constant and $\tau := d\mu - 2Bg(\Lambda, \cdot)$ is identically zero on U. For this, suppose that a nonzero vector Z is in the B-nullity of the curvature and insert $\nabla\Lambda = \mu\,\mathrm{Id} + BA$ into (18) to obtain

$$J\mathcal{N}_B(X, J\Lambda) + \tau(X)\,\mathrm{Id} + dB(X)A = 0, \tag{20}$$

and hence, by applying this identity to Z, $\tau(X)Z + dB(X)AZ = 0$. If Z is not an eigenvector of A, we have $\tau(X) = dB(X) = 0$ for all $X \in TU$, which is what we wanted to show. We may thus assume $AZ = \xi Z$ for some function ξ, so that $\tau = -\xi\, dB$ and

$$\begin{aligned}\mathcal{N}_B(X, \Lambda) &= dB(JX)(A - \xi\,\mathrm{Id})J \\ &= ((A - \xi\,\mathrm{Id})X)^\flat \otimes (dB)^\sharp - dB \otimes (A - \xi\,\mathrm{Id})X,\end{aligned} \tag{21}$$

where $\alpha^\sharp$ denotes the metric dual of a 1-form α, and the second line follows from the Bianchi symmetry satisfied by $g(\mathcal{N}_B(X, \Lambda)Y, W) = g(\mathcal{N}_B(Y, W)X, \Lambda)$. Comparing the second and third lines, it follows that $A - \xi\,\mathrm{Id}$ has complex rank at most one. It remains to show the following lemma.

Lemma 4. *Suppose that $A \in Sol(g, J)$ and that, on an open subset U, A is not parallel with exactly two (distinct) eigenvalues, both smooth, and Z is an eigenvector of A in the B-nullity of g for smooth B. Then $dB = 0$ on U.*

Given this lemma, proven below, we obtain also $\tau = -\xi\, dB = 0$ on U, and hence the system (8) holds in a neighbourhood of every point of an open dense subset for a (local) constant B and a smooth function μ. On the other hand, it was proven in [FKMR12, § 2.5] that the constants B are the same for each such neighbourhood. Taking the trace of the second equation in (8), we obtain $2m\mu = \operatorname{tr}\nabla\Lambda - B\operatorname{tr}(A)$, so that the functions μ coincide on overlaps and patch together to a globally defined function. Hence the system (8) holds everywhere on M for a constant B and a smooth function μ.

(1)–(5) ⇒ (6). Since $\mathcal{N}_B(X, Y)\Lambda = 0$, (7) implies

$$0 = \sum_{i=1}^{l} \mathcal{N}_B(X, Y)\operatorname{grad}_g \xi_i, \tag{22}$$

where $\xi_1, \ldots, \xi_l$ are the eigenvalues of A. It was shown in [ACG06, Proposition 14] and [MR12, Proposition 1] that the gradient $\operatorname{grad}_g \xi_i$ is contained in the eigenspace of A corresponding to ξ_i. Since $[\mathcal{N}_B(X, Y), A] = 0$, $\mathcal{N}_B(X, Y)$ leaves the eigenspaces of A invariant. Then at any point where $\mathcal{N}_B(X, Y)\operatorname{grad}_g \xi_i$ is nonzero, it is an eigenvector of A corresponding to the eigenvalue ξ_i and (22) shows that $\mathcal{N}_B(X, Y)\operatorname{grad}_g \xi_i = 0$.

(6) ⇒ (5). It was shown in [ACG06, Proposition 14] that every nonconstant eigenvalue ξ of $A \in Sol(g, J)$ has nonvanishing differential on an open and dense subset.

The final observation of the theorem follows because condition (5) is independent of $A \in Sol(g, J)$, and if A is ∇-parallel (i.e., the corresponding Λ is zero), then (17) and Lemma 3 imply that A is a multiple of the identity or $B = 0$. □

Proof of Lemma 4. Since A is nonparallel (i.e., $\Lambda \neq 0$), it has at least one nonconstant eigenvalue. We consider first the case that A has one nonconstant eigenvalue ξ and one constant eigenvalue, which we may assume to be zero. The ξ-eigenspace is therefore spanned by Λ, and if this is in the nullity, then (20) implies $dB = 0$. Thus we may assume $AZ = 0$, hence $d\mu = 2Bg(\Lambda, \cdot) = B\, d\xi$, so that μ and B are functions of ξ. For any X with $AX = 0$ we have

$$\xi\mu X = -(A - \xi\ \mathrm{Id})\mu X = -(A - \xi\ \mathrm{Id})\nabla_X\Lambda = (\nabla_X A - d\xi(X))\Lambda = g(\Lambda, \Lambda)X$$

since $\nabla\Lambda = \mu\,\mathrm{Id} + BA$, $d\xi(X) = 0$ and ∇A is given by (4). Hence $\xi\mu = g(\Lambda, \Lambda)$.

On the other hand, using the Gray–O'Neill formulae [ACG06, Gra67] or the explicit form

$$g = -\xi g_0 + \frac{d\xi^2}{\Theta(\xi)} + \Theta(\xi)\theta^2$$

of the metric, we obtain that

$$-4BK(X, Y)Z = R(X, Y)Z = R_0(X, Y)Z - \frac{4g(\Lambda, \Lambda)}{\xi^2}K(X, Y)Z, \tag{23}$$

for X, Y in the zero eigenspace of A, where R_0 denotes the curvature of g_0 (the Kähler quotient by $J\Lambda$), lifted to the zero eigenspace. Thus $R_0(X, Y)Z = 4(B\xi - \mu)K_0(X, Y)Z$, where $K_0 = -K/\xi$

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is the algebraic constant holomorphic sectional curvature tensor of g_0. Taking the trace over X (on the zero eigenspace), $Ric_0(Z) = 2(m-1)(B\xi - \mu)Z$ and so $B\xi - \mu$ is independent of ξ, hence constant. This combines with $d\mu = B\, d\xi$ to give $dB = 0$ as required.

