RENEWAL THEORY FOR ITERATED PERTURBED RANDOM WALKS ON A GENERAL BRANCHING PROCESS TREE: EARLY GENERATIONS

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Abstract

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be independent identically distributed random vectors with arbitrarily dependent positive components. We call a (globally) perturbed random walk a random sequence $T := (T_k)_{k\in\mathbb{N}}$ defined by $T_k := \xi_1 + \cdots + \xi_{k-1} + \eta_k$ for $k \in \mathbb{N}$. Consider a general branching process generated by T and let $N_j(t)$ denote the number of the jth generation individuals with birth times $\leq t$. We treat early generations, that is, fixed generations j which do not depend on t. In this setting we prove counterparts for $\mathbb{E}N_j$ of the Blackwell theorem and the key renewal theorem, prove a strong law of large numbers for N_j, and find the first-order asymptotics for the variance of N_j. Also, we prove a functional limit theorem for the vector-valued process $(N_1(ut), \ldots, N_j(ut))_{u\geq0}$, properly normalized and centered, as $t \to \infty$. The limit is a vector-valued Gaussian process whose components are integrated Brownian motions.

Keywords: Functional limit theorem; general branching process; key renewal theorem; perturbed random walk; renewal theory; strong law of large numbers

2020 Mathematics Subject Classification: Primary 60K05; 60J80
Secondary 60G05

1. Introduction

Let $(\xi_i, \eta_i)_{i\in\mathbb{N}}$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ, η) with arbitrarily dependent components. Let $(S_i)_{i\in\mathbb{N}_0}$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) denote the standard random walk with increments ξ_i for $i \in \mathbb{N}$, i.e. $S_0 := 0$ and $S_i := \xi_1 + \cdots + \xi_i$ for $i \in \mathbb{N}$. Define

$$T_i := S_{i-1} + \eta_i, \quad i \in \mathbb{N}.$$

The sequence $T := (T_i)_{i\in\mathbb{N}}$ is called *perturbed random walk*.

In the following we assume that ξ and η are almost surely (a.s.) positive. Now we define a general branching process generated by T. At time 0 there is one individual, the ancestor. The ancestor produces offspring (the first generation) with birth times given by the points of T. The first generation produces the second generation. The shifts of birth times of the second generation individuals with respect to their mothers' birth times are distributed according

to copies of T, and for different mothers these copies are independent. The second generation produces the third one, and so on. All individuals act independently of each other. For $t \geq 0$ and $j \in \mathbb{N}$, let $N_j(t)$ denote the number of the jth generation individuals with birth times $\leq t$ and put $V_j(t) := \mathbb{E}[N_j(t)]$, and $V_j(t) := 0$ for $t < 0$. Then $N_1(t) = \sum_{r \geq 1} \mathbb{1}_{\{T_r \leq t\}}$ and

$$N_j(t) = \sum_{r \geq 1} N_{j-1}^{(r)}(t - T_r) \mathbb{1}_{\{T_r \leq t\}} = \sum_{k \geq 1} N_{1,j}^{(k)}\big(t - T_k^{(j-1)}\big) \mathbb{1}_{\{T_k^{(j-1)} \leq t\}}, \quad j \geq 2, \ t \geq 0, \quad (1)$$

where $N_{j-1}^{(r)}(t)$ is the number of the jth generation individuals who are descendants of the first-generation individual with birth time T_r, and whose birth times fall in $[T_r, T_r + t]$; $T^{(j-1)} := \big(T_k^{(j-1)}\big)_{k \geq 1}$ is some enumeration of the birth times in the $(j-1)$th generation; $N_{1,j}^{(k)}(t)$ is the number of the jth generation individuals that are children of the $(j-1)$th generation individual with birth time $T_k^{(j-1)}$, and whose birth times fall in

$$\big[T_k^{(j-1)}, T_k^{(j-1)} + t\big].$$

By the branching property,

$$\big(N_{j-1}^{(1)}(t)\big)_{t \geq 0}, \quad \big(N_{j-1}^{(2)}(t)\big)_{t \geq 0}, \ldots$$

are independent copies of N_{j-1} which are also independent of T, and

$$\big(N_{1,j}^{(1)}(t)\big)_{t \geq 0}, \quad \big(N_{1,j}^{(2)}(t)\big)_{t \geq 0}, \ldots$$

are independent copies of $(N_1(t))_{t \geq 0}$ which are also independent of $T^{(j-1)}$. In what follows we write N for N_1 and V for V_1. Passing in (1) to expectations we infer, for $j \geq 2$ and $t \geq 0$,

$$V_j(t) = V^{*(j)}(t) = (V_{j-1} * V)(t) = \int_{[0, t]} V_{j-1}(t - y) \, dV(y) = \int_{[0, t]} V(t - y) \, dV_{j-1}(y), \quad (2)$$

where $V^{*(j)}$ is the j-fold convolution of V with itself. We call the sequence $\mathcal{T} := (T^{(j)})_{j \in \mathbb{N}}$ *iterated perturbed random walk on a general branching process tree*.

The motivation behind the study of $\mathcal{T}$ is at least threefold.

(i) Originally, the iterated perturbed random walks were introduced in Section 3 of [4] as an auxiliary tool for investigating the nested occupancy scheme in random environment generated by stick-breaking. This scheme is a hierarchical extension of the Karlin [9] infinite balls-in-boxes occupancy scheme, in which, for $k \in \mathbb{N}$, the hitting (random) probability of the box k is equal to $e^{-T_k^*}$. Here $(T_k^*)_{k \in \mathbb{N}}$ is a particular perturbed random walk defined by

$$T_k^* := |\log W_1| + \cdots + |\log W_{k-1}| + \log |1 - W_k|, \quad k \in \mathbb{N},$$

and $(W_k)_{k \in \mathbb{N}}$ are independent identically distributed random variables taking values in $(0, 1)$. The sequence $(e^{-T_k^*})_{k \in \mathbb{N}}$ is called a stick-breaking sequence or residual allocation model.

(ii) Assuming that j is fixed, the sequence $T^{(j)}$ and the process N_j are a natural generalization of the perturbed random walk T and the counting process N. The reader who is interested

in standard random walks and not interested in perturbed random walks may put $\eta = \xi$. Then T and N become a standard random walk and the corresponding renewal process, respectively. Thus any results obtained for $T^{(j)}$ and N_j can be thought of as a contribution to an interesting extension of renewal theory.

(iii) The sequence $\mathcal{T}$ is a particular yet non-trivial instance of a branching random walk; see [13] for a survey. As a consequence, an outcome of the analysis of $\mathcal{T}$ is a contribution to the theory of branching random walks and the theory of general branching processes.

Following [3], we call the jth generation *early*, *intermediate*, or *late* depending on whether j is fixed, $j = j(t) \to \infty$, and $j(t) = o(t)$ as $t \to \infty$, or $j = j(t)$ is of order t. In [3], Bohun, Iksanov, Marynych, and Rashytov prove counterparts of the elementary renewal theorem, the Blackwell theorem, and the key renewal theorem for some intermediate generations. In the present work we investigate early generations. Although the analysis of early generations is simpler than that of intermediate generations, we solve here a larger collection of problems. More precisely, we prove a strong law of large numbers (Theorem 5) for $N_j(t)$ and a functional limit theorem (Theorem 6) for the vector-valued process $(N_1(ut), N_2(ut), \ldots, N_k(ut))_{u \geq 0}$ for each $k \in \mathbb{N}$, properly normalized and centered as $t \to \infty$, investigate the rate of convergence (Theorem 1) in a counterpart for V_j of the elementary renewal theorem, and find the asymptotics of the variance $\mathrm{Var}[N_j(t)]$ (Theorem 4). Also, counterparts for V_j of the key renewal theorem (Theorem 2) and the Blackwell theorem (Theorem 3) are given. These two results which follow from a modified version of the proof of Theorem 2.7(a) of [3] are included for completeness. Theorem 1 is an early generations analogue of Proposition 2.3 of [3] which treats intermediate generations. In the latter result the asymptotic relation (4) is stated under rather strong assumptions: the distribution of ξ has an absolutely continuous component, the random variables ξ and η have some finite exponential moments, and $\mathbb{E}[\xi^2] > 2\mathbb{E}[\xi]\mathbb{E}[\eta]$. In contrast, the assumptions of Theorem 1 are optimal. No intermediate generation versions of Theorems 4, 5, and 6 are available. The reason is that here we exploit rather precise asymptotics of various involved functions. Such asymptotics are not known in the setting of intermediate generations. To be more specific, we only mention that Theorems 4 and 6 rely heavily upon Lemma 3 and Assertion 3, respectively. At the moment, intermediate generation counterparts of these are beyond our reach.

