Bull. Aust. Math. Soc. **107** (2023), 416–431
doi:10.1017/S0004972722000910

ON THE WARING–GOLDBACH PROBLEM FOR ONE SQUARE, FOUR CUBES AND ONE BIQUADRATE

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(Received 25 June 2022; accepted 12 July 2022; first published online 15 September 2022)

Abstract

Let N be a sufficiently large integer. We prove that, with at most $O(N^{23/48+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers.

2020 *Mathematics subject classification*: primary 11P05; secondary 11P32, 11P55.

Keywords and phrases: Waring–Goldbach problem, Hardy–Littlewood method, exceptional set.

1. Introduction and main result

Additive representations of Waring's problem over natural numbers by mixtures of squares, cubes and biquadrates are among the more interesting special cases for testing the general expectation that any sufficiently large natural number n is representable in the form $n = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}$, as soon as $\sum_{j=1}^{s} k_j^{-1}$ is reasonably large. With the exception of a handful of very special problems, in the current state of knowledge, this sum must exceed 2, at the very least, to successfully apply the Hardy–Littlewood method.

In 1999, Brüdern and Wooley [1] removed a case from the list of those combinations of exponents which have defied treatment. Let $\nu(n)$ denote the number of representations of the natural number n as the sum of a square, four cubes and a biquadrate. Then $\nu(n) \gg n^{13/12}$. It is remarkable that this lower bound is of the same order of magnitude as the main term of the conjectured asymptotic formula for $\nu(n)$ predicted by a formal application of the circle method. This result should be compared with the work of Vaughan [6], who obtained a theorem of similar strength for the sum of one square and five cubes. The result established by Vaughan [6] was strengthened by Cai [2], Li and Zhang [4] and Xue *et al.* [10].

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11901566, 12001047, 11971476 and 12071238) and the Fundamental Research Funds for the Central Universities (Grant No. 2022YQLX05).

Based on the result of Brüdern and Wooley [1], it is reasonable to conjecture that every sufficiently large even integer n can be represented as

$$n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4,$$

where $p_1, \ldots, p_6$ are prime numbers. This conjecture is probably far outside the reach of current analytic number theory techniques. We shall investigate the exceptional set in the above representation and establish the following result.

THEOREM 1.1. *Let $E(N)$ denote the number of positive even integers n up to N, which cannot be represented as $n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4$. Then, for any $\varepsilon > 0$,*

$$E(N) \ll N^{23/48+\varepsilon}.$$

2. Outline of the proof of Theorem 1.1

Let N be a sufficiently large positive integer. By a splitting argument, it is sufficient to consider the even integers $n \in (N/2, N]$. For the application of the Hardy–Littlewood method, we need to define the Farey dissection. Let $A > 0$ be a sufficiently large fixed number, which will be determined at the end of the proof. We set

$$Q_0 = \log^A N, \quad Q_1 = N^{1/48}, \quad Q_2 = N^{47/48}, \quad \mathfrak{I}_0 = \Big[-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\Big].$$

By Dirichlet's lemma on rational approximation (see, for example, [7, Lemma 2.1]), each $\alpha \in [-1/Q_2, 1 - 1/Q_2]$ can be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leqslant \frac{1}{qQ_2}, \tag{2.1}$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q_2$ and $(a, q) = 1$. Define

$$\mathfrak{M}(q, a) = \Big[\frac{a}{q} - \frac{Q_1}{qN}, \frac{a}{q} + \frac{Q_1}{qN}\Big], \quad \mathfrak{M} = \bigcup_{1 \leqslant q \leqslant Q_1} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(q, a),$$

$$\mathfrak{M}_0(q, a) = \Big[\frac{a}{q} - \frac{Q_0^{200}}{qN}, \frac{a}{q} + \frac{Q_0^{200}}{qN}\Big], \quad \mathfrak{M}_0 = \bigcup_{1 \leqslant q \leqslant Q_0^{100}} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_0(q, a),$$

$$\mathfrak{m}_1 = \mathfrak{I}_0 \backslash \mathfrak{M}, \quad \mathfrak{m}_2 = \mathfrak{M} \backslash \mathfrak{M}_0.$$

Then we obtain the Farey dissection

$$\mathfrak{I}_0 = \mathfrak{M}_0 \cup \mathfrak{m}_1 \cup \mathfrak{m}_2. \tag{2.2}$$

For $k = 2, 3, 4$, we define

$$f_k(\alpha) = \sum_{X_k < p \leqslant 2X_k} e(p^k \alpha),$$

https://doi.org/10.1017/S0004972722000910 Published online by Cambridge University Press

where $X_k = (N/16)^{1/k}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n=p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4\\ X_3<p_2,\dots,p_5\leqslant 2X_3\\ X_2<p_1\leqslant 2X_2\\ X_4<p_6\leqslant 2X_4}} 1.$$

From (2.2),

$$\begin{aligned}\mathscr{R}(n) &= \int_0^1 f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha = \int_{-1/Q_2}^{1-1/Q_2} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha\\ &= \Big\{\int_{\mathfrak{M}_0}+\int_{\mathfrak{m}_1}+\int_{\mathfrak{m}_2}\Big\}f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha.\end{aligned}$$

To prove Theorem 1.1, we need the following two propositions.