We now turn to the case where A has two nonconstant eigenvalues ξ_1 and ξ_2. Note that in this case, M is necessarily real four-dimensional. Let $V_1 = \operatorname{grad}_g \xi_1$, $V_2 = \operatorname{grad}_g \xi_2$ and suppose that V_2 is contained in the B-nullity of the curvature R. To compute B and μ, we apply V_1 and V_2 to the equation $\nabla\Lambda = \mu \operatorname{Id} + BA$ to obtain the linear system $m_1 = \mu + \xi_1 B$, $m_2 = \mu + \xi_2 B$, where m_1, m_2 are the eigenvalues of $\nabla\Lambda$, i.e., $\nabla_{V_1}\Lambda = m_1 V_1$ and $\nabla_{V_2}\Lambda = m_2 V_2$. Hence,

$$\mu = \frac{\xi_2 m_1 - \xi_1 m_2}{\xi_2 - \xi_1}, \quad B = \frac{m_1 - m_2}{\xi_1 - \xi_2}.$$

To calculate m_1, m_2, we recall that $\Lambda = (1/2)(V_1 + V_2)$ and so $\nabla_\Lambda \Lambda = (1/2)(m_1 V_1 + m_2 V_2)$, or, dually, $d(g(\Lambda, \Lambda)) = m_1\, d\xi_1 + m_2\, d\xi_2$. The classification of hamiltonian 2-forms from § 1.2 shows that, in a neighbourhood of almost every point, g takes the form

$$g = \frac{\xi_1 - \xi_2}{F_1(\xi_1)}\, d\xi_1^2 + \frac{\xi_2 - \xi_1}{F_2(\xi_2)}\, d\xi_2^2 + \frac{F_1(\xi_1)}{\xi_1 - \xi_2}(dt_1 + \xi_2\, dt_2)^2 + \frac{F_2(\xi_2)}{\xi_2 - \xi_1}(dt_1 + \xi_1\, dt_2)^2$$

in local coordinates ξ_1, ξ_2, t_1, t_2. From this, we obtain $g(\Lambda, \Lambda)$ in terms of the functions F_1, F_2. Calculating $d(g(\Lambda, \Lambda))$ and comparing coefficients, we obtain

$$B = \frac{m_1 - m_2}{\xi_1 - \xi_2} = \frac{(F_1'(\xi_1) + F_2'(\xi_2))(\xi_1 - \xi_2) - 2(F_1(\xi_1) - F_2(\xi_2))}{4(\xi_1 - \xi_2)^3}. \tag{24}$$

Replacing X in (21) by the vector JV_2 in the nullity, we see that $dB(V_2) = 0$, i.e., B does not depend on the variable ξ_2. Using (24), it is straightforward to show that the condition $dB/d\xi_2 = 0$ is equivalent to

$$0 = F_2''(\xi_2)(\xi_1 - \xi_2)^2 + 2(F_1'(\xi_1) + 2F_2'(\xi_2))(\xi_1 - \xi_2) - 6(F_1(\xi_1) - F_2(\xi_2)). \tag{25}$$

Taking three derivatives of this equation with respect to ξ_1 yields $F_1^{(4)}(\xi_1) = 0$, hence $F_1(\xi_1)$ is a polynomial of degree at most three. Inserting this condition back into (25), a straightforward calculation shows $F_1 = F_2$. Inserting these polynomials into (24) shows that B is a constant. This also follows from [ACG06] where it is shown that (g, J) has constant holomorphic sectional curvature (and hence B is constant) if $F_1 = F_2$ is a polynomial of degree at most three. □

Remark 7. Recall from Remark 1 that A not being parallel is necessary for (5). All other conditions are automatically fulfilled for parallel A, in which case we have $\mu = B = 0$.

To relate this result to the local classification of metrics with hamiltonian 2-forms (see § 1.2), observe that at each point in a dense open set, the J-linear span of the gradients of the eigenvalues of A is the tangent space to the orthotoric fibres of the metric g.

Corollary 1. *$A \in Sol(g, J)$ satisfies the extended system (8) for $B \in \mathbb{R}$ if and only if A is parallel (in which case, we may assume $B = 0$) or the orthotoric fibres of A are in the B-nullity of g. In particular, since these fibres are totally geodesic, they have constant holomorphic sectional curvature $-4B$.*

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By Definition 4, a Kähler metric is $C_{\mathbb{C}}(B)$ for a constant B if one of the conditions in Theorem 2 is satisfied for some nonparallel $A \in \mathit{Sol}(g, J)$. We next describe the conditions on the parameters in formula (9) under which a Kähler metric is $C_{\mathbb{C}}(B)$.

For this, we first observe that the extended system (8) is equivalent to the special case $\tau_0 = 0$, $\tau_1 = -4B$, $\tau_2 = -4\mu$ of the system [ACG06, § 2.3, Equation (30)], with the term '$W^{\mathcal{K}}(\phi)$' omitted. Hence (the proof of) [ACG06, Proposition 5] applies to show that the polynomial

$$F(t) = -4(Bt + \mu)p_A(t) - g(K, K(t)) \tag{26}$$

has constant coefficients, where $p_A(t)$ is the characteristic polynomial of A, $K = J\,\mathrm{grad}_g\,\sigma_1$ and $K(t) = J\,\mathrm{grad}_g\,p_A(t)$ (thus K coincides with the Killing vector field $2J\Lambda$). To interpret this fact geometrically, we next observe that any triple $(A, \Lambda, \mu) \in \mathfrak{gl}(TM) \oplus TM \oplus M \times \mathbb{R}$, with A hermitian, defines a hermitian (bundle) metric on $\{(\sigma, \rho) \in \mathrm{Hom}(TM, \mathbb{C}) \oplus M \times \mathbb{C} : \sigma(JX) = i\sigma(X)\}$, via the expression

$$\begin{bmatrix}\rho\\ \sigma\end{bmatrix}^{\dagger}\begin{bmatrix}\mu & \Lambda\\ \Lambda & A\end{bmatrix}\begin{bmatrix}\rho'\\ \sigma'\end{bmatrix} := \mu\overline{\rho}\rho' + \overline{\sigma}(\Lambda)\rho' + \overline{\rho}\sigma'(\Lambda) + g(\overline{\sigma}\circ A, \sigma'). \tag{27}$$