The remainder of the paper is structured as follows. In Section 2 we state our main results. Some auxiliary statements are discussed in Section 3. The proofs of the main results are given in Section 4. Finally, the Appendix collects a couple of assertions borrowed from other articles.

2. Results

In Sections 2 and 3 we work under the assumption $\mathfrak{m} = \mathbb{E}[\xi] < \infty$. In view of the standing assumption $\xi > 0$ a.s. (see Section 1), we actually have $\mathfrak{m} \in (0, \infty)$.

2.1. A counterpart of the elementary renewal theorem and the rate of convergence result

Assertion 1 is a counterpart for $V_j = \mathbb{E}[N_j]$ of the elementary renewal theorem.

Assertion 1. *Assume that* $\mathfrak{m} = \mathbb{E}[\xi] < \infty$. *Then, for fixed* $j \in \mathbb{N}$,

$$\lim_{t \to \infty} \frac{V_j(t)}{t^j} = \frac{1}{j! \mathfrak{m}^j}. \tag{3}$$

The standard version of the elementary renewal theorem can be found, for instance, in Theorem 3.3.3 of [11].

Theorem 1 quantifies the rate of convergence in (3) under the assumptions $\mathbb{E}[\xi^2] < \infty$ and $\mathbb{E}[\eta] < \infty$. As usual, $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t \to \infty}(f(t)/g(t)) = 1$. Recall that the distribution of a positive random variable is called *nonlattice* if it is not concentrated on any lattice $(nd)_{n \in \mathbb{N}_0}$, $d > 0$.

Theorem 1. *Assume that the distribution of ξ is nonlattice, and that $\mathbb{E}[\xi^2] < \infty$ and $\mathbb{E}[\eta] < \infty$. Then, for any fixed $j \in \mathbb{N}$,*

$$V_j(t) - \frac{t^j}{j!\mathfrak{m}^j} \sim \frac{b_V\, jt^{j-1}}{(j-1)!\mathfrak{m}^{j-1}}, \quad t \to \infty, \tag{4}$$

where $\mathfrak{m} = \mathbb{E}[\xi] < \infty$ and $b_V := \mathfrak{m}^{-1}(\mathbb{E}[\xi^2]/(2\mathfrak{m}) - \mathbb{E}[\eta]) \in \mathbb{R}$.

Theorem 1 is an extension of the previously known result concerning renewal functions, which can be obtained, for instance, as a specialization of Theorem 9.2 of [6].

2.2. Counterparts of the key renewal theorem and the Blackwell theorem

Theorem 2 is a counterpart for V_j of the key renewal theorem; see Theorem 1.12 of [10].

Theorem 2. *Let $f : [0, \infty) \to [0, \infty)$ be a directly Riemann integrable function on $[0, \infty)$. Assume that the distribution of ξ is nonlattice and that $\mathfrak{m} < \infty$. Then, for fixed $j \in \mathbb{N}$,*

$$\int_{[0,\,t]} f(t-y)\, \mathrm{d}V_j(y) \sim \left(\frac{1}{\mathfrak{m}} \int_0^\infty f(y)\, \mathrm{d}y\right) V_{j-1}(t) \sim \int_0^\infty f(y)\, \mathrm{d}y\, \frac{t^{j-1}}{(j-1)!\mathfrak{m}^j}, \quad t \to \infty.$$

Theorem 3, a counterpart for V_j of the Blackwell theorem, is just a specialization of Theorem 2 with $f(y) = \mathbb{1}_{[0,\,h)}(y)$ for $y \geq 0$. Nevertheless, we find it instructive to provide an alternative proof. The reason is that the proof given in Section 4.2 nicely illustrates basic concepts of the renewal theory and may be adapted to other settings.

Theorem 3. *Assume that the distribution of ξ is nonlattice and $\mathfrak{m} = \mathbb{E}[\xi] < \infty$. Then, for fixed $j \in \mathbb{N}$ and fixed $h > 0$,*

$$V_j(t+h) - V_j(t) \sim \frac{ht^{j-1}}{(j-1)!\mathfrak{m}^j}, \quad t \to \infty. \tag{5}$$

The standard version of the Blackwell theorem is given, for instance, in Theorem 1.10 of [10]. In the case $\eta = 0$ a.s. limit relation (5) can be found in Theorem 1.16 of [10].

Remark 1. Assume that the distributions of ξ and η are d-arithmetic for some $d > 0$. This means that these are concentrated on the lattice $(nd)_{n \in \mathbb{N}_0}$ for some $d > 0$ and not concentrated on $(nd_1)_{n \in \mathbb{N}_0}$ for any $d_1 > d$. Then one may expect that Theorems 1, 2, and 3 admit natural counterparts in which the limit should be taken along (nd) rather than continuously. Further, the constant b_V in Theorem 1 should take a slightly different form, the integral in Theorem 2 should be replaced by a sum, and the parameter h in Theorem 3 should be restricted to the lattice (nd). The case when the distribution of ξ is d-arithmetic, whereas the distribution of η is not, looks more challenging, and we refrain from discussing it here.

2.3. Asymptotics of the variance

In this section we find, for fixed $j \in \mathbb{N}$, the asymptotics of $\mathrm{Var}[N_j(t)]$ as $t \to \infty$ under the assumption $\eta = \xi$ a.s., so that $T_k = S_k$ for $k \in \mathbb{N}$. In other words, below we treat *iterated standard random walks*. Theorem 4 is a strengthening of Lemma 4.2 of [8] in which an asymptotic

upper bound for $\mathrm{Var}[N_j(t)]$ was obtained. Although we do not know the asymptotic behavior of $\mathrm{Var}[N_j(t)]$ for (genuine) iterated perturbed random walks, Remark 2 contains a partial result in that direction.

Theorem 4. *Assume that $\eta = \xi$ a.s., that the distribution of ξ is nonlattice, and $\mathrm{s}^2 := \mathrm{Var}[\xi] \in (0, \infty)$. Then, for any $j \in \mathbb{N}$,*

$$\lim_{t \to \infty} \frac{\mathrm{Var}[N_j(t)]}{t^{2j-1}} = \frac{\mathrm{s}^2}{(2j-1)((j-1)!)^2 \mathrm{m}^{2j+1}}, \tag{6}$$

where $\mathrm{m} = \mathbb{E}[\xi] < \infty$.

Theorem 4 is an extension of the previously known result for $\mathrm{Var}\, N_1$; see e.g. Theorem 9.1(ii) of [6].

2.4. Strong law of large numbers

Theorem 5 is a strong law of large numbers for N_j.

Theorem 5. *Assume that $\mathrm{m} = \mathbb{E}[\xi] < \infty$. Then, for fixed $j \in \mathbb{N}$,*

$$\lim_{t \to \infty} \frac{N_j(t)}{t^j} = \frac{1}{\mathrm{m}^j j!} \quad a.s. \tag{7}$$

The strong law of large numbers for renewal processes is given, for instance, in Theorem 5.1 of [6].