PROPOSITION 2.1. *For $n \in (N/2, N]$,*

$$\int_{\mathfrak{M}_0} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O\Big(\frac{n^{13/12}}{\log^7 n}\Big), \tag{2.3}$$

where $\mathfrak{S}(n)$ is the singular series defined in (4.1), which is absolutely convergent and satisfies

$$0 < c^* \leqslant \mathfrak{S}(n) \ll 1 \tag{2.4}$$

for any integer n satisfying $n \equiv 0 \pmod 2$ and some fixed constant $c^* > 0$; while $\mathfrak{J}(n)$ is defined by (4.4) and satisfies

$$\mathfrak{J}(n) \asymp \frac{n^{13/12}}{\log^6 n}.$$

The proof of (2.3) in Proposition 2.1 will be demonstrated in Section 4. For the property (2.4) of the singular series, we shall give the proof in Section 5.

PROPOSITION 2.2. *Let $\mathcal{Z}(N)$ denote the number of integers $n \in (N/2, N]$ satisfying $n \equiv 0 \pmod 2$ such that*

$$\sum_{j=1}^{2}\Big|\int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha\Big| \gg \frac{n^{13/12}}{\log^7 n}.$$

Then

$$\mathcal{Z}(N) \ll N^{23/48+\varepsilon}.$$

The proof of Proposition 2.2 will be given in Section 6. The rest of this section is devoted to establishing Theorem 1.1 by using Propositions 2.1 and 2.2.

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PROOF OF THEOREM 1.1. From Proposition 2.2, we deduce that, with at most $O(N^{23/48+\varepsilon})$ exceptions, all even integers $n \in (N/2, N]$ satisfy

$$\sum_{j=1}^{2} \left| \int_{\mathfrak{m}_j} f_2(\alpha) f_3^4(\alpha) f_4(\alpha) e(-n\alpha)\, d\alpha \right| \ll \frac{n^{13/12}}{\log^7 n}.$$

By Proposition 2.1, we conclude that, with at most $O(N^{23/48+\varepsilon})$ exceptions, all even integers $n \in (N/2, N]$ can be represented in the form $p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers. By a splitting argument, we get

$$E(N) \ll \sum_{0 \leqslant \ell \ll \log N} \mathcal{Z}\Big(\frac{N}{2^\ell}\Big) \ll \sum_{0 \leqslant \ell \ll \log N} \Big(\frac{N}{2^\ell}\Big)^{23/48+\varepsilon} \ll N^{23/48+\varepsilon}.$$

This completes the proof of Theorem 1.1. □

3. Some auxiliary lemmas

LEMMA 3.1 [5, Theorem 1.1]. *Suppose that α is a real number and that $|\alpha - a/q| \leqslant q^{-2}$ with $(a, q) = 1$. Let $\beta = \alpha - a/q$. Then*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log N)^c \Big(X_k^{1/2} \sqrt{q(1 + N|\beta|)} + X_k^{4/5} + \frac{X_k}{\sqrt{q(1 + N|\beta|)}} \Big),$$

where $\delta_k = 1/2 + \log k/\log 2$, $d(q)$ is the Dirichlet divisor function and c is a constant.

LEMMA 3.2 [11, Lemma 2.4]. *Suppose that α is a real number and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \leqslant q \leqslant Q$ and $|q\alpha - a| \leqslant Q^{-1}$. If $P^{1/12} \leqslant Q \leqslant P^{47/12}$, then*

$$\sum_{P < p \leqslant 2P} e(p^4 \alpha) \ll P^{1-1/24+\varepsilon} + \frac{P^{1+\varepsilon}}{\sqrt{q(1 + P^4 |\alpha - a/q|)}}.$$

LEMMA 3.3. *For $\alpha \in \mathfrak{m}_1$, we have $f_4(\alpha) \ll N^{23/96+\varepsilon}$.*

PROOF. By Dirichlet's rational approximation (2.1), for $\alpha \in \mathfrak{m}_1$, one has $Q_1 < q \leqslant Q_2$ and $qX_4^4|\alpha - a/q| \leqslant Q_1$. From Lemma 3.2,

$$f_4(\alpha) \ll X_4^{23/24+\varepsilon} + X_4^{1+\varepsilon} Q_1^{-1/2} \ll N^{23/96+\varepsilon}.$$ □

For $1 \leqslant a \leqslant q$ with $(a, q) = 1$, set

$$\mathcal{I}(q, a) = \Big[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \Big], \quad \mathcal{I} = \bigcup_{1 \leqslant q \leqslant Q_0} \bigcup_{\substack{a=-q \\ (a,q)=1}}^{2q} \mathcal{I}(q, a). \tag{3.1}$$

For $\alpha \in \mathfrak{m}_2$, by Lemma 3.1,

$$f_4(\alpha) \ll \frac{N^{1/4} \log^c N}{q^{1/2-\varepsilon}(1 + N|\lambda|)^{1/2}} + N^{1/5+\varepsilon} = V_4(\alpha) + N^{1/5+\varepsilon}, \tag{3.2}$$

say. Then we obtain the following Lemma.