When $B = -1$, this bundle may be identified with the (holomorphic) tangent bundle of the complex cone over (M, g, J) studied in [MR15], which we discuss in the appendix. For any $B \in \mathbb{R}$, the bundle carries a connection $\mathcal{D}$ defined by

$$\mathcal{D}_X\begin{bmatrix}\rho\\ \sigma\end{bmatrix} = \begin{bmatrix}\nabla_X\rho + \sigma(X)\\ \nabla_X\sigma + Bg(X + iJX, \cdot)\rho\end{bmatrix}. \tag{28}$$

This connection induces the extended system (8) in the following sense (cf. [FKMR12, §§ 4.1–4.2] in the case $B \neq 0$).

Lemma 5. *For any sections (A, Λ, μ) and (σ, ρ) as above, we have*

$$\begin{aligned}&\partial_X\left(\begin{bmatrix}\rho\\ \sigma\end{bmatrix}^{\dagger}\begin{bmatrix}\mu & \Lambda\\ \Lambda & A\end{bmatrix}\begin{bmatrix}\rho'\\ \sigma'\end{bmatrix}\right) - \left(\mathcal{D}_X\begin{bmatrix}\rho\\ \sigma\end{bmatrix}\right)^{\dagger}\begin{bmatrix}\mu & \Lambda\\ \Lambda & A\end{bmatrix}\begin{bmatrix}\rho'\\ \sigma'\end{bmatrix} - \begin{bmatrix}\rho\\ \sigma\end{bmatrix}^{\dagger}\begin{bmatrix}\mu & \Lambda\\ \Lambda & A\end{bmatrix}\mathcal{D}_X\begin{bmatrix}\rho'\\ \sigma'\end{bmatrix}\\ &= \begin{bmatrix}\rho\\ \sigma\end{bmatrix}^{\dagger}\begin{bmatrix}\nabla_X\mu - 2Bg(\Lambda, X) & \nabla_X\Lambda - \mu X - BAX\\ \nabla_X\Lambda - \mu X - BAX & \nabla_X A - \left(\begin{smallmatrix}X^{\flat}\otimes\Lambda + \Lambda^{\flat}\otimes X\\ +JX^{\flat}\otimes J\Lambda + J\Lambda^{\flat}\otimes JX\end{smallmatrix}\right)\end{bmatrix}\begin{bmatrix}\rho'\\ \sigma'\end{bmatrix}.\end{aligned}$$

The proof is a straightforward computation. Up to a normalization constant, the function $F(t)$ is the (complex) determinant of the hermitian form on $\mathrm{Hom}(TM, \mathbb{C}) \oplus M \times \mathbb{C}$ defined by $(A - t\,\mathrm{Id}, \Lambda, \mu + Bt)$, so its roots are the relative eigenvalues of the hermitian forms defined by (A, Λ, μ) and $(\mathrm{Id}, 0, -B)$. This gives another proof that $F(t)$ has constant coefficients when (A, Λ, μ) solves (8), and further shows that the relative eigenspaces are $\mathcal{D}$-parallel subbundles of $\mathrm{Hom}(TM, \mathbb{C}) \oplus M \times \mathbb{C}$.

Theorem 3. *Let (g, J, ω) be a Kähler metric with a nonparallel hamiltonian 2-form, given explicitly by (9) on a dense open set. Then g is $C_{\mathbb{C}}(B)$ if and only if $\Theta_j(t) = \Theta(t)$, a polynomial of degree at most $\ell + 1$ (independent of j) with leading coefficient $-4B$, and $\Theta(\eta) = 0$ for all constant eigenvalues η.*

Proof. If g is $C_{\mathbb{C}}(B)$ then the (totally geodesic) orthotoric fibres have CHSC. The hamiltonian 2-form restricts to a hamiltonian 2-form on each fibre whose characteristic polynomial is

$p_A(t)/p_c(t)$. Applying [ACG06, Proposition 18] fibrewise, we thus have $\Theta_j(t) = \Theta(t) := F(t)/p_c(t)$ for all j (where we recall that $p_c(t)$ is the monic polynomial whose roots are the constant eigenvalues η of A). It remains to show that any root η of $p_c(t)$ is a root of $\Theta(t)$, i.e., the multiplicity of η as a root of $F(t)$ is greater than its multiplicity as a root of $p_c(t)$. The latter is the dimension of the kernel of $A - \eta\,\mathrm{Id}$ in $\mathrm{Hom}(TM, \mathbb{C})$ which is a subspace U of the relative η-eigenspace, i.e., the kernel of the hermitian form defined by $(A - \eta\,\mathrm{Id}, \Lambda, \mu + B\eta)$. However, by (28), U cannot be $\mathcal{D}$-parallel, so the dimension of the relative η-eigenspace is strictly larger, hence so is the multiplicity of η as a root of $F(t)$.

Conversely, if $\Theta_j(t) = \Theta(t)$ as stated, then the orthotoric fibres belong to the B-nullity of g. To see this, observe that the Gray–O'Neill curvature formulae [Gra67, O'Ne66] (with Gray–O'Neill tensor (12)) imply that all components of the curvature of g, apart from the purely horizontal part, depend on the base metrics g_η in (9) only to first order at each point. Hence, to compute $R(X,Y)Z$ for Z vertical, we may use a metric $\tilde{g}$ which agrees with g at a given point, but where we replace the base metrics g_η with metrics $\tilde{g}_\eta$ which have CHSC equal to $\Theta'(\eta)$ at that point. By [ACG06, Proposition 17], $\tilde{g}$ has CHSC given by a multiple of B, hence the fibres are in the B-nullity. Consequently, the same holds for g. $\square$

Corollary 2. *Let (g, J) be a $C_{\mathbb{C}}(B)$ Kähler metric (e.g., with $D(g, J) \geqslant 3$) which is weakly Bochner flat (or is Bochner flat). Then g is Kähler–Einstein (or has constant holomorphic sectional curvature, respectively).*

Recall from [ACG06] that a Kähler metric (g, J) of dimension $2m$ is *orthotoric* if it admits a hamiltonian 2-form having m nonconstant eigenvalues $\xi_1, \ldots, \xi_m$ (these metrics are also 'Kähler–Liouville'; see [KT11]).

