*Example* 1: Let $f(x) = \sin x$. Since $F(x) = -\cos x$, from (2) we conclude that

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

*Example* 2: For $f(x) = \tan x$ we have $F(x) = \ln \sec x$. Since

$$\sec\left(\tan^{-1} x\right) = \sqrt{x^2 + 1},$$

by using (2) we obtain

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \ln \sqrt{x^2 + 1} + C.$$

*Example* 3: Let $f(x) = e^x$. We have $F(x) = e^x$. Thus, from (2) we get

$$\int \ln x\, dx = x \ln x - e^{\ln x} + C = x \ln x - x + C.$$

*Example* 4: The Lambert $W$ function is defined as the inverse of the function $f(x) = xe^x$. Thus, it satisfies $x = W(x)e^{W(x)}$. We get $e^{W(x)} = x/W(x)$. Also, since $F(x) = (x - 1)e^x$, from (2) we deduce that

$$\int W(x)\, dx = xW(x) - \left(W(x) - 1\right)e^{W(x)} + C$$

$$= xW(x) - x + \frac{x}{W(x)} + C.$$

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SOHEILA EMAMYARI
*Department of Physics,*
*Institute for Advanced Studies*
*in Basic Sciences (IASBS),*
*Zanjan, Iran*
e-mail: *emamyari@iasbs.ac.ir*
MEHDI HASSANI
*Department of Mathematics, University of Zanjan,*
*University Blvd., 45371-38791, Zanjan, Iran*
e-mail: *mehdi.hassani@znu.ac.ir*

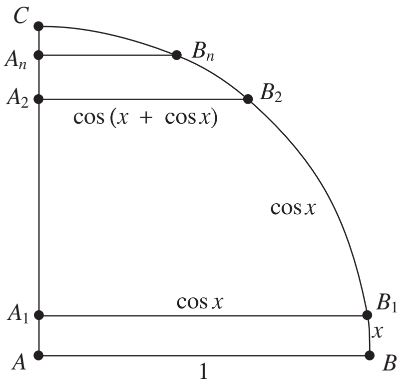
## 107.34 On a staircase function

Let $x \in \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ and consider the series defined as follows: the first term is equal to $x$ and the cosine of the sum $S_n(x)$ of the first $n$ terms equals the $(n + 1)$th term. We regard $|S_n|$ as the length of an arc on the unit circle. In this Note, by using an elementary geometric argument, we show that $S_n$ is a monotone sequence that converges to $\frac{1}{2}\pi$. Due to periodicity of the cosine function, we also have convergence to $2k\pi + \frac{1}{2}\pi$ for any

$x \in \left((2k - \frac{1}{2})\pi, (2k + \frac{3}{2})\pi\right)$, $k \in \mathbb{Z}$. As a consequence, the series turns out to represent a staircase function. More precisely we have the following.

*Theorem*: For any $k \in \mathbb{Z}$ and any $x \in \left((2k - \frac{1}{2})\pi, (2k + \frac{3}{2})\pi\right)$ we have the following absolutely convergent series

$$\left(2k + \tfrac{1}{2}\right)\pi = x + \cos(x) + \cos\left(x + \cos(x)\right) + \cos\left(x + \cos(x) + \cos\left(x + \cos(x)\right)\right) + \ldots$$




FIGURE

*Proof*: It suffices to consider the case of $k = 0$. Let $(BC)$ be a circular arc of length $d(BC)$ equal to $\pi/2$, defined by two points $B$ and $C$ lying on a circle of radius 1 that is centred at the point $A$ as in the Figure. Assume that $x \in [0, \pi/2)$ and choose $B_1$ on $(BC)$ such that $d(BB_1) = x$. Denote by $[AC]$ the line segment defined by $A$ and $C$ and let $A_1$ be the projection of $B_1$ onto $[AC]$. Since the hypotenuse of the right triangle $\Delta AA_1B_1$ defined by $A$, $A_1$ and $B_1$ is equal to 1, we have $d[A_1B_1] = \cos x$, where $d[A_1B_1]$ denotes the length of $[A_1B_1]$. From the right triangle $\Delta A_1B_1C$ we have $d[A_1B_1] < d[B_1C]$, so that $d[A_1B_1] < d(B_1C)$. Let $B_2$ be a point on $(B_1C)$ such that $d(B_1B_2) = d[A_1B_1]$ and let $A_2$ be the projection of $B_2$ onto the segment $[A_1C]$. We have $d[A_2B_2] = \cos\left(x + \cos x\right)$.