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LEMMA 3.4. *We have*

$$\int_{\mathcal{I}} |V_4(\alpha)|^2\, d\alpha = \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |V_4(\alpha)|^2\, d\alpha \ll N^{-1/2}\log^{2A} N.$$

PROOF. We have

$$\begin{aligned}\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |V_4(\alpha)|^2\, d\alpha &\ll \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{|\lambda|\leqslant 1/Q_0} \frac{N^{1/2}\log^{2c} N}{1+N|\lambda|}\, d\lambda \\ &\ll \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \Big(\int_{|\lambda|\leqslant 1/N} N^{1/2}\log^{2c} N\, d\lambda + \int_{1/N\leqslant|\lambda|\leqslant 1/Q_0} \frac{N^{1/2}\log^{2c} N}{N|\lambda|}\, d\lambda\Big) \\ &\ll N^{-1/2}\log^{3c} N \cdot \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon}\varphi(q) \ll N^{-1/2}Q_0^{1+\varepsilon}\log^{3c} N \ll N^{-1/2}\log^{2A} N.\end{aligned}$$

This completes the proof of Lemma 3.4. □

4. Proof of Proposition 2.1

In this section, we establish Proposition 2.1. We first introduce some notation. For a Dirichlet character χ mod q and $k \in \{2, 3, 4\}$, we define

$$C_k(\chi, a) = \sum_{h=1}^{q} \overline{\chi(h)}\, e\Big(\frac{ah^k}{q}\Big), \quad C_k(q, a) = C_k(\chi^0, a),$$

where χ^0 is the principal character modulo q. Let $\chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}, \chi_3^{(4)}, \chi_4$ be Dirichlet characters modulo q. Define

$$B(n, q, \chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}, \chi_3^{(4)}, \chi_4) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \Big(C_2(\chi_2, a)\Big(\prod_{i=1}^{4} C_3(\chi_3^{(i)}, a)\Big)C_4(\chi_4, a)\Big)e\Big(-\frac{an}{q}\Big),$$

$$B(n, q) = B(n, q, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0),$$

and write

$$A(n, q) = \frac{B(n, q)}{\varphi^6(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{4.1}$$

LEMMA 4.1 [8, Ch. VI, Problem 14]. *For $(a, q) = 1$ and any Dirichlet character χ mod q,*

$$|C_k(\chi, a)| \leqslant 2q^{1/2} d^{\beta_k}(q)$$

with $\beta_k = \log k/\log 2$, where $d(q)$ is the Dirichlet divisor function.

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LEMMA 4.2. *Let $C_k(q,a)$ be defined as above. Then*

$$\sum_{q\leqslant x}\frac{|B(n,q)|}{\varphi^6(q)} \ll \log x. \tag{4.2}$$

PROOF. By Lemma 4.1,

$$B(n,q) \ll \sum_{\substack{a=1\\(a,q)=1}}^{q} |C_2(q,a)C_3^4(q,a)C_4(q,a)| \ll q^3\varphi(q)d^{10}(q).$$

Therefore, the left-hand side of (4.2) is

$$\ll \sum_{q\leqslant x}\frac{q^3\varphi(q)d^{10}(q)}{\varphi^6(q)} \ll \sum_{q\leqslant x}\frac{(\log\log q)^5 d^{10}(q)}{q^2} \ll (\log\log x)^5\sum_{q\leqslant x}\frac{d^{10}(q)}{q^2} \ll \log x.$$

This completes the proof of Lemma 4.2. □

To treat the integral on the major arcs, we write $f_k(\alpha)$ as follows:

$$f_k(\alpha) = \sum_{\substack{X_k<p\leqslant 2X_k\\(p,q)=1}} e\Big(p^k\Big(\frac{a}{q}+\lambda\Big)\Big) + O(\log q) = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e\Big(\frac{a\ell^k}{q}\Big)\sum_{\substack{X_k<p\leqslant 2X_k\\p\equiv\ell \ (\mathrm{mod}\ q)}} e(p^k\lambda) + O(\log N).$$

For the innermost sum on the right-hand side of the above equation, by the Siegel–Walfisz theorem, we have

$$\begin{aligned}\sum_{\substack{X_k<p\leqslant 2X_k\\p\equiv\ell \ (\mathrm{mod}\ q)}} e(p^k\lambda) &= \int_{X_k}^{2X_k} e(u^k\lambda)\,d\pi(u,q,\ell)\\ &= \int_{X_k}^{2X_k} e(u^k\lambda)\,d\Big(\frac{1}{\varphi(q)}\int_2^u\frac{dt}{\log t} + O(ue^{-c\sqrt{\log u}})\Big)\\ &= \frac{1}{\varphi(q)}\int_{X_k}^{2X_k}\frac{e(u^k\lambda)}{\log u}\,du + O(X_k e^{-c\sqrt{\log N}})\\ &= \frac{v_k(\lambda)}{\varphi(q)} + O(X_k e^{-c\sqrt{\log N}}),\end{aligned}$$

say. Therefore,

$$f_k(\alpha) = \frac{C_k(q,a)}{\varphi(q)}v_k(\lambda) + O(X_k e^{-c\sqrt{\log N}}),$$

and thus

$$f_2(\alpha)f_3^4(\alpha)f_4(\alpha) = \frac{C_2(q,a)C_3^4(q,a)C_4(q,a)}{\varphi^6(q)}v_2(\lambda)v_3^4(\lambda)v_4(\lambda) + O(N^{25/12}e^{-c\sqrt{\log N}}).$$