Corollary 3. *Let (g, J) be a $C_{\mathbb{C}}(B)$ Kähler metric (e.g., with $D(g, J) \geqslant 3$) which is orthotoric. Then g has constant holomorphic sectional curvature.*

Remark 8. An analogue of the corollary in real projective geometry, which is also true under more general assumptions, can be found in [BKM09].

Theorem 3 has the following global consequence.

Theorem 4. *Let M be a closed connected $2m$-orbifold ($2m \geqslant 4$) and suppose (g, J) is a $C_{\mathbb{C}}(B)$ Kähler metric on M. Then (M, g, J) is an orbifold quotient of $\mathbb{C}P^m$ with a Fubini–Study metric.*

Proof. By assumption, M admits a hamiltonian 2-form of order $\ell \geqslant 1$. The theory of [ACGT04, § 2], which extends to orbifolds following [LT97], shows that the universal orbifold cover of M has a blow-up $\hat{M}$ which is a bundle of (connected) toric orbifolds over an orbifold S which is a complete Kähler product over the constant eigenvalues η of A. Since blow-up does not change the orbifold fundamental group, $\hat{M}$ is a simply connected orbifold, hence so is S (since the fibres of $\hat{M} \to S$ are connected). Now, since every constant eigenvalue η is a root of the function Θ of Theorem 3, it follows from [ACGT04, Proposition 6] (or rather, its proof, extended straightforwardly to orbifolds) that S is a Kähler product of complex projective spaces where the Kähler metric on the factor corresponding to a root η has CHSC $-\Theta'(\eta)$ (see [ACGT04, Theorem 5(iv)–(v)]). As discussed in § 1.2(i), these are precisely the conditions (given that Θ is a polynomial of degree at most $\ell + 1$ vanishing on the constant eigenvalues η) which ensure that the metric on M has CHSC [ACG06]. (This is not a coincidence: the Fubini–Study metric on

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$\mathbb{C}P^m$ admits hamiltonian 2-forms of any order $0 \leqslant \ell \leqslant m$.) Since $K = J\Lambda$ is a nonparallel Killing vector field on M (it is hamiltonian, hence has zeros), the curvature of g must be positive by Bochner's argument. Hence the universal cover of M is isometric to $\mathbb{C}P^m$ with a Fubini–Study (positive CHSC) metric. □

Corollary 4. *Let (M, g, J) be a closed connected Kähler orbifold of dimension $2m \geqslant 4$ and mobility $D(g, J) \geqslant 3$. Then either M is an orbifold quotient of $\mathbb{C}P^m$ with a Fubini–Study metric, or every Kähler metric c-projectively equivalent to g is affinely equivalent to g.*

Remark 9. This corollary is immediate from Theorems 4 and 1 (i.e., [FKMR12, § 2]). In the manifold case, it is the main result of [FKMR12], where it was established for metrics of arbitrary signature. Indeed, on manifolds, the analogue of Theorem 4 for metrics of arbitrary signature was obtained in [FKMR12, Remark 12]. Furthermore, the proof in [FKMR12] proceeds by first reducing to the case that $-Bg$ is positive definite, and this part of the argument extends straightforwardly to orbifolds. Hence Theorem 4 is actually valid in all signatures.

On the other hand, in the remaining case, where (without loss of generality) $B = -1$ and g is positive definite, [FKMR12, Lemma 8] shows that the extended system (8) yields a nontrivial solution of the kählerian *Tanno equation*, and so the manifold case of Theorem 4 follows from [Tan78, Theorem 10.1]. In fact, as shown in [FR11, § 3, see (4)], the Tanno equation is equivalent to the extended system in this case, and so Theorem 4 may be regarded as providing a natural generalization of [Tan78, Theorem 10.1] to orbifolds of arbitrary signature. Note that our method of proof for Theorem 4 is very different from [Tan78].

The corollary is a rigidity result for closed connected Kähler orbifolds (M, g, J) which are not quotients of $\mathbb{C}P^m$, but admit a c-projectively equivalent metric which is not affine equivalent (i.e., a hamiltonian 2-form of order $\ell > 0$). This has several consequences. First, as observed in [FKMR12], the isometry group of g has codimension at most one in the group of c-projective transformations of M: this is because the latter group acts on the projectivization of $Sol(M, g)$, with the isometry group of g as a point stabilizer. Secondly, since the hamiltonian 2-form is essentially unique (i.e., $A \in Sol(M, g)$ is unique up to a linear combination with the identity solution), the ℓ-torus action it defines must be central.

3. Classification of metrics with c-projective mobility at least three

Let us recall the following result.

Theorem 5 [ACGT04, FKMR12]. *Let (M, g, J) be a connected Kähler manifold of real dimension four. Then $D(g, J) \geqslant 3$ if and only if the holomorphic sectional curvature is constant.*

Remark 10. In [ACGT04, Proposition 10] and [FKMR12, Lemma 7], it was shown that a Kähler manifold of real dimension four and of mobility at least three has constant holomorphic sectional curvature. The fact that every CHSC Kähler manifold of any dimension $2m$ has mobility $(m + 1)^2 \geqslant 3$ is a standard result; see, for example, [ACG06, Mik98].

By Theorem 1, the condition $D(g, J) \geqslant 3$ implies either that all $A \in Sol(M, g)$ are parallel, or that the metric is $C_{\mathbb{C}}(B)$, i.e., the equivalent conditions of Theorem 2 hold. Conversely, we now find the metrics satisfying $D(g, J) \geqslant 3$ among those that are $C_{\mathbb{C}}(B)$.