By continuing the above procedure, we construct a sequence of points $\{A_n\}_{n \in \mathbb{N}}$ on $[AC]$ and $\{B_n\}_{n \in \mathbb{N}}$ on $(BC)$ satisfying $S_n(x) = d\left(BB_n\right) < d\left(BB_{n+1}\right) < \pi/2$ for each $n \in \mathbb{N}$, where $S_n(x)$ denotes the sum of the first $n$ terms of the series. Let $y$ be the limit of $d(BB_n)$ as $n \to \infty$. If $y \neq \pi/2$, then let $D \in (BC)$ such that $d(BD) = y$. Since $\cos\left(S_n(x)\right) = d[A_n, B_n] > \cos y$ for each $n \in \mathbb{N}$, we have

$$S_{n+1}(x) = S_n(x) + \cos\left(S_n(x)\right) > S_n(x) + \cos y,$$

$n \in \mathbb{N}$. On the other hand, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n \geqslant N$ implies $0 < y - S_n(x) < \varepsilon$. Therefore, if $n \geqslant N$ we have $y > S_{n+1}(x) > y + \cos y - \varepsilon$, which, for $\varepsilon > 0$ sufficiently small, gives us a contradiction.

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The case of general $x \in (-\pi/2,\ 3\pi/2)$ can be treated similarly; in particular, when $x \in (-\pi/2,\ \pi/2)/(\pi/2,\ 3\pi/2)$ the sequence $S_n(x)$, $n \in \mathbb{N}$, increases/decreases monotonically.

As an alternative perspective of the above calculation, we mention that the iterative procedure $S_1(x) = x$, $S_{n+1}(x) = S_n(x) + \cos(S_n(x))$, $n \in \mathbb{N}$, is just the fixed-point iteration for the map $x \to x + \cos x$. We conclude by noting that, due to the above result, the finite sum $S_n(\cdot)$ serves as a smooth approximation of a staircase function. Consequently, the derivative of $S_n(\cdot)$ can be viewed as a smooth approximation of a *Dirac comb*, i.e. of a periodic pulse wave consisting of Dirac delta functions.

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NIKOLAOS ROIDOS
*Department of Mathematics,*
*University of Patras,*
*26504 Rio Patras, Greece*
e-mail: *roidos@math.upatras.gr*

## 107.35 Two definite integrals that are (not surprisingly) equal

1. *Introduction*

In their recent note Ekhad, Zeilberger and Zudilin [1] gave a clever proof of the identity

$$\int_0^1 \frac{x^n(1-x)^n}{((x+a)(x+b))^{n+1}}dx = \int_0^1 \frac{x^n(1-x)^n}{((a-b)x+(a+1)b)^{n+1}}dx, \qquad (1)$$

for $n = 0, 1, 2, \ldots$ and $a > b > 0$, using the Almkvist–Zeilberger creative telescoping algorithm. If $L(n)$ and $R(n)$ denote the integrals on the left and right sides, respectively, for fixed $a$ and $b$, then they show that $L(n)$ and $R(n)$ satisfy the same linear recursive formula of order two. Confirming that $L(0) = R(0)$ and $L(1) = R(1)$, the identity follows by mathematical induction. The authors mentioned that three other proofs of (1) exist. Bostan, Chamizo and Sundqvist [2] recognized in identity (1) a particular case of a known relation for Appell's bivariate hypergeometric function and gave three different proofs of (1).

The authors of [1, Remark 3] mention that the right-hand side $R(n)$ covers a famous sequence of rational approximations to $\log\left(1 + \frac{a-b}{(a+1)b}\right)$, and hence the left-hand side $L(n)$ does, too, and cite [3]. The estimate of the irrationality measure is based on considering certain integrals involving the $n$th Legendre type polynomial $L_n(x) = (n!)^{-1}\left(x^n(1-x)^n\right)^{(n)}$.

In the following we consider a more natural representation of (1). We rewrite identity (1) by replacing $a, b > 0$ with their reciprocals, in the form

$$\int_0^1 \frac{x^n(1-x)^n}{((1+ax)(1+bx))^{n+1}}dx = \int_0^1 \frac{x^n(1-x)^n}{(1+a(1-x)+bx)^{n+1}}\,dx, \qquad (2)$$