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From this, we derive

$$\begin{aligned}\int_{\mathfrak{M}_0} & f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha \\ &= \sum_{1\leqslant q\leqslant Q_0^{100}} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big)\int_{-Q_0^{200}/qN}^{Q_0^{200}/qN}\Big(\frac{C_2(q,a)C_3^4(q,a)C_4(q,a)}{\varphi^6(q)}v_2(\lambda)v_3^4(\lambda)v_4(\lambda) \\ &\qquad + O(N^{25/12}e^{-c\sqrt{\log N}})\Big)e(-n\lambda)\,d\lambda \\ &= \sum_{1\leqslant q\leqslant Q_0^{100}} \frac{B(n,q)}{\varphi^6(q)}\int_{-Q_0^{200}/qN}^{Q_0^{200}/qN} v_2(\lambda)v_3^4(\lambda)v_4(\lambda)e(-n\lambda)\,d\lambda + O(N^{13/12}e^{-c\sqrt{\log N}}). \end{aligned} \tag{4.3}$$

Note that

$$v_k(\lambda) = \int_{X_k^k}^{(2X_k)^k} \frac{x^{1/k-1}e(\lambda x)}{\log x}\,dx \ll \frac{N^{1/k-1}}{\log N}\cdot \min\Big(N, \frac{1}{|\lambda|}\Big),$$

using an elementary estimate. If we extend the interval of the innermost integral over λ in (4.3) to $[-1/2, 1/2]$, then the resulting error is

$$\ll \int_{Q_0^{200}/qN}^{1/2} \frac{N^{-47/12}}{\log^6 N}\cdot\frac{d\lambda}{\lambda^6} \ll \frac{N^{-47/12}}{\log^6 N}\cdot\frac{q^5N^5}{Q_0^{1000}} \ll \frac{N^{13/12}}{(\log N)^{500A}} \ll \frac{n^{13/12}}{(\log n)^{500A}}.$$

Hence, we obtain

$$\int_{-Q_0^{200}/qN}^{Q_0^{200}/qN} v_2(\lambda)v_3^4(\lambda)v_4(\lambda)e(-n\lambda)\,d\lambda = \mathfrak{J}(n) + O\Big(\frac{n^{13/12}}{(\log n)^{500A}}\Big),$$

where

$$\begin{aligned}\mathfrak{J}(n) &= \int_{-1/2}^{1/2}\Big(\int_{X_2^2}^{(2X_2)^2}\frac{x^{-1/2}e(\lambda x)}{\log x}\,dx\Big)\Big(\int_{X_3^3}^{(2X_3)^3}\frac{x^{-2/3}e(\lambda x)}{\log x}\,dx\Big)^4 \\ &\qquad\times\Big(\int_{X_4^4}^{(2X_4)^4}\frac{x^{-3/4}e(\lambda x)}{\log x}\,dx\Big)e(-n\lambda)\,d\lambda \\ &= \int_{X_2^2}^{(2X_2)^2}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_4^4}^{(2X_4)^4}\int_{-1/2}^{1/2}\frac{x_1^{-1/2}(x_2x_3x_4x_5)^{-2/3}x_6^{-3/4}}{(\log x_1)(\log x_2)\cdots(\log x_6)} \\ &\qquad\times e((x_1+x_2+x_3+x_4+x_5+x_6-n)\lambda)\,d\lambda\,dx_1\cdots dx_6 \\ &\asymp \frac{X_2^{-1}X_3^{-8}X_4^{-3}}{(\log N)^6}\int_{X_2^2}^{(2X_2)^2}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_4^4}^{(2X_4)^4}\int_{-1/2}^{1/2} \\ &\qquad\times e((x_1+x_2+x_3+x_4+x_5+x_6-n)\lambda)\,d\lambda\,dx_1\cdots dx_6 \\ &\asymp \frac{X_2^{-1}X_3^{-8}X_4^{-3}}{(\log N)^6}N^5 \asymp \frac{N^{13/12}}{(\log N)^6} \asymp \frac{n^{13/12}}{(\log n)^6}. \end{aligned}\tag{4.4}$$

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Therefore, by (5.4) and Lemma 4.2, (4.3) becomes

$$\int_{\mathfrak{M}_0} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O\Big(\frac{n^{13/12}}{\log^7 n}\Big),$$

which completes the proof of Proposition 2.1.

5. The singular series

In this section, we investigate the properties of the singular series which appears in Proposition 2.1.

LEMMA 5.1 [3, Lemma 8.3]. *Let p be a prime and $p^\alpha \| k$. For $(a,p)=1$, if $\ell \geqslant \gamma(p)$, then $C_k(p^\ell, a) = 0$, where*

$$\gamma(p) = \begin{cases} \alpha+2 & \text{if } p \neq 2 \text{ or } p=2,\ \alpha=0 \\ \alpha+3 & \text{if } p=2,\ \alpha>0. \end{cases}$$

For $k \geqslant 1$, we define

$$S_k(q,a) = \sum_{m=1}^{q} e\Big(\frac{am^k}{q}\Big).$$

LEMMA 5.2 [7, Lemma 4.3]. *Suppose that $(p,a)=1$. Then*

$$S_k(p,a) = \sum_{\chi\in\mathscr{A}_k} \overline{\chi(a)}\tau(\chi),$$

where $\mathscr{A}_k$ denotes the set of nonprincipal characters χ modulo p for which χ^k is principal and $\tau(\chi)$ denotes the Gauss sum

$$\sum_{m=1}^{p} \chi(m)e\Big(\frac{m}{p}\Big).$$

Also, $|\tau(\chi)| = p^{1/2}$ and $|\mathscr{A}_k| = (k,p-1)-1$.