Theorem 6. *Let (M, g, J) be a connected Kähler manifold of real dimension $2m \geqslant 4$ which is $C_{\mathbb{C}}(B)$. Suppose in addition that there exists $A \in Sol(g, J)$ such that:*

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- *either the number of nonconstant eigenvalues of A is two or more;*
- *or the number of constant eigenvalues of A is three or more.*

Then $D(g, J) \geqslant 3$.

Proof. Let us choose $A \in Sol(g, J)$ satisfying one of the two conditions on the eigenvalues.

First suppose that the corresponding vector field Λ is identically zero. Then A is covariantly constant and all eigenvalues of A are constant (see Remark 1). The endomorphism $\tilde{A} = A^2$ is covariantly constant and hence contained in $Sol(g, J)$. It follows that $\tilde{A}, A$ and Id are linearly independent and therefore $D(g, J) \geqslant 3$, since otherwise, A would be annihilated by a polynomial with constant coefficients of order two or lower, and this contradicts the assumption that the number of constant eigenvalues is at least three. We have proven Theorem 6 under the assumption $\Lambda \equiv 0$.

Let us now suppose that Λ is not identically zero.

First case: $B = 0$. A straightforward computation (using the equations in (8)) shows

$$\tilde{A} = \Lambda^\flat \otimes \Lambda + J\Lambda^\flat \otimes J\Lambda$$

is contained in $Sol(g, J)$, where the corresponding vector field is $\tilde{\Lambda} = \mu\Lambda$ and μ is a constant.

Clearly, $\tilde{A}$ is not proportional to Id (since it is multiplication with $g(\Lambda, \Lambda)$ on $\operatorname{span}\{\Lambda, J\Lambda\}$ and multiplication with zero on $\operatorname{span}\{\Lambda, J\Lambda\}^\perp$). If $D(g, J) = 2$, we have $A = \alpha\tilde{A} + \beta \operatorname{Id}$ for certain constants α and β, but this contradicts the assumptions on the eigenvalues of A. Theorem 6 is proven in the case $B = 0$.

Second case: $B \neq 0$. Let us multiply the metric with $-B$, such that the system (8) for the new metric (which we again denote by the symbol g) holds with $B = -1$. Note that the mobility remains unchanged by this procedure. A straightforward computation (one may also compare [Mik98, p. 1338], [FKMR12, Equation (88) in the proof of Lemma 10] or the cone construction [MR15, Theorem 9]; see the appendix below) using the equations in (8) shows

$$\tilde{A} = A^2 + \Lambda^\flat \otimes \Lambda + J\Lambda^\flat \otimes J\Lambda$$

is contained in $Sol(g, J)$ with corresponding vector field $\tilde{\Lambda} = (A + \mu \operatorname{Id})\Lambda$. Assuming $D(g, J) = 2$, we obtain (up to rescaling) $A = \tilde{A} + \alpha \operatorname{Id}$ for a certain constant α. Taking the covariant derivative of this equation shows $\Lambda = (A + \mu \operatorname{Id})\Lambda$. Hence, Λ is an eigenvector of A corresponding to the nonconstant eigenvalue $1 - \mu$. Equation (7) (together with the fact that for each nonconstant eigenvalue ξ_i of A, $\operatorname{grad}_g \xi_i$ is contained in the corresponding eigenspace) implies that A has exactly one nonconstant eigenvalue. Restricting $A = \tilde{A} + \alpha \operatorname{Id}$ to the orthogonal complement $U := \operatorname{span}\{\Lambda, J\Lambda\}^\perp$ shows that the restriction $A|_U$ is annihilated by a quadratic polynomial. Then the number of nonconstant eigenvalues is at most two. We obtain a contradiction to any of the two conditions on the eigenvalues of A. Hence, $D(g, J) \geqslant 3$ and Theorem 6 is proven. □

Appendix. Cone construction for $C_{\mathbb{C}}(-1)$ metrics

A.1 The cone construction

If g is a $C_{\mathbb{C}}(-1)$ Kähler metric then the space $Sol(g, J)$ is isomorphic to the space of solutions (A, Λ, μ) of the PDE system

$$\begin{aligned} \nabla_X A &= X^\flat \otimes \Lambda + \Lambda^\flat \otimes X + JX^\flat \otimes J\Lambda + J\Lambda^\flat \otimes JX, \\ \nabla\Lambda &= \mu \operatorname{Id} - A, \\ \nabla\mu &= -2\Lambda^\flat. \end{aligned} \tag{A.1}$$

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The cone construction [MR15, Theorem 9] (see also the formulae in [Mik98, pp. 1338–1339] for the same statement, though the formula for $\hat{A}$ appearing there seems to have a misprint) asserts that the space of solutions (A, Λ, μ) of this system is isomorphic to the space of parallel hermitian endomorphisms $\hat{A} \in \mathrm{End}(T\hat{M})$ on the cone

$$\hat{M} = \mathbb{R}_{>0} \times \mathbb{R} \times M, \quad \hat{g} = dr^2 + r^2(\phi^2 + g), \quad \hat{J} = \frac{1}{r}\partial_t \otimes dr - r\partial_r \otimes \phi + J, \tag{A.2}$$

where $\phi = dt - \tau$ and τ is a 1-form on M satisfying $d\tau = 2\omega$ ($\omega = g(J\cdot, \cdot)$ denotes the Kähler form on M). The construction is local, but this is sufficient for our purposes. The correspondence between solutions (A, Λ, μ) of (A.1) and parallel hermitian endomorphisms $\hat{A} \in \mathrm{End}(T\hat{M})$ is given by

$$\hat{g}(\hat{A}\cdot, \cdot) = \mu dr^2 - r\, dr \odot \Lambda^\flat + r^2(\mu\phi^2 + \phi \odot \Lambda^\flat(J\cdot) + g(A\cdot, \cdot)). \tag{A.3}$$

Further, we view the manifold $N = \mathbb{R} \times M$ with metric $h = \phi^2 + g$ as naturally embedded into $\hat{M}$ as the hypersurface $N = \{r = 1\}$. The manifold (M, g, J) is recovered from $(\hat{M}, \hat{g}, \hat{J})$ as the Kähler quotient with respect to the action of the hamiltonian Killing vector field $K := (1/2)\hat{J}\, \mathrm{grad}_{\hat{g}}\, r^2$ on the level set N, where the function $(1/2)r^2$ serves as the moment map for the (local) hamiltonian S^1-action induced by K.