2.5. A functional limit theorem

Let $B := (B(s))_{s \geq 0}$ be a standard Brownian motion and, for $q \geq 0$, let

$$B_q(s) := \int_{[0,\, s]} (s - y)^q \, \mathrm{d}B(y), \quad s \geq 0.$$

The process $B_q := (B_q(s))_{s \geq 0}$ is a centered Gaussian process called the *fractionally integrated Brownian motion* or the *Riemann–Liouville process*. Plainly $B = B_0$, $B_1(s) = \int_0^s B(y) \, \mathrm{d}y$, $s \geq 0$ and, for integer $q \geq 2$,

$$B_q(s) = q! \int_0^s \int_0^{s_2} \cdots \int_0^{s_q} B(y) \, \mathrm{d}y \, \mathrm{d}s_q \cdots \mathrm{d}s_2, \quad s \geq 0.$$

In the following we write $\Rightarrow$ and $\xrightarrow{\mathrm{d}}$ to denote weak convergence in a function space and weak convergence of one-dimensional distributions, respectively. As usual, we let D denote the Skorokhod space of right-continuous functions defined on $[0, \infty)$ with finite limits from the left at positive points. We prefer to use $(X_t(u))_{u \geq 0} \Rightarrow (X(u))_{u \geq 0}$ in place of the formally correct notation $X_t(\cdot) \Rightarrow X(\cdot)$.

Given next is a functional limit theorem for $(N_1(ut), N_2(ut), \ldots)_{u \geq 0}$, properly normalized and centered, as $t \to \infty$.

Theorem 6. *Assume that $\mathrm{m} = \mathbb{E}[\xi] < \infty$, $\mathrm{s}^2 = \mathrm{Var}[\xi] \in (0, \infty)$, and $\mathbb{E}[\eta^a] < \infty$ for some $a > 0$. Then*

$$\left((j-1)! \left(\frac{N_j(ut) - V_j(ut)}{\sqrt{\mathrm{m}^{-2j-1} \mathrm{s}^2 t^{2j-1}}} \right)_{u \geq 0} \right)_{j \in \mathbb{N}} \Rightarrow ((B_{j-1}(u))_{u \geq 0})_{j \in \mathbb{N}}, \quad t \to \infty \tag{8}$$

in the product J_1-topology on $D^{\mathbb{N}}$.

If $\mathbb{E}[\eta^{1/2}] < \infty$, then the centering $V_j(ut)$ can be replaced by $(ut)^j/(j!\mathfrak{m}^j)$. If $\mathbb{E}[\eta^{1/2}] = \infty$, the centering $V_j(ut)$ can be replaced by

$$
\frac{\mathbb{E}[(ut - (\eta_1 + \cdots + \eta_j))^j \mathbb{1}_{\{\eta_1 + \cdots + \eta_j \leq ut\}}]}{j!\mathfrak{m}^j}
$$
$$
= \frac{\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{\eta_1 + \cdots + \eta_j \leq y\} \, \mathrm{d}y \, \mathrm{d}t_j \cdots \mathrm{d}t_2}{\mathfrak{m}^j}, \tag{9}
$$

where $t_1 = ut$.

In the renewal-theoretic context (the cases $\eta = 0$ or $\eta = \xi$ a.s.), under the assumption that the distribution of ξ belongs to the domain of attraction of a stable distribution, various functional limit theorems for N_1 can be found, for instance, in Section 5 of [6], on page 115 of [7], and in Section 7.3.2 of [14].

Now we derive a one-dimensional central limit theorem for N_j. To this end, it is enough to restrict attention to just one coordinate in (8), put $u = 1$ there, and note that $B_{j-1}(1)$ has the same distribution as $(2j - 1)^{-1/2}B(1)$.

Corollary 1. *Under the assumptions of Theorem 6, for fixed $j \in \mathbb{N}$,*

$$
\frac{(j-1)!(2j-1)^{1/2}\mathfrak{m}^{j+1/2}}{\mathfrak{s}t^{j-1/2}} \big(N_j(t) - V_j(t)\big) \xrightarrow{\mathrm{d}} B(1), \quad t \to \infty. \tag{10}
$$

Remark 2. In view of Corollary 1 the result of Theorem 4 is expected, yet the complete proof requires some effort. Of course, the relation

$$
\liminf_{t \to \infty} \frac{\mathrm{Var}[N_j(t)]}{t^{2j-1}} \geq \frac{\mathfrak{s}^2}{(2j-1)((j-1)!)^2\mathfrak{m}^{2j+1}}
$$

is an immediate consequence of (10) and Fatou's lemma.

3. Auxiliary tools

For $t \geq 0$, put $U(t) := \sum_{n \geq 0} \mathbb{P}\{S_n \leq t\}$, so that U is the renewal function of $(S_n)_{n \in \mathbb{N}_0}$. Whenever $\mathbb{E}\xi^2 < \infty$, we have

$$
0 \leq U(t) - \mathfrak{m}^{-1}t \leq c_U, \quad t \geq 0, \tag{11}
$$

where $c_U := \mathfrak{m}^{-2}\mathbb{E}[\xi^2]$. The right-hand side is called *Lorden's inequality*. Perhaps it is not commonly known that Lorden's inequality takes the same form for nonlattice and lattice distributions of ξ, and we refer to Section 3 of [3] for an explanation of this fact. The left-hand inequality in (11) is a consequence of Wald's identity $t \leq \mathbb{E}S_{\nu(t)} = \mathfrak{m}U(t)$, where $\nu(t) := \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$.

Let us show that the left-hand inequality in (11) extends to the convolution powers $U_j = U^{*(j)}$ (see (2) for the definition) in the following sense.

Lemma 1. *Let $j \in \mathbb{N}$ and $\mathfrak{m} = \mathbb{E}[\xi] < \infty$. Then*

$$
U_j(t) \geq \frac{t^j}{j!\mathfrak{m}^j}, \quad t \geq 0. \tag{12}
$$

Proof. We use mathematical induction. When $j = 1$, (12) reduces to the left-hand inequality in (11). Assuming that (12) holds for $j \leq k$, we infer

$$U_{k+1}(t) - \frac{t^{k+1}}{(k+1)!\mathrm{m}^{k+1}} = \int_{[0,\,t]} \left(U(t-y) - \frac{t-y}{\mathrm{m}} \right) dU_k(y) + \frac{1}{\mathrm{m}} \int_0^t \left(U_k(y) - \frac{y^k}{k!\mathrm{m}^k} \right) dy$$
$$\geq 0,$$

that is, (12) holds with $j = k + 1$. $\qquad\square$

Put $G(t) := \mathbb{P}\{\eta \leq t\}$ for $t \in \mathbb{R}$. Observe that $G(t) = 0$ for $t < 0$. Since $V(t) \leq U(t)$ for $t \geq 0$, we conclude that

$$V(t) - \mathrm{m}^{-1}t \leq c_U, \quad t \geq 0$$

for any distribution of η. On the other hand, assuming that $\mathbb{E}[\eta] < \infty$,

$$V(t) - \mathrm{m}^{-1}t = \int_{[0,\,t]} (U(t-y) - \mathrm{m}^{-1}(t-y))\, dG(y) - \mathrm{m}^{-1} \int_0^t (1 - G(y))\, dy$$
$$\geq -\mathrm{m}^{-1} \int_0^t (1 - G(y))\, dy$$
$$\geq -\mathrm{m}^{-1} \mathbb{E}[\eta],$$

having utilized $U(t) \geq \mathrm{m}^{-1}t$ for $t \geq 0$. Assuming that $\mathbb{E}[\eta^a] < \infty$ for some $a \in (0, 1)$, which in particular implies that $\lim_{t \to \infty} t^a \mathbb{P}\{\eta > t\} = 0$, we infer

$$V(t) - \mathrm{m}^{-1}t \geq -\mathrm{m}^{-1} \int_0^t (1 - G(y))\, dy \geq -c_1 - c_2 t^{1-a}, \quad t \geq 0.$$

Thus we have proved the following.