LEMMA 5.3. *For $(p,n)=1$,*

$$\Big|\sum_{a=1}^{p-1} \frac{S_2(p,a)S_3^4(p,a)S_4(p,a)}{p^6} e\Big(-\frac{an}{p}\Big)\Big| \leqslant 48p^{-5/2}. \tag{5.1}$$

PROOF. We denote by $\mathcal{S}$ the sum in the absolute value signs on the left-hand side of (5.1). By Lemma 5.2,

$$\mathcal{S} = \frac{1}{p^6}\sum_{a=1}^{p-1}\Big(\sum_{\chi_2\in\mathscr{A}_2}\overline{\chi_2(a)}\tau(\chi_2)\Big)\Big(\sum_{\chi_3\in\mathscr{A}_3}\overline{\chi_3(a)}\tau(\chi_3)\Big)^4\Big(\sum_{\chi_4\in\mathscr{A}_4}\overline{\chi_4(a)}\tau(\chi_4)\Big)e\Big(-\frac{an}{p}\Big).$$

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If $|\mathscr{A}_k| = 0$ for some $k \in \{2, 3, 4\}$, then $\mathcal{S} = 0$. If this is not the case, then

$$\mathcal{S} = \frac{1}{p^6} \sum_{\chi_2 \in \mathscr{A}_2} \sum_{\chi_3^{(1)} \in \mathscr{A}_3} \sum_{\chi_3^{(2)} \in \mathscr{A}_3} \sum_{\chi_3^{(3)} \in \mathscr{A}_3} \sum_{\chi_3^{(4)} \in \mathscr{A}_3} \sum_{\chi_4 \in \mathscr{A}_4} \tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})\tau(\chi_3^{(4)})\tau(\chi_4)$$
$$\times \sum_{a=1}^{p-1} \overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)\chi_3^{(4)}(a)\chi_4(a)}\, e\Big(-\frac{an}{p}\Big).$$

From Lemma 5.2, the sextuple outer sums have not more than

$$((2, p-1) - 1) \times ((3, p-1) - 1)^4 \times ((4, p-1) - 1) \leqslant 1 \times 2^4 \times 3 = 48$$

terms. In each of these terms, $|\tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})\tau(\chi_3^{(4)})\tau(\chi_4)| = p^3$. Since in any one of these terms $\overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)\chi_3^{(4)}(a)\chi_4(a)}$ is a Dirichlet character $\chi (\mathrm{mod}\ p)$, the inner sum is

$$\sum_{a=1}^{p-1} \chi(a) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)} \sum_{a=1}^{p-1} \chi(-an) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)}\tau(\chi).$$

Because $\tau(\chi^0) = -1$ for the principal character χ^0 mod p, we have $|\overline{\chi(-n)}\tau(\chi)| \leqslant p^{1/2}$. By the above arguments, we obtain

$$|\mathcal{S}| \leqslant \frac{1}{p^6} \cdot 48 \cdot p^3 \cdot p^{1/2} = 48 p^{-5/2}.$$

This completes the proof of Lemma 5.3. □

LEMMA 5.4. *Let $\mathcal{L}(p, n)$ denote the number of solutions of the congruence*

$$x_1^2 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^4 \equiv n \ (\mathrm{mod}\ p), \qquad 1 \leqslant x_1, x_2, \ldots, x_6 \leqslant p-1.$$

Then, for $n \equiv 0 (\mathrm{mod}\ 2)$, we have $\mathcal{L}(p, n) > 0$.

PROOF. We have

$$p \cdot \mathcal{L}(p, n) = \sum_{a=1}^{p} C_2(p, a) C_3^4(p, a) C_4(p, a) e\Big(-\frac{an}{p}\Big) = (p-1)^6 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_2(p, a) C_3^4(p, a) C_4(p, a) e\Big(-\frac{an}{p}\Big).$$

By Lemma 5.2, $|E_p| \leqslant (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)^4(3\sqrt{p}+1)$. It is easy to check that $|E_p| < (p-1)^6$ for $p \geqslant 13$. Therefore, we obtain $\mathcal{L}(p, n) > 0$ for $p \geqslant 13$. For $p = 2, 3, 5, 7, 11$, we can check $\mathcal{L}(p, n) > 0$ directly provided that $n \equiv 0 \ (\mathrm{mod}\ 2)$. □

LEMMA 5.5. *$A(n, q)$ is multiplicative in q.*

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PROOF. By the definition of $A(n,q)$ in (4.1), we only need to show that $B(n,q)$ is multiplicative in q. Suppose $q=q_1q_2$ with $(q_1,q_2)=1$. Then

$$\begin{aligned}B(n,q_1q_2) &= \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} C_2(q_1q_2,a)C_3^4(q_1q_2,a)C_4(q_1q_2,a)e\Big(-\frac{an}{q_1q_2}\Big)\\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_1q_2,a_1q_2+a_2q_1)C_3^4(q_1q_2,a_1q_2+a_2q_1)\\ &\qquad\times C_4(q_1q_2,a_1q_2+a_2q_1)e\Big(-\frac{a_1n}{q_1}\Big)e\Big(-\frac{a_2n}{q_2}\Big).\end{aligned}\tag{5.2}$$