A.2 The Kähler quotient in the presence of a decomposition of the cone into a direct product

By the decomposition theorem for riemannian manifolds [Eis23], the parallel hermitian endomorphisms on a manifold are classified by all the ways the manifold can be decomposed into a direct product of Kähler manifolds. Let $(\hat{M}, \hat{g}, \hat{J})$ be the cone over a Kähler manifold (M, g, J) given by (A.2). Suppose $\hat{g}$ decomposes into a direct product

$$M = \prod_i M_i, \quad \hat{g} = \sum_i \hat{g}_i, \quad \hat{J} = \sum_i \hat{J}_i \tag{A.4}$$

of Kähler manifolds $(\hat{M}_i, \hat{g}_i, \hat{J}_i)$. Recall that the cone structure on $(\hat{M}, \hat{g}, \hat{J})$ gives rise to the cone vector field $\mathcal{C} = r\partial_r$ satisfying $\hat{\nabla}\mathcal{C} = \mathrm{Id}$. Conversely, a vector field satisfying this equation induces a cone structure by defining the radial coordinate to be

$$r := \sqrt{\hat{g}(\mathcal{C}, \mathcal{C})}.$$

The decomposition $\mathcal{C} = \sum_{i=0}^{\ell} \mathcal{C}_i$ of the cone vector field with respect to (A.4) defines cone vector fields $\mathcal{C}_i$ on each component $(\hat{M}_i, \hat{g}_i, \hat{J}_i)$ making them into cones over certain Kähler manifolds (M_i, g_i, J_i). Hence, having a decomposition as in (A.4), we may write

$$\hat{g} = \sum_{i=0}^{\ell} \underbrace{(dr_i^2 + r_i^2(\phi_i^2 + g_i))}_{=\hat{g}_i}. \tag{A.5}$$

Here we allow some of the g_i to be zero, meaning that the corresponding cone $(\hat{M}_i, \hat{g}_i, \hat{J}_i)$ is (complex) one-dimensional over a base of dimension zero. In particular, $\hat{g}_i$ is flat.

As a Kähler riemannian cone, $\hat{g}$ is of the form (A.2). Using $\mathcal{C} = \sum_{i=0}^{\ell} \mathcal{C}_i$ and $\mathcal{C}_i = r_i\partial_{r_i}$, we see that

$$r, K = \tfrac{1}{2}\hat{J}\, \mathrm{grad}_{\hat{g}}\, r^2, \partial_r, dr \text{ and } \phi$$

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relate to the corresponding objects on the components $\hat{g}_i$ of $\hat{g}$ in (A.5) by the equations

$$r^2 = \sum_i r_i^2, \quad K = \sum_i K_i, \quad \partial_r = \frac{1}{r}\sum_i r_i \partial_{r_i},$$
$$dr = \frac{1}{r}\sum_i r_i dr_i \quad \text{and} \quad \phi = \frac{1}{r^2}\sum_i r_i^2 \phi_i. \tag{A.6}$$

Next we describe the Kähler quotient of the direct product metric $\hat{g}$ in (A.5) with respect to the action of the hamiltonian Killing vector field $K := (1/2)\hat{J}\,\mathrm{grad}_{\hat{g}}\, r^2$ on the level set $r = 1$.

Theorem A.1. *The Kähler quotient metric g of the metric $\hat{g}$ is given by the formula*

$$g = \sum_{i=0}^{\ell} dr_i^2 + \frac{1}{2}\sum_{i,j=0}^{\ell} r_i^2 r_j^2 (\phi_i - \phi_j)^2 + \sum_{i=0}^{\ell} r_i^2 g_i. \tag{A.7}$$

Remark A.1. The forms $\phi_i - \phi_j$ are basic, i.e., they can be written as the pullback of forms defined on the quotient. Indeed, these forms vanish upon insertion of ∂_r and K, they do not depend on r and they are K-invariant (that is, invariant with respect to the (local) S^1-action).

Remark A.2. Recall that the metrics g_i in (A.7) are zero if $\hat{g}_i = dr_i^2 + r_i^2(\phi_i^2 + g_i)$ is (complex) one-dimensional.

Proof of Theorem A.1. Restricted to the level set $r = 1$, the quotient metric g is given by

$$g = \hat{g} - \phi^2 = \sum_{i=0}^{\ell} dr_i^2 + \sum_{i=0}^{\ell} r_i^2 \phi_i^2 - \phi^2 + \sum_{i=0}^{\ell} r_i^2 g_i. \tag{A.8}$$

Using (A.6), we obtain

$$\sum_{i=0}^{\ell} r_i^2\phi_i^2 - \phi^2 = \sum_{i=0}^{\ell} r_i^2\phi_i^2 - \sum_{i,j=0}^{\ell} r_i^2 r_j^2 \phi_i \otimes \phi_j = \frac{1}{2}\sum_{i,j=0}^{\ell} r_i^2 r_j^2(\phi_i - \phi_j)^2$$

which gives us formula (A.7). □

In what follows, let $\hat{A}$ be a parallel hermitian endomorphism for $\hat{g}$ with distinct eigenvalues $C_0 < \cdots < C_\ell$ of multiplicities $m_0, \ldots, m_\ell$. Let (A.5) be the decomposition of $\hat{g}$ with respect to the parallel eigenspace distributions of $\hat{A}$. If we consider $\hat{A}$ as a parallel symmetric $(0,2)$-tensor field (by lowering one index with respect to the metric $\hat{g}$), it is given by the formula

$$\hat{A} = \sum_{I=0}^{\ell} C_I (dr_I^2 + r_I^2(\phi_I^2 + h_I)). \tag{A.9}$$

Let us relate the (constant) eigenvalues $C_0, \ldots, C_\ell$ of $\hat{A}$ to the (generically nonconstant) eigenvalues $\xi_1, \ldots, \xi_\ell$ of $A \in Sol(g, J)$ corresponding to $\hat{A}$.