Lemma 2. *Assume that* $\mathbb{E}[\xi^2] < \infty$. *If* $\mathbb{E}[\eta] < \infty$, *then*

$$|V(t) - \mathrm{m}^{-1}t| \leq c_V, \quad t \geq 0, \tag{13}$$

where $c_V := \max(c_U, \mathrm{m}^{-1}\mathbb{E}[\eta])$ *and* $c_U = \mathrm{m}^{-2}\mathbb{E}[\xi^2]$. *If* $\mathbb{E}[\eta^a] < \infty$ *for some* $a \in (0, 1)$, *then*

$$-c_1 - c_2 t^{1-a} \leq V(t) - \mathrm{m}^{-1}t \leq c_V, \quad t \geq 0 \tag{14}$$

for appropriate positive constants c_1 *and* c_2.

Lemma 3 is needed for the proof of Theorem 4. The convolution integral in (15), with $2j - 1$ replacing j, is one of the summands appearing in a representation of Var N_j. It is important that (15) is a two-term expansion of the integral, because the coefficient $b_U - 1$ contributes to the leading coefficient in the asymptotics of Var N_j. Put

$$\tilde{U}(t) := \sum_{r \geq 1} \mathbb{P}\{S_r \leq t\} = U(t) - 1 \quad \text{for } t \geq 0.$$

Lemma 3. *Assume that the distribution of* ξ *is nonlattice and* $\mathbb{E}[\xi^2] < \infty$. *Then, for fixed* $j \in \mathbb{N}$,

$$\int_{[0,\,t]} (t-y)^j\, d\tilde{U}(y) = \frac{t^{j+1}}{(j+1)\mathrm{m}} + (b_U - 1)t^j + \mathrm{o}(t^j), \quad t \to \infty, \tag{15}$$

where $b_U := \mathbb{E}[\xi^2]/(2\mathrm{m}^2)$.

Proof. Using the easily checked formula

$$\int_{[0,\,t]} (t-y)^j \, \mathrm{d}\tilde{U}(y) = j \int_0^t \int_{[0,\,s]} (s-y)^{j-1} \, \mathrm{d}\tilde{U}(y) \, \mathrm{d}s, \quad j \in \mathbb{N}, \ t \geq 0$$

and mathematical induction, one can show that

$$\int_{[0,\,t]} (t-y)^j \, \mathrm{d}\tilde{U}(y) = j! \int_0^t \int_0^{y_j} \cdots \int_0^{y_2} \tilde{U}(y_1) \, \mathrm{d}y_1 \cdots \mathrm{d}y_j, \quad j \in \mathbb{N}, \ t \geq 0.$$

Here the right-hand side reads $\int_0^t \tilde{U}(y) \, \mathrm{d}y$ when $j = 1$. Now (15) follows from the latter equality and the relation $\tilde{U}(t) = \mathrm{m}^{-1}t + b_U - 1 + \mathrm{o}(1)$ as $t \to \infty$, which is simply formula (4) with $j = 1$ and $\eta = \xi$ a.s. To obtain the asymptotic relation

$$\int_0^t \int_0^{y_j} \cdots \int_0^{y_2} \mathrm{o}(1) \, \mathrm{d}y_1 \cdots \mathrm{d}y_j = \mathrm{o}(t^j), \quad t \to \infty,$$

we have used L'Hôpital's rule $(j - 1)$ times in the situation that the o(1)-term is integrable and j times in the situation that it is not. $\square$

Lemma 4 will be essential to the proof of Theorem 5 when showing that one of the two summands in a representation of N_j is asymptotically negligible.

Lemma 4. *Assume that* $\mathrm{m} = \mathbb{E}[\xi] < \infty$. *Then*

$$\lim_{t \to \infty} \frac{\mathbb{E}[(N(t))^2]}{t^2} = \frac{1}{\mathrm{m}^2}.$$

Proof. The relation

$$\liminf_{t \to \infty} \frac{\mathbb{E}[(N(t))^2]}{t^2} \geq \frac{1}{\mathrm{m}^2}$$

follows from $\mathbb{E}[(N(t))^2] \geq (V(t))^2$ and Assertion 1. The converse inequality for the limit superior is implied by the inequality

$$N(t) \leq \nu(t) = \sum_{i \geq 0} \mathbb{1}_{\{S_i \leq t\}}, \quad t \geq 0 \quad \text{a.s.}$$

and $\lim_{t \to \infty} t^{-2} \mathbb{E}[(\nu(t))^2] = \mathrm{m}^{-2}$ (see Theorem 5.1(ii) of [6]). $\square$

Assertions 2 and 3 are important ingredients in the proof of Theorem 6, which is based on an application of Theorem 7 given in the Appendix. While the former provides a functional limit theorem for the counting process in the first generation, thereby ensuring that condition (37) of Theorem 7 is satisfied, the latter is responsible for checking condition (36) of the same theorem.

Assertion 2. *Assume that* $\mathrm{m} = \mathbb{E}[\xi] < \infty$, $\mathrm{s}^2 = \mathrm{Var}[\xi] \in (0, \infty)$, *and* $\mathbb{E}[\eta^a] < \infty$ *for some* $a > 0$. *Then*

$$\left(\frac{N(ut) - V(ut)}{\sqrt{\mathrm{m}^{-3}\mathrm{s}^2 t}} \right)_{u \geq 0} \Rightarrow (B(u))_{u \geq 0}, \quad t \to \infty$$

in the J_1-*topology on* D, *where* $(B(u))_{u \geq 0}$ *is a standard Brownian motion.*

Proof. According to part (B1) of Theorem 3.2 of [1],

$$\left(\frac{N(ut) - \mathrm{m}^{-1} \int_0^{ut} G(y) \, \mathrm{d}y}{\sqrt{\mathrm{m}^{-3} \mathrm{s}^2 t}} \right)_{u \geq 0} \Rightarrow (B(u))_{u \geq 0}, \quad t \to \infty$$

in the J_1-topology on D, where, as before, G is the distribution function of η. Thus it is enough to show that, for all $T > 0$,

$$\lim_{t \to \infty} t^{-1/2} \sup_{u \in [0, T]} \left| V(ut) - \mathrm{m}^{-1} \int_0^{ut} G(y) \, \mathrm{d}y \right| = 0. \tag{16}$$

According to (11), for $u \in [0, T]$ and $t \geq 0$,

$$0 \leq V(ut) - \mathrm{m}^{-1} \int_0^{ut} G(y) \, \mathrm{d}y = \int_{[0, ut]} (U(tu - y) - \mathrm{m}^{-1}(tu - y)) \, \mathrm{d}G(y) \leq c_U,$$

and (16) follows. $\square$

Assertion 3. *Assume that* $\mathrm{m} = \mathbb{E}[\xi] < \infty$, $\mathrm{s}^2 = \mathrm{Var}[\xi] \in (0, \infty)$, *and* $\mathbb{E}[\eta^a] < \infty$ *for some* $a > 0$. *Then*

$$\mathbb{E}\left[\sup_{s \in [0, t]} (N(s) - V(s))^2 \right] = \mathrm{O}(t), \quad t \to \infty.$$

Proof. In the case $\mathbb{E}[\eta] < \infty$, this limit relation is proved in Lemma 4.2(b) of [5].

From now on we assume that $\mathbb{E}[\eta^a] < \infty$ for some $a \in (0, 1)$ and $\mathbb{E}[\eta] = \infty$. As in the proof of Lemma 4.2(b) of [5], we shall use a decomposition

$$N(t) - V(t) = \sum_{k \geq 0} (\mathbb{1}_{\{S_k + \eta_{k+1} \leq t\}} - G(t - S_k)) + \sum_{k \geq 0} G(t - S_k) - V(t),$$

where G is the distribution function of η. It suffices to prove that

$$\mathbb{E}\left[\sup_{s \in [0, t]} \left(\sum_{k \geq 0} (\mathbb{1}_{\{S_k + \eta_{k+1} \leq s\}} - G(s - S_k)) \right)^2 \right] = \mathrm{O}(t), \quad t \to \infty \tag{17}$$

and

$$\mathbb{E}\left[\sup_{s \in [0, t]} \left(\sum_{k \geq 0} G(t - S_k) - V(t) \right)^2 \right] = \mathrm{O}(t), \quad t \to \infty. \tag{18}$$

The proof of (18) given in [5] goes through for any distribution of η and as such applies without changes under the present assumptions. On the other hand, the proof of (17) given in [5] depends crucially on the assumption $\mathbb{E}[\eta] < \infty$ imposed in [5, Lemma 4.2(b)]. Thus another argument has to be found.