For $(q_1,q_2)=1$ and $k\in\{2,3,4\}$,

$$\begin{aligned}C_k(q_1q_2,a_1q_2+a_2q_1) &= \sum_{\substack{m=1\\(m,q_1q_2)=1}}^{q_1q_2} e\Big(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\Big)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\Big(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\Big)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1} e\Big(\frac{a_1(m_1q_2)^k}{q_1}\Big)\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\Big(\frac{a_2(m_2q_1)^k}{q_2}\Big)\\ &= C_k(q_1,a_1)C_k(q_2,a_2).\end{aligned}\tag{5.3}$$

Putting (5.3) into (5.2), we deduce that

$$\begin{aligned}B(n,q_1q_2) &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} C_2(q_1,a_1)C_3^4(q_1,a_1)C_4(q_1,a_1)e\Big(-\frac{a_1n}{q_1}\Big)\\ &\qquad\times\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_2,a_2)C_3^4(q_2,a_2)C_4(q_2,a_2)e\Big(-\frac{a_2n}{q_2}\Big)\\ &= B(n,q_1)B(n,q_2).\end{aligned}$$

This completes the proof of Lemma 5.5. □

LEMMA 5.6. *Let $A(n,q)$ be as defined in (4.1).*

(i) *We have*

$$\sum_{q>Z}|A(n,q)| \ll Z^{-3/2+\varepsilon}d(n).\tag{5.4}$$

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(ii) *There exists an absolute positive constant* $c^* > 0$, *such that, for* $n \equiv 0 \pmod 2$,

$$0 < c^* \leqslant \mathfrak{S}(n) \ll 1.$$

PROOF. From Lemma 5.5, $B(n,q)$ is multiplicative in q. Therefore,

$$B(n,q) = \prod_{p^t \| q} B(n,p^t) = \prod_{p^t \| q} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^t} C_2(p^t,a)C_3^4(p^t,a)C_4(p^t,a)e\Big(-\frac{an}{p^t}\Big). \tag{5.5}$$

From (5.5) and Lemma 5.1, $B(n,q) = \prod_{p\|q} B(n,p)$ or 0 according to if q is square-free or not. Thus,

$$\sum_{q=1}^{\infty} A(n,q) = \sum_{\substack{q=1 \\ q \text{ square-free}}}^{\infty} A(n,q). \tag{5.6}$$

Write

$$\mathcal{R}(p,a) := C_2(p,a)C_3^4(p,a)C_4(p,a) - S_2(p,a)S_3^4(p,a)S_4(p,a).$$

Then

$$\begin{aligned} A(n,p) &= \frac{1}{(p-1)^6} \sum_{a=1}^{p-1} S_2(p,a)S_3^4(p,a)S_4(p,a)e\Big(-\frac{an}{p}\Big) \\ &\quad + \frac{1}{(p-1)^6} \sum_{a=1}^{p-1} \mathcal{R}(p,a)e\Big(-\frac{an}{p}\Big). \end{aligned} \tag{5.7}$$

Applying Lemma 4.1 and noticing that $S_k(p,a) = C_k(p,a) + 1$, we get $S_k(p,a) \ll p^{1/2}$, and thus $\mathcal{R}(p,a) \ll p^{5/2}$. Therefore, the absolute value of the second term in (5.7) is $\leqslant c_1 p^{-5/2}$. However, from Lemma 5.3, we can see that the absolute value of the first term in (5.7) is $\leqslant 2^6 \cdot 48 p^{-5/2} = 3072 p^{-5/2}$. Let $c_2 = c_1 + 3072$. Then we have proved that for $p \nmid n$,

$$|A(n,p)| \leqslant c_2 p^{-5/2}. \tag{5.8}$$

Moreover, if we use Lemma 4.1 directly, it follows that

$$\begin{aligned} |B(n,p)| &= \Big| \sum_{a=1}^{p-1} C_2(p,a)C_3^4(p,a)C_4(p,a)e\Big(-\frac{an}{p}\Big) \Big| \leqslant \sum_{a=1}^{p-1} |C_2(p,a)C_3^4(p,a)C_4(p,a)| \\ &\leqslant (p-1) \cdot 2^6 \cdot p^3 \cdot 648 = 41472 p^3 (p-1), \end{aligned}$$

and therefore

$$|A(n,p)| = \frac{|B(n,p)|}{\varphi^6(p)} \leqslant \frac{41472 p^3}{(p-1)^5} \leqslant \frac{2^5 \cdot 41472 p^3}{p^5} = \frac{1327104}{p^2}. \tag{5.9}$$

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Let $c_3 = \max(c_2, 1327104)$. Then for square-free q,

$$|A(n,q)| = \Big(\prod_{\substack{p|q\\ p\nmid n}} |A(n,p)|\Big)\Big(\prod_{\substack{p|q\\ p|n}} |A(n,p)|\Big) \leqslant \Big(\prod_{\substack{p|q\\ p\nmid n}} (c_3 p^{-5/2})\Big)\Big(\prod_{\substack{p|q\\ p|n}} (c_3 p^{-2})\Big)$$
$$= c_3^{\omega(q)}\Big(\prod_{p|q} p^{-5/2}\Big)\Big(\prod_{p|(n,q)} p^{1/2}\Big) \ll q^{-5/2+\varepsilon}(n,q)^{1/2}.$$

Hence, by (5.6), we obtain

$$\sum_{q>Z} |A(n,q)| \ll \sum_{q>Z} q^{-5/2+\varepsilon}(n,q)^{1/2} = \sum_{d|n}\sum_{q>\frac{Z}{d}} (dq)^{-5/2+\varepsilon} d^{1/2} = \sum_{d|n} d^{-2+\varepsilon} \sum_{q>\frac{Z}{d}} q^{-5/2+\varepsilon}$$
$$\ll \sum_{d|n} d^{-2+\varepsilon}\Big(\frac{Z}{d}\Big)^{-3/2+\varepsilon} = Z^{-3/2+\varepsilon}\sum_{d|n} d^{-1/2} \ll Z^{-3/2+\varepsilon} d(n),$$

which proves (5.4), and hence gives the absolute convergence of $\mathfrak{S}(n)$.