Lemma A.1. *Let $\hat{A}$ be given by (A.9) for numbers $C_0 < \cdots < C_\ell$ and let the cone metric $\hat{g}$ over g be given by (A.5). Let $A \in Sol(g, J)$ correspond to $\hat{A}$ via the isomorphism (A.3). Then the function $p_A \colon \hat{M} \times \mathbb{R} \to \mathbb{R}$, given by*

$$p_A(t) = \frac{1}{r^2}\prod_{i=0}^{\ell}(t - C_i)^{m_i - 1}\sum_{i=0}^{\ell} r_i^2 \prod_{j\neq i}(t - C_j), \tag{A.10}$$

is the characteristic polynomial of A. Moreover, we have

$$C_0 \leqslant \xi_1 \leqslant C_1 \leqslant \cdots \leqslant \xi_\ell \leqslant C_\ell, \tag{A.11}$$

where ξ_i are the ordered nonconstant eigenvalues of A. In particular, ℓ is the number of nonconstant eigenvalues of $A \in Sol(g, J)$ on the base M and the eigenvalues of $\hat{A}$ occurring with multiplicity two or higher are the constant eigenvalues of A.

Remark A.3. The calculations in the proof of Lemma A.1 below are analogous to the derivation of elliptic separation coordinates on the n-sphere; see [Sch14, § 7].

Proof. Recall that A is the horizontal part of $\hat{A}$. Its action on the horizontal distribution

$$\mathcal{H} = \{X \in T\hat{M} : \hat{g}(X, \partial_r) = \hat{g}(X, \hat{J}\partial_r) = 0\}$$

is then given by

$$AX = \hat{A}X - \hat{g}(\hat{A}X, \partial_r)\partial_r - \hat{g}(\hat{A}X, \hat{J}\partial_r)\hat{J}\partial_r.$$

In particular, if ξ is an eigenvalue of A, i.e., $AX = \xi X$ for some nonzero $X \in \mathcal{H}$, we have $(\hat{A} - \xi\,\mathrm{Id})X = \langle \hat{A}X, \partial_r\rangle\partial_r$, where $\langle\cdot,\cdot\rangle = \hat{g} - i\hat{g}(\hat{J}\cdot,\cdot)$ denotes the hermitian inner product associated to $\hat{g}$. Thus, ξ is an eigenvalue of A if and only if there exists $X \neq 0$ such that

$$\langle X, \partial_r\rangle = 0 \quad \text{and} \quad (\hat{A} - \xi\,\mathrm{Id})X = c\partial_r \quad \text{for some } c \in \mathbb{C}.$$

If ξ is not an eigenvalue of $\hat{A}$, this condition is equivalent to $\langle(\hat{A} - \xi\,\mathrm{Id})^{-1}\partial_r, \partial_r\rangle = 0$. Inserting $\hat{A}$ given by (A.9) and $\partial_r = \sum_{i=0}^{\ell}(r_i/r)\partial_{r_i}$, this equation becomes equal to

$$\sum_{i=0}^{\ell} \frac{r_i^2}{C_i - \xi} = 0. \tag{A.12}$$

We obtain that each eigenvalue ξ of A which is not an eigenvalue of $\hat{A}$ must be a solution to this equation. For fixed $r_0, \ldots, r_\ell$, the function $h(\xi) = \sum_{i=0}^{\ell} r_i^2/(C_i - \xi)$ has $\ell + 1$ poles at $C_0, \ldots, C_\ell$ and is monotonously increasing within the intervals (C_i, C_{i+1}). Hence, it has ℓ zeros $\xi_1, \ldots, \xi_\ell$ which are the ℓ nonconstant eigenvalues of A depending on $r_0, \ldots, r_\ell$. We have just seen that these eigenvalues have to satisfy the relation (A.11).

On the other hand, if an eigenvalue C_i of $\hat{A}$ has multiplicity $m_i \geqslant 2$, the corresponding eigenspace must have an $(m_i - 1)$-dimensional intersection with $\mathcal{H}$, hence, C_i is also a constant eigenvalue of A of multiplicity $m_i - 1$. The number of eigenvalues of A found so far is

$$\ell + \sum_{i=0}^{\ell}(m_i - 1) = -1 + \sum_{i=0}^{\ell} m_i = -1 + \dim\hat{M} = \dim M.$$

Thus, we have certainly found all eigenvalues of A.

Multiplying (A.12) with $\prod_{i=0}^{\ell}(C_i - \xi)$, we obtain $\sum_{i=0}^{\ell} r_i^2 \prod_{j\neq i}(C_j - \xi) = 0$. The left-hand side is a polynomial in ξ of degree ℓ, and, since the nonconstant eigenvalues $\xi_1, \ldots, \xi_\ell$ are the roots of this polynomial, we obtain

$$p_{\mathrm{nc}}(t) = \frac{1}{r^2}\sum_{i=0}^{\ell} r_i^2 \prod_{j\neq i}(t - C_j),$$

where $p_{\mathrm{nc}}(t) = \prod_{i=1}^{\ell}(t - \xi_i)$ is the nonconstant part of the characteristic polynomial of A. The characteristic polynomial of A is then given by formula (A.10). □