For $u, t \geq 0$, put

$$Z_t(u) := \sum_{k \geq 0} \left(\mathbb{1}_{\{S_k + \eta_{k+1} \leq ut\}} - G(ut - S_k) \right) \mathbb{1}_{\{S_k \leq ut\}},$$

so that

$$\mathbb{E}\left[\sup_{s \in [0, t]} \left(\sum_{k \geq 0} (\mathbb{1}_{\{S_k + \eta_{k+1} \leq s\}} - G(s - S_k)) \right)^2 \right] = \mathbb{E}\left[\sup_{u \in [0, 1]} (Z_t(u))^2 \right].$$

In what follows we write $\sup_{u \in K}$ when the supremum is taken over an uncountable set K and $\max_{m \leq k \leq n}$ when the maximum is taken over the discrete set $\{m, m+1, \ldots, n\}$. We start by observing that, for positive integer $I = I(t)$ to be chosen later in (25),

$$\sup_{u \in [0, 1]} |Z_t(u)| = \max_{0 \leq k \leq 2^I - 1} \sup_{u \in [0, 2^{-I}]} \left| Z_t(k2^{-I} + u) - Z_t(k2^{-I}) + Z_t(k2^{-I}) \right|$$

$$\leq \max_{0 \leq k \leq 2^I} |Z_t(k2^{-I})| + \max_{0 \leq k \leq 2^I - 1} \sup_{u \in [0, 2^{-I}]} |Z_t(k2^{-I} + u) - Z_t(k2^{-I})|.$$

We have used subadditivity of the supremum for the last inequality. We proceed as on page 764 of [12]. Put $F_j := \{k2^{-j} \colon 0 \leq k \leq 2^j\}$ for $j \in \mathbb{N}_0$ and fix $u \in F_I$. Now define $u_j := \max\{w \in F_j \colon w \leq u\}$ for non-negative integer $j \leq I$. Then $u_{j-1} = u_j$ or $u_{j-1} = u_j - 2^{-j}$. With this at hand,

$$|Z_t(u)| = \left| \sum_{j=1}^{I} (Z_t(u_j) - Z_t(u_{j-1})) + Z_t(u_0) \right| \leq \sum_{j=0}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|.$$

Combining the fragments together, we arrive at the inequality which is a starting point of our subsequent work:

$$\sup_{u \in [0, 1]} |Z_t(u)|$$

$$\leq \sum_{j=0}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| + \max_{0 \leq k \leq 2^I - 1} \sup_{u \in [0, 2^{-I}]} |Z_t(k2^{-I} + u) - Z_t(k2^{-I})|.$$

Thus (17) follows if we can show that

$$\mathbb{E}\left[\left(\sum_{j=0}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| \right)^2 \right] = \mathrm{O}(t), \quad t \to \infty \tag{19}$$

and that

$$\mathbb{E}\left[\max_{0 \leq k \leq 2^I - 1} \sup_{u \in [0, 2^{-I}]} (Z_t(k2^{-I} + u) - Z_t(k2^{-I}))^2 \right] = \mathrm{O}(t), \quad t \to \infty. \tag{20}$$

We intend to prove that, for $u, v \geq 0$, $u > v$, and $t \geq 0$,

$$\mathbb{E}[(Z_t(u) - Z_t(v))^2] \leq 2\mathbb{E}[v(1)]a((u - v)t), \tag{21}$$

where, for $t \geq 0$,

$$a(t) := \sum_{k=0}^{[t]+1} (1 - G(k))$$

and

$$v(t) = \inf\{k \in \mathbb{N} \colon S_k > t\} = \sum_{k \geq 0} \mathbb{1}_{\{S_k \leq t\}}.$$

Indeed,

$$
\begin{aligned}
&\mathbb{E}[(Z_t(u) - Z_t(v))^2] \\
&= \mathbb{E}\left[\int_{(vt,\, ut]} G(ut - y)(1 - G(ut - y))\, d\nu(y)\right] \\
&\quad + \mathbb{E}\left[\int_{[0,\, vt]} (G(ut - y) - G(vt - y))(1 - G(ut - y) + G(vt - y))\, d\nu(y)\right] \\
&\leq \mathbb{E}\left[\int_{(vt,\, ut]} (1 - G(ut - y))\, d\nu(y)\right] + \mathbb{E}\left[\int_{[0,\, vt]} (G(ut - y) - G(vt - y))\, d\nu(y)\right].
\end{aligned}
$$

Using Lemma 5 with $f(y) = (1 - G(y))\mathbb{1}_{[0,\,(u-v)t)}(y)$ and $f(y) = G((u - v)t + y) - G(y)$, respectively, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\int_{(vt,\, ut]} (1 - G(ut - y))\, d\nu(y)\right] &= \mathbb{E}\left[\int_{[0,\, ut]} (1 - G(ut - y))\mathbb{1}_{[0,\,(u-v)t)}(ut - y)\, d\nu(y)\right] \\
&\leq \mathbb{E}[\nu(1)] \sum_{n=0}^{[ut]} \sup_{y \in [n,\, n+1)} ((1 - G(y))\mathbb{1}_{[0,\,(u-v)t)}(y)) \\
&\leq \mathbb{E}[\nu(1)] \sum_{n=0}^{[(u-v)t]} (1 - G(n)) \leq \mathbb{E}[\nu(1)]a((u - v)t) \quad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\int_{[0,\, vt]} (G(ut - y) - G(vt - y))\, d\nu(y)\right] \\
&\leq \mathbb{E}[\nu(1)] \sum_{n=0}^{[vt]} \sup_{y \in [n,\, n+1)} (G((u - v)t + y) - G(y)) \\
&\leq \mathbb{E}[\nu(1)]\left(\sum_{n=0}^{[vt]} (1 - G(n)) - \sum_{n=0}^{[vt]} (1 - G((u - v)t + n + 1))\right) \\
&\leq \mathbb{E}[\nu(1)]\left(\sum_{n=0}^{[vt]} (1 - G(n)) - \sum_{n=0}^{[ut]+1} (1 - G(n)) + \sum_{n=0}^{[(u-v)t]+1} (1 - G(n))\right) \\
&\leq \mathbb{E}[\nu(1)]a((u - v)t). \quad (23)
\end{aligned}
$$

Combining (22) and (23) yields (21).

Proof of (19). The assumption $\mathbb{E}[\eta^a] < \infty$ entails $\lim_{t\to\infty} t^a(1 - G(t)) = 0$ and thereupon, given $C > 0$, there exists $t_1 > 0$ such that $a(t) \leq Ct^{1-a}$ whenever $t \geq t_1$. Using this in combination with (21) yields

$$
\mathbb{E}[(Z_t(k2^{-j}) - Z_t((k-1)2^{-j}))^2] \leq 2C\mathbb{E}[\nu(1)]2^{-j(1-a)} =: C_1 2^{-j(1-a)} \quad (24)
$$

whenever $2^{-j}t \geq t_1$. Let $I = I(t)$ denote the integer number satisfying

$$
2^{-I}t \geq t_1 > 2^{-I-1}t. \quad (25)
$$

Then the inequalities (24) and

$$\mathbb{E}\left[\max_{1 \le k \le 2^j} (Z_t(k2^{-j}) - Z_t((k-1)2^{-j}))^2\right] \le \sum_{k=1}^{2^j} \mathbb{E}[(Z_t(k2^{-j}) - Z_t((k-1)2^{-j}))^2]$$
$$\le C_1 2^{aj}$$

hold whenever $j \le I$. Invoking the triangle inequality for the L_2-norm yields

$$\mathbb{E}\left[\left(\sum_{j=0}^{I} \max_{1 \le k \le 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|\right)^2\right]$$
$$\le \left(\sum_{j=0}^{I} \left(\mathbb{E}\left[\max_{1 \le k \le 2^j} (Z_t(k2^{-j}) - Z_t((k-1)2^{-j}))^2\right]\right)^{1/2}\right)^2$$
$$\le C_1 \left(\sum_{j=0}^{I} 2^{aj/2}\right)^2$$
$$= \left(\mathrm{O}(2^{aI/2})\right)^2$$
$$= \mathrm{O}(t^a), \quad t \to \infty.$$

Here the last equality is ensured by the choice of I.

Proof of (20). We shall use the decomposition

$$Z_t(k2^{-I} + u) - Z_t(k2^{-I})$$
$$= \sum_{j \ge 0} \left(\mathbb{1}_{\{S_j + \eta_{j+1} \le (k2^{-I} + u)t\}} - G((k2^{-I} + u)t - S_j)\right)\mathbb{1}_{\{k2^{-I}t < S_j \le (k2^{-I} + u)t\}}$$
$$+ \sum_{j \ge 0} \left(\mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \le (k2^{-I} + u)t\}} - (G((k2^{-I} + u)t - S_j) - G(k2^{-I}t - S_j))\right)\mathbb{1}_{\{S_j \le k2^{-I}t\}}$$
$$=: J_1(t, k, u) + J_2(t, k, u).$$

For $i = 1, 2$ we have

$$\mathbb{E}\left[\max_{0 \le k \le 2^I - 1} \sup_{u \in [0, \, 2^{-I}]} (J_i(t, k, u))^2\right] = \mathrm{O}(t), \quad t \to \infty, \tag{26}$$

completing the proof.

Proof of (26) *for* $i = 1$. Since $|J_1(t, k, u)| \le \nu((k2^{-I} + u)t) - \nu(k2^{-I}t)$ and $t \mapsto \nu(t)$ is a.s. non-decreasing, we infer $\sup_{u \in [0, \, 2^{-I}]} |J_1(t, k, u)| \le \nu((k+1)2^{-I}t) - \nu(k2^{-I}t)$ a.s. Hence

$$\mathbb{E}\left[\max_{0 \le k \le 2^I - 1} \sup_{u \in [0, \, 2^{-I}]} (J_i(t, k, u))^2\right]$$
$$\le \mathbb{E}\left[\max_{0 \le k \le 2^I - 1} (\nu((k+1)2^{-I}t) - \nu(k2^{-I}t))^2\right]$$
$$\le \sum_{k=0}^{2^I - 1} \mathbb{E}[(\nu((k+1)2^{-I}t) - \nu(k2^{-I}t))^2] \le 2^I \mathbb{E}[(\nu(2^{-I}t))^2]$$
$$\le (t/t_1)\mathbb{E}[(\nu(2t_1))^2] = \mathrm{O}(t), \quad t \to \infty.$$

Here the second inequality follows from distributional subadditivity of $\nu(t)$ (see e.g. formula (5.7) of [6]), and the third inequality is secured by the choice of I.

Proof of (26) *for* $i = 2$. We have

$$\sup_{u \in [0,\, 2^{-I}]} |J_2(t, k, u)|$$

$$\leq \sup_{u \in [0,\, 2^{-I}]} \left(\sum_{j \geq 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k2^{-I}+u)t\}} \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right.$$

$$\left. + \sum_{j \geq 0} \left(G((k2^{-I}+u)t - S_j) - G(k2^{-I}t - S_j) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right)$$

$$\leq \sum_{j \geq 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k+1)2^{-I}t\}} \mathbb{1}_{\{S_j \leq k2^{-I}t\}}$$

$$+ \sum_{j \geq 0} \left(G(((k+1)2^{-I})t - S_j) - G(k2^{-I}t - S_j) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}}$$

$$\leq \left| \sum_{j \geq 0} \left(\mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k+1)2^{-I}t\}} - \left(G(((k+1)2^{-I})t - S_j) - G(k2^{-I}t - S_j) \right) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right|$$

$$+ 2 \sum_{j \geq 0} \left(G(((k+1)2^{-I})t - S_j) - G(k2^{-I}t - S_j) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}}$$

$$=: J_{21}(t, k) + 2 J_{22}(t, k).$$

Using (23) with $u = (k+1)2^{-I}$ and $v = k2^{-I}$, we obtain

$$\mathbb{E}[(J_{22}(t, k))^2] \leq \mathbb{E}[(\nu(1))^2](a(2^{-I}t))^2 \leq \mathbb{E}[(\nu(1))^2](a(2t_1))^2,$$

which implies

$$\mathbb{E}\left[\left(\max_{0 \leq k \leq 2^I - 1} J_{22}(t, k) \right)^2 \right] \leq 2^I \max_{0 \leq k \leq 2^I - 1} \mathbb{E}[(J_{22}(t, k))^2]$$

$$\leq (t/t_1) \mathbb{E}[(\nu(1))^2](a(2t_1))^2$$

$$= \mathrm{O}(t), \quad t \to \infty.$$

Further, by (23),

$$\mathbb{E}[(J_{21}(t, k))^2] = \mathbb{E}\left[\int_{[0,\, k2^{-I}t]} \left(G((k+1)2^{-I}t - y) - G(k2^{-I}t - y) \right) \mathrm{d}\nu(y) \right] \leq \mathbb{E}[\nu(1)]a(2^{-I}t).$$

Hence $\mathbb{E}[(\max_{0 \leq k \leq 2^I - 1} J_{21}(t, k))^2] = \mathrm{O}(t)$ by the same reasoning as above, and (26) for $i = 2$ follows. The proof of Assertion 3 is complete. $\square$

4. Proofs of the main results

4.1. Proofs of Assertion 1 and Theorem 1

Proof of Assertion 1. The simplest way to prove this is to use Laplace transforms. Indeed, for fixed $j \in \mathbb{N}$,

$$\int_{[0,\, \infty)} \mathrm{e}^{-st} \, \mathrm{d}V_j(t) = \left(\frac{\mathbb{E}[\mathrm{e}^{-s\eta}]}{1 - \mathbb{E}[\mathrm{e}^{-s\xi}]} \right)^j \sim \frac{1}{\mathtt{m}^j s^j}, \quad s \to 0+.$$

By Karamata's Tauberian theorem (Theorem 1.7.1 of [2]), (3) holds. $\square$

Proof of Theorem 1. We use induction on j. Let $j = 1$. Write

$$V(t) - \mathrm{m}^{-1}t = \int_{[0,\,t]} (U(t-y) - \mathrm{m}^{-1}(t-y))\,\mathrm{d}G(y) - \mathrm{m}^{-1} \int_0^t (1 - G(y))\,\mathrm{d}y. \qquad (27)$$

Plainly, the second term converges to $-\mathrm{m}^{-1}\mathbb{E}[\eta]$ as $t \to \infty$. It is a simple consequence of the Blackwell theorem that

$$\lim_{t \to \infty} (U(t) - \mathrm{m}^{-1}t) = (2\mathrm{m}^2)^{-1}\mathbb{E}[\xi^2] = b_U.$$

Using this in combination with (11), we invoke Lebesgue's dominated convergence theorem to infer that the first term in (27) converges to b_U as $t \to \infty$. Thus we have shown that (4) with $j = 1$ holds true.

Assume that (4) holds for $j = k$. In view of (4) with $j = 1$, given $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$|V(t) - \mathrm{m}^{-1}t - b_V| \le \varepsilon \qquad (28)$$

whenever $t \ge t_0$. Write, for $t \ge t_0$,

$$
\begin{aligned}
V_{k+1}(t) &- \frac{t^{k+1}}{(k+1)!\mathrm{m}^{k+1}} \\
&= \int_{[0,\,t-t_0]} (V(t-y) - \mathrm{m}^{-1}(t-y))\,\mathrm{d}V_k(y) \\
&\quad + \int_{(t-t_0,\,t]} (V(t-y) - \mathrm{m}^{-1}(t-y))\,\mathrm{d}V_k(y) + \mathrm{m}^{-1} \int_0^t \left(V_k(y) - \frac{y^k}{k!\mathrm{m}^k} \right) \mathrm{d}y \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned}
$$

In view of (28),

$$(b_V - \varepsilon)V_k(t - t_0) \le I_1(t) \le (b_V + \varepsilon)V_k(t - t_0),$$

whence

$$\frac{b_V - \varepsilon}{k!\mathrm{m}^k} \le \liminf_{t \to \infty} \frac{I_1(t)}{t^k} \le \limsup_{t \to \infty} \frac{I_1(t)}{t^k} \le \frac{b_V + \varepsilon}{k!\mathrm{m}^k}$$

by Assertion 1.