To prove item (ii) of Lemma 5.6, by Lemma 5.5, we first note that

$$\mathfrak{S}(n) = \prod_p \Big(1 + \sum_{t=1}^{\infty} A(n,p^t)\Big) = \prod_p (1 + A(n,p))$$
$$= \Big(\prod_{p\leqslant c_3} (1 + A(n,p))\Big)\Big(\prod_{\substack{p>c_3\\ p\nmid n}} (1 + A(n,p))\Big)\Big(\prod_{\substack{p>c_3\\ p|n}} (1 + A(n,p))\Big). \tag{5.10}$$

From (5.8),

$$\prod_{\substack{p>c_3\\ p\nmid n}} (1 + A(n,p)) \geqslant \prod_{p>c_3}\Big(1 - \frac{c_3}{p^{5/2}}\Big) \geqslant c_4 > 0.$$

By (5.9), we obtain

$$\prod_{\substack{p>c_3\\ p|n}} (1 + A(n,p)) \geqslant \prod_{p>c_3}\Big(1 - \frac{c_3}{p^{2}}\Big) \geqslant c_5 > 0.$$

However, it is easy to see that

$$1 + A(n,p) = \frac{p\cdot \mathcal{L}(p,n)}{\varphi^6(p)}.$$

By Lemma 5.4, $\mathcal{L}(p,n) > 0$ for all p with $n \equiv 0 \pmod 2$, and thus $1 + A(n,p) > 0$. Therefore,

$$\prod_{p\leqslant c_3} (1 + A(n,p)) \geqslant c_6 > 0. \tag{5.11}$$

Combining the estimates (5.10)–(5.11), and taking $c^* = c_4c_5c_6 > 0$, we derive

$$\mathfrak{S}(n) \geqslant c^* > 0.$$

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Moreover, by (5.8) and (5.9),

$$\mathfrak{S}(n) \leqslant \prod_{p\nmid n}\Big(1+\frac{c_3}{p^{5/2}}\Big)\cdot\prod_{p\mid n}\Big(1+\frac{c_3}{p^2}\Big) \ll 1.$$

This completes the proof Lemma 5.6. □

6. Proof of Proposition 2.2

In this section, we shall give the proof of Proposition 2.2. We denote by $\mathcal{Z}_j(N)$ the set of integers n satisfying $n \in (N/2, N]$ and $n \equiv 0 \pmod 2$ for which

$$\Big|\int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha\Big| \gg \frac{n^{13/12}}{\log^7 n}. \tag{6.1}$$

For convenience, we use $\mathcal{Z}_j$ to denote the cardinality of $\mathcal{Z}_j(N)$. Also, we define the complex number $\xi_j(n)$ by taking $\xi_j(n) = 0$ for $n \notin \mathcal{Z}_j(N)$, and

$$\Big|\int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha\Big| = \xi_j(n)\int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha$$

for $n \in \mathcal{Z}_j(N)$. Plainly, $|\xi_j(n)| = 1$ whenever $\xi_j(n)$ is nonzero. Therefore, we obtain

$$\sum_{n\in\mathcal{Z}_j(N)} \xi_j(n)\int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)\mathcal{K}_j(\alpha)\,d\alpha, \tag{6.2}$$

where the exponential sum $\mathcal{K}_j(\alpha)$ is defined by

$$\mathcal{K}_j(\alpha) = \sum_{n\in\mathcal{Z}_j(N)} \xi_j(n)e(-n\alpha).$$

For $j = 1, 2$, set

$$I_j = \int_{\mathfrak{m}_j} f_2(\alpha)f_3^4(\alpha)f_4(\alpha)\mathcal{K}_j(\alpha)\,d\alpha.$$

From (6.1)–(6.2),

$$I_j \gg \sum_{n\in\mathcal{Z}_j(N)} \frac{n^{13/12}}{\log^7 n} \gg \frac{\mathcal{Z}_j N^{13/12}}{\log^7 N}, \quad j = 1, 2. \tag{6.3}$$

By [9, Lemma 2.1] with $k = 2$,

$$\int_0^1 |f_2(\alpha)\mathcal{K}_j(\alpha)|^2 d\alpha \ll N^{\varepsilon}(\mathcal{Z}_j N^{1/2} + \mathcal{Z}_j^2), \quad j = 1, 2. \tag{6.4}$$