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Denote by $\xi_1,\dots,\xi_\ell$ the nonconstant eigenvalues of A and by η its constant eigenvalues of multiplicity m_η. The characteristic polynomial $p_A(t)$, expressed in terms of the radial coordinates r_i, is given by (A.10); hence, we obtain the relation

$$\prod_{i=1}^{\ell}(t-\xi_i)=\frac{1}{r^2}\sum_{I=0}^{\ell} r_I^2\prod_{J\neq I}(t-C_J), \tag{A.13}$$

between the two sets of functions $\{\xi_1,\dots,\xi_\ell\}$ and $\{r_0,\dots,r_\ell\}$. Inserting $t=C_I$ into formula (A.13), we obtain the functions r_I explicitly as functions of the ξ_i:

$$r_I^2=\frac{\prod_{i=1}^{\ell}(C_I-\xi_i)}{\prod_{J\neq I}(C_I-C_J)}. \tag{A.14}$$

Differentiating yields

$$2r_I\,dr_I=-\sum_{i=1}^{\ell}\frac{\prod_{j\neq i}(C_I-\xi_j)}{\prod_{J\neq I}(C_I-C_J)}\,d\xi_i. \tag{A.15}$$

A.3 A local description of $C_{\mathbb{C}}(-1)$-metrics

We rederive the part of Theorem 3 stating necessary conditions on the parameters from formula (9) for g being $C_{\mathbb{C}}(-1)$.

Proposition A.1. *Consider a $C_{\mathbb{C}}(-1)$ metric g given by formula (9) with respect to some $A\in Sol(g,J)$ with nonconstant eigenvalues $\xi_1,\dots,\xi_\ell$. Let $C_0<\cdots<C_\ell$ be the distinct eigenvalues of the corresponding parallel hermitian endomorphism $\hat{A}$ on the cone. Then $\Theta_j(t)=-4\prod_{I=0}^{\ell}(t-C_I)$ for $j=1,\dots,\ell$.*

Proof. The part of the metric g in (9) involving the $d\xi_i$ corresponds to the part $\sum_{I=0}^{\ell}dr_I^2$ of g in (A.7). Using (A.14) and (A.15), we obtain

$$4\,dr_I^2=\frac{\sum_{i_1,i_2=1}^{\ell}\prod_{j_1\neq i_1}(C_I-\xi_{j_1})\prod_{j_2\neq i_2}(C_I-\xi_{j_2})\,d\xi_{i_1}\otimes d\xi_{i_2}}{\prod_{J\neq I}(C_I-C_J)\prod_{i=1}^{\ell}(C_I-\xi_i)}. \tag{A.16}$$

Let $4\sum_{I=0}^{\ell}dr_I^2=:A_{i_1i_2}\,d\xi_{i_1}\otimes d\xi_{i_2}$. For $i_1\neq i_2$, (A.16) implies that

$$A_{i_1i_2}=\sum_{I=0}^{\ell}\frac{\prod_{j\neq i_1,i_2}(C_I-\xi_j)}{\prod_{J\neq I}(C_I-C_J)}.$$

The numerator of each term in this sum is a polynomial of degree $\ell-2$ in C_I, hence, applying a Vandermonde identity (see, for instance, the appendix of [ACG06]) in the $\ell+1$ variables $C_0,\dots,C_\ell$, we see that $A_{i_1i_2}=0$ for $i_1\neq i_2$. For the case $i=i_1=i_2$, we obtain

$$A_{ii}=\sum_{I=0}^{\ell}\frac{\prod_{j\neq i}(C_I-\xi_j)}{\prod_{J\neq I}(C_I-C_J)(C_I-\xi_i)}. \tag{A.17}$$

The numerator of each term in this sum is a polynomial of degree $\ell-1$ in C_I. Applying Vandermonde identities with respect to the $\ell+2$ variables $C_0,\dots,C_\ell,\xi_i$, we obtain that

$$A_{ii}=-\frac{\prod_{j\neq i}(\xi_i-\xi_j)}{\prod_{I=0}^{\ell}(\xi_i-C_I)}.$$

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Thus we have

$$\sum_{i=0}^{\ell} dr_i^2 = -\sum_{i=1}^{\ell} \frac{\prod_{j\neq i}(\xi_i - \xi_j)}{4\prod_{I=0}^{\ell}(\xi_i - C_I)} d\xi_i^2.$$

Comparing this with (9), we see that $\Theta_i(t) = -4\prod_{I=0}^{\ell}(t - C_I)$ as we claimed. □

A.4 $C_{\mathbb{C}}(-1)$-metrics with mobility at least three

The cone construction provides a more geometric explanation why the conditions on the eigenvalues in Theorem 6 imply that the mobility is at least three: since for a $C_{\mathbb{C}}(-1)$ metric g the space $Sol(g, J)$ is isomorphic to the space of parallel hermitian endomorphisms on the cone $(\hat{M}, \hat{g}, \hat{J})$, the decomposition theorem for riemannian manifolds [Eis23] implies that the mobility $D(g, J) = \dim Sol(g, J)$ is given by

$$D(g, J) = f^2 + i, \tag{A.18}$$

where f is the complex dimension of the flat part and i is the number of irreducible (nonflat) components of $\hat{g}$ (see also [MR15]). Let $C_0 \leqslant \cdots \leqslant C_n$ denote the (not necessarily distinct) eigenvalues of a parallel hermitian endomorphism $\hat{A}$ on $\hat{M}$, and let A be the corresponding element of $Sol(g, J)$. Lemma A.1 shows that each repeated eigenvalue $C_{i-1} = C_i$ of $\hat{A}$ gives rise to a constant eigenvalue of A, while each gap $C_{j-1} < C_j$ gives rise to a nonconstant eigenvalue of A taking values in the interval $[C_{j-1}, C_j]$. This explains the assumptions in Theorem 6: if the number of nonconstant eigenvalues of A is two or more or the number of constant eigenvalues of A is three or more, then the number of distinct eigenvalues of $\hat{A}$ must be three or more. Now, given a parallel hermitian endomorphism $\hat{A}$ on the cone with at least three distinct eigenvalues, the decomposition theorem, together with formula (A.18), shows that the mobility is at least three.

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