Using (13) we obtain $|I_2(t)| \le c_V(V_k(t) - V_k(t - t_0))$ for all $t \ge t_0$, whence $\lim_{t \to \infty} t^{-k}I_2(t) = 0$ by Theorem 3. A combination of this with the last centered formula and sending $\varepsilon \to 0+$, we infer

$$I_1(t) + I_2(t) \sim \frac{b_V t^k}{k!\mathrm{m}^k}, \qquad t \to \infty.$$

Finally, by the induction assumption and L'Hôpital's rule,

$$I_3(t) \sim \frac{b_V t^k}{(k-1)!\mathrm{m}^k}, \qquad t \to \infty.$$

Combining fragments together, we arrive at (4) with $j = k + 1$. □

4.2. Proofs of Theorems 2 and 3

Proof of Theorem 2. The proof of Theorem 2.7(a) of [3] applies, with obvious simplifications. Note that for early generations (j is fixed), asymptotic relation (3) holds under the sole assumption $\mathtt{m} < \infty$. This is not the case for intermediate generations ($j = j(t) \to \infty$, $j(t) = \mathrm{o}(t)$ as $t \to \infty$) treated in [3], which explains the appearance of the additional assumption $\mathbb{E}[\xi^r] < \infty$ for some $r \in (1, 2]$ in [3, Theorem 2.7]. □

Proof of Theorem 3. When $j = 1$, relation (5) holds by Lemma 4.2(a) of [3]. Write

$$V_j(t+h) - V_j(t) = \int_{[0,\,t]} (V(t+h-y) - V(t-y))\, \mathrm{d}V_{j-1}(y) + \int_{(t,\,t+h]} V(t+h-y)\, \mathrm{d}V_{j-1}(y)$$
$$=: A_j(t) + B_j(t).$$

We first show that the contribution of $B_j(t)$ is negligible. Indeed, using monotonicity of V and $\lim_{t\to\infty} (V_j(t+h)/V_j(t)) = 1$ (see Assertion 1), we obtain

$$B_j(t) \le V(h)(V_{j-1}(t+h) - V_{j-1}(t)) = \mathrm{o}(V_{j-1}(t)), \quad t \to \infty.$$

In view of (5) with $j = 1$, given $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$|V(t+h) - V(t) - \mathtt{m}^{-1}h| \le \varepsilon$$

whenever $t \ge t_0$. Thus we have, for $t \ge t_0$,

$$A_j(t) = \int_{[0,\,t-t_0]} (V(t+h-y) - V(t-y))\, \mathrm{d}V_{j-1}(y) + \int_{(t-t_0,\,t]} V(t+h-y)\, \mathrm{d}V_{j-1}(y)$$
$$=: A_{j,1}(t) + A_{j,2}(t).$$

By the argument used for $B_j(t)$, we infer $A_{j,2}(t) = \mathrm{o}(V_{j-1}(t))$. Further,

$$A_{j,1}(t) \le (\mathtt{m}^{-1}h + \varepsilon)V_{j-1}(t-t_0),$$

whence

$$\limsup_{t\to\infty} (A_j(t)/V_{j-1}(t)) \le \mathtt{m}^{-1}h.$$

A symmetric argument proves the converse inequality for the limit inferior. Invoking Assertion 1 completes the proof of Theorem 3. □

4.3. Proof of Theorem 4

For $j \in \mathbb{N}$ and $t \ge 0$, put $D_j(t) := \mathrm{Var}[N_j(t)]$. Recall that, as a consequence of the assumption $\eta = \xi$ a.s., $T_r = S_r$ for $r \in \mathbb{N}$ and $V(t) = \tilde{U}(t) = \sum_{r\ge1} \mathbb{P}\{S_r \le t\}$ for $t \ge 0$. However, we prefer to write V_j rather than $\tilde{U}_j$.

Let $j \in \mathbb{N}$, $j \ge 2$. We obtain with the help of (1)

$$N_j(t) - V_j(t) = \sum_{r\ge1} \left(N_{j-1}^{(r)}(t-S_r) - V_{j-1}(t-S_r)\right)\mathbb{1}_{\{S_r \le t\}} + \left(\sum_{r\ge1} V_{j-1}(t-S_r)\mathbb{1}_{\{S_r \le t\}} - V_j(t)\right)$$
$$=: N_{j,1}(t) + N_{j,2}(t), \quad t \ge 0.$$

Since $N_{j-1}^{(1)}, N_{j-1}^{(2)}, \ldots$ are independent of $(S_r)_{r \in \mathbb{N}}$ and $\mathbb{E}N_{j-1}^{(r)}(t) = V_{j-1}(t)$, $r \in \mathbb{N}$, $t \geq 0$, we infer $\mathbb{E}(N_{j,1}(t)N_{j,2}(t) \mid (S_r)_{r \in \mathbb{N}}) = 0$ a.s., whence

$$D_j(t) = \mathbb{E}[(N_{j,1}(t))^2] + \mathbb{E}[(N_{j,2}(t))^2]. \tag{29}$$

We start by showing that the following asymptotic relations hold, for $j \geq 2$, as $t \to \infty$:

$$\mathbb{E}[(N_{j,2}(t))^2] = \mathrm{Var}\left[\sum_{r \geq 1} V_{j-1}(t - S_r)\mathbb{1}_{\{S_r \leq t\}}\right]$$

$$= \mathbb{E}\left[\left(\sum_{r \geq 1} V_{j-1}(t - S_r)\mathbb{1}_{\{S_r \leq t\}}\right)^2\right] - V_j^2(t) \sim \frac{\mathsf{s}^2}{(2j-1)((j-1)!)^2\mathsf{m}^{2j+1}}t^{2j-1}, \quad t \to \infty. \tag{30}$$

Proof of (30). We shall use the equality (see formula (4.9) of [8])

$$\mathbb{E}\left[\left(\sum_{r \geq 1} V_{j-1}(t - S_r)\mathbb{1}_{\{S_r \leq t\}}\right)^2\right]$$

$$= 2\int_{[0,\,t]} V_{j-1}(t - y)V_j(t - y)\,\mathrm{d}\tilde{U}(y) + \int_{[0,\,t]} V_{j-1}^2(t - y)\,\mathrm{d}\tilde{U}(y). \tag{31}$$

In view of (4),

$$\int_{[0,\,t]} V_{j-1}(t - y)V_j(t - y)\,\mathrm{d}\tilde{U}(y)$$

$$= \int_{[0,\,t]}\left(\frac{(t-y)^{j-1}}{(j-1)!\mathsf{m}^{j-1}} + \frac{b_V(j-1)(t-y)^{j-2}}{(j-2)!\mathsf{m}^{j-2}} + \mathrm{o}((t-y)^{j-2})\right)$$

$$\times \left(\frac{(t-y)^j}{j!\mathsf{m}^j} + \frac{b_V\,j(t-y)^{j-1}}{(j-1)!\mathsf{m}^{j-1}} + \mathrm{o}((t-y)^{j-1})\right)\mathrm{d}\tilde{U}(y)$$

$$= \frac{1}{(j-1)!j!\mathsf{m}^{2j-1}}\int_{[0,\,t]}(t-y)^{2j-1}\,\mathrm{d}\tilde{U}(y)$$

$$+ \frac{b_V(2j^2 - 2j + 1)}{(j-1)!j!\mathsf{m}^{2j-2}}\int_{[0,\,t]}(t-y)^{2j-2}\,\mathrm{d}\tilde{U}(y) + \int_{[0,\,t]}\mathrm{o}((t-y)^{2j-2})\,\mathrm{d}\tilde{U}(y),$$

where $b_V = \mathbb{E}[\xi^2]/(2\mathsf{m}^2) - 1$ because under the present assumption $\mathbb{E}[\eta] = \mathsf{m}$. According to (15),

$$\int_{[0,\,t]}(t-y)^{2j-1}\,\mathrm{d}\tilde{U}(y) = \frac{t^{2j}}{2j\mathsf{m}} + b_V\,t^{2j-1} + \mathrm{o}(t^{2j-1}), \quad t \to \infty,$$

$$\int_{[0,\,t]}(t-y)^{2j-2}\,\mathrm{d}\tilde{U}(y) = \frac{t^{2j-1}}{(2j-1)\mathsf{m}} + \mathrm{o}(t^{2j-1}), \quad t \to \infty.$$

Also, it can be checked that

$$\int_{[0,\,t]}\mathrm{o}((t-y)^{2j-2})\,\mathrm{d}\tilde{U}(y) = \mathrm{o}(t^{2j-1}), \quad t \to \infty.$$

By Assertion 1,

$$V_{j-1}^2(t) \sim \frac{t^{2j-2}}{((j-1)!)^2 \mathtt{m}^{2j-2}} \quad \text{and} \quad V(t) = \tilde{U}(t) \sim \frac{t}{\mathtt{m}}, \quad t \to \infty.$$

As in the proof of Assertion 1 we now invoke Karamata's Tauberian theorem (Theorem 1.7.1 of [2]) to obtain

$$\int_{[0, t]} V_{j-1}^2(t - y) \, \mathrm{d}\tilde{U}(y) \sim \frac{t^{2j-1}}{(2j-1)((j-1)!)^2 \mathtt{m}^{2j-1}}, \quad t \to \infty.$$

Using the aforementioned asymptotic relations and recalling (31), we infer that

$$\mathbb{E}\left[\left(\sum_{r \geq 1} V_{j-1}(t - S_r)\mathbb{1}_{\{S_r \leq t\}}\right)^2\right] = \frac{t^{2j}}{(j!)^2 \mathtt{m}^{2j}} + \frac{2b_V \, t^{2j-1}}{(j-1)!j!\mathtt{m}^{2j-1}} + \frac{2b_V(2j^2 - 2j + 1)t^{2j-1}}{(2j-1)(j-1)!j!\mathtt{m}^{2j-1}}$$

$$+ \frac{t^{2j-1}}{(2j-1)((j-1)!)^2 \mathtt{m}^{2j-1}} + \mathrm{o}(t^{2j-1}), \quad t \to \infty.$$

Further, as $t \to \infty$,

$$V_j^2(t) = \frac{t^{2j}}{(j!)^2 \mathtt{m}^{2j}} + \frac{2t^j}{j!\mathtt{m}^j}\left(V_j(t) - \frac{t^j}{j!\mathtt{m}^j}\right) + \left(V_j(t) - \frac{t^j}{j!\mathtt{m}^j}\right)^2$$

$$= \frac{t^{2j}}{(j!)^2 \mathtt{m}^{2j}} + \frac{2b_V \, t^{2j-1}}{((j-1)!)^2 \mathtt{m}^{2j-1}} + \mathrm{o}(t^{2j-1}),$$

having utilized (4). The last two asymptotic relations entail (30).

With (30) at hand we are ready to prove (6). To this end, we shall use mathematical induction. If $j = 1$, (6) takes the form $D_1(t) = \mathbb{E}[(N(t) - \tilde{U}(t))^2] \sim \mathtt{s}^2 \mathtt{m}^{-3} t$ as $t \to \infty$. This can be checked along the lines of the proof of (30). Alternatively, this relation follows from Theorem 3.8.4 of [6], where the assumption that the distribution of ξ is nonlattice is not made. Assume that (6) holds for $j = k - 1 \geq 1$, that is,

$$D_{k-1}(t) \sim \frac{\mathtt{s}^2}{(2k-3)((k-2)!)^2 \mathtt{m}^{2k-1}} t^{2k-3}, \quad t \to \infty.$$

Using this and the equality

$$\mathbb{E}[(N_{k,1}(t))^2] = \mathbb{E}\left[\sum_{r \geq 1} D_{k-1}(t - S_r)\mathbb{1}_{\{S_r \leq t\}}\right] = \int_{[0, t]} D_{k-1}(t - y) \, \mathrm{d}\tilde{U}(y)$$

in combination with Karamata's Tauberian theorem (Theorem 1.7.1 of [2]) or, more simply, a bare hands calculation, we infer

$$\mathbb{E}[(N_{k, 1}(t))^2] \sim \frac{\mathtt{s}^2}{(2k-3)(2k-2)((k-2)!)^2 \mathtt{m}^{2k}} t^{2k-2}, \quad t \to \infty.$$

By virtue of (29) and (30) we conclude that (6) holds for $j = k$. The proof of Theorem 4 is complete. □

4.4. Proof of Theorem 5

We use mathematical induction. When $j = 1$, (7) holds true by formula (24) of [1]. Assuming that it holds for $j = k$, we intend to show that (7) also holds for $j = k + 1$. To this end, we write with the help of (1) for $j = k + 1$

$$N_{k+1}(t) = \sum_{r \geq 1} \left(N_{1,k+1}^{(r)}\left(t - T_r^{(k)}\right) - V\left(t - T_r^{(k)}\right) \right) \mathbb{1}_{\{T_r^{(k)} \leq t\}} + \int_{[0,t]} V(t - y)\,\mathrm{d}N_k(y), \quad t \geq 0.$$

By Assertion 1, given $\varepsilon > 0$ there exists $t_0 > 0$ such that $|t^{-1}V(t) - \mathrm{m}^{-1}| \leq \varepsilon$ whenever $t \geq t_0$. We have

$$\int_{(t-t_0,\,t]} V(t - y)\,\mathrm{d}N_k(y) \leq V(t_0)(N_k(t) - N_k(t - t_0)) = \mathrm{o}(t^k) \quad \text{a.s.} \quad \text{as } t \to \infty$$

by the induction assumption. Analogously,

$$\int_{(t-t_0,\,t]} (t - y)\,\mathrm{d}N_k(y) = \mathrm{o}(t^k) \quad \text{a.s.} \quad \text{as } t \to \infty.$$

Further,

$$\int_{[0,\,t-t_0]} V(t - y)\,\mathrm{d}N_k(y) \geq (\mathrm{m}^{-1} - \varepsilon) \int_{[0,\,t-t_0]} (t - y)\,\mathrm{d}N_k(y)$$

$$= (\mathrm{m}^{-1} - \varepsilon)\left(\int_{[0,\,t]} (t - y)\,\mathrm{d}N_k(y) - \int_{(t-t_0,\,t]} (t - y)\,\mathrm{d}N_k(y) \right).$$

Using

$$\int_{[0,\,t]} (t - y)\,\mathrm{d}N_k(y) = \int_0^t N_k(y)\,\mathrm{d}y$$

and applying L'Hôpital's rule together with the induction assumption, we infer

$$\lim_{t \to \infty} \frac{\int_{[0,\,t]} (t - y)\,\mathrm{d}N_k(y)}{t^{k+1}} = \frac{1}{\mathrm{m}^k(k + 1)!} \quad \text{a.s.}$$

Combining the pieces together, we arrive at

$$\liminf_{t \to \infty} \frac{\int_{[0,\,t]} V(t - y)\,\mathrm{d}N_k(y)}{t^{k+1}} \geq \frac{1}{\mathrm{m}^{k+1}(k + 1)!} \quad \text{a.s.}$$

The converse inequality for the limit superior follows similarly, whence

$$\lim_{t \to \infty} \frac{\int_{[0,\,t]} V(t - y)\,\mathrm{d}N_k(y)}{t^{k+1}} = \frac{1