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It follows from Cauchy's inequality, [7, Lemma 2.5], Lemma 3.3 and (6.4) that

$$\begin{aligned} I_1 &\ll \Big(\sup_{\alpha\in\mathfrak{m}_1}|f_4(\alpha)|\Big)\times\Big(\int_0^1|f_3(\alpha)|^8\,d\alpha\Big)^{1/2}\Big(\int_0^1|f_2(\alpha)\mathcal{K}_1(\alpha)|^2\,d\alpha\Big)^{1/2}\\ &\ll N^{23/96+\varepsilon}\cdot(N^{5/3+\varepsilon})^{1/2}\cdot(N^{\varepsilon}(\mathcal{Z}_1N^{1/2}+\mathcal{Z}_1^2))^{1/2}\\ &\ll N^{103/96+\varepsilon}(\mathcal{Z}_1^{1/2}N^{1/4}+\mathcal{Z}_1)\ll \mathcal{Z}_1^{1/2}N^{127/96+\varepsilon}+\mathcal{Z}_1N^{103/96+\varepsilon}. \end{aligned} \tag{6.5}$$

Combining (6.3) and (6.5), we get

$$\mathcal{Z}_1N^{13/12}\log^{-7}N\ll I_1\ll \mathcal{Z}_1^{1/2}N^{127/96+\varepsilon}+\mathcal{Z}_1N^{103/96+\varepsilon},$$

which implies

$$\mathcal{Z}_1\ll N^{23/48+\varepsilon}. \tag{6.6}$$

Next, we give the upper bound for $\mathcal{Z}_2$. By (3.2),

$$I_2\ll\int_{\mathfrak{m}_2}|f_2(\alpha)f_3^4(\alpha)V_4(\alpha)\mathcal{K}_2(\alpha)|\,d\alpha+N^{1/5+\varepsilon}\int_{\mathfrak{m}_2}|f_2(\alpha)f_3^4(\alpha)\mathcal{K}_2(\alpha)|\,d\alpha=I_{21}+I_{22}, \tag{6.7}$$

say. For $\alpha\in\mathfrak{m}_2$, either $Q_0^{100}<q\leqslant Q_1$ or $Q_0^{100}<N|q\alpha-a|\leqslant NQ_2^{-1}=Q_1$. Therefore, by Lemma 3.1,

$$\sup_{\alpha\in\mathfrak{m}_2}|f_2(\alpha)|\ll X_2^{4/5+\varepsilon}+\frac{X_2(\log N)^c}{(q(1+N|\alpha-a/q|))^{1/2-\varepsilon}}\ll\frac{X_2(\log N)^c}{Q_0^{50-\varepsilon}}\ll\frac{N^{1/2}}{\log^{40A}N}. \tag{6.8}$$

In view of the fact that $\mathfrak{m}_2\subseteq\mathcal{I}$, where $\mathcal{I}$ is defined by (3.1), Cauchy's inequality, the trivial estimate $\mathcal{K}_2(\alpha)\ll\mathcal{Z}_2$, [3, Theorem 4, page 19], Lemma 3.4 and (6.8), we obtain

$$\begin{aligned} I_{21}&\ll\mathcal{Z}_2\cdot\sup_{\alpha\in\mathfrak{m}_2}|f_2(\alpha)|\times\Big(\int_0^1|f_3(\alpha)|^8\,d\alpha\Big)^{1/2}\Big(\int_{\mathcal{I}}|V_3(\alpha)|^2\,d\alpha\Big)^{1/2}\\ &\ll\mathcal{Z}_2\cdot\Big(\frac{N^{1/2}}{\log^{40A}N}\Big)\cdot(N^{5/3}\log^cN)^{1/2}\cdot(N^{-1/2}\log^{2A}N)^{1/2}\ll\frac{\mathcal{Z}_2N^{13/12}}{\log^{30A}N}, \end{aligned} \tag{6.9}$$

where the parameter A is chosen sufficiently large for the bounds (6.8) and (6.9) to work. Moreover, it follows from Cauchy's inequality, (6.4) and [3, Theorem 4, page 19] that

$$\begin{aligned} I_{22}&\ll N^{1/5+\varepsilon}\times\Big(\int_0^1|f_3(\alpha)|^8\,d\alpha\Big)^{1/2}\Big(\int_0^1|f_2(\alpha)\mathcal{K}_2(\alpha)|^2\,d\alpha\Big)^{1/2}\\ &\ll N^{1/5+\varepsilon}\cdot(N^{5/3+\varepsilon})^{1/2}\cdot(N^{\varepsilon}(\mathcal{Z}_2N^{1/2}+\mathcal{Z}_2^2))^{1/2}\\ &\ll N^{31/30+\varepsilon}(\mathcal{Z}_2^{1/2}N^{1/4}+\mathcal{Z}_2)\ll\mathcal{Z}_2^{1/2}N^{77/60+\varepsilon}+\mathcal{Z}_2N^{31/30+\varepsilon}. \end{aligned} \tag{6.10}$$

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Combining (6.3), (6.7), (6.9) and (6.10), we deduce that

$$\frac{\mathcal{Z}_2 N^{13/12}}{\log^7 N} \ll I_2 = I_{21} + I_{22} \ll \frac{\mathcal{Z}_2 N^{13/12}}{\log^{30A} N} + \mathcal{Z}_2^{1/2} N^{77/60+\varepsilon} + \mathcal{Z}_2 N^{31/30+\varepsilon},$$

which implies

$$\mathcal{Z}_2 \ll N^{2/5+\varepsilon}. \tag{6.11}$$

From (6.6) and (6.11), $\mathcal{Z}(N) \ll \mathcal{Z}_1 + \mathcal{Z}_2 \ll N^{23/48+\varepsilon}$. This completes the proof of Proposition 2.2.

Acknowledgement

The authors appreciate the contributions of the referee in reviewing this paper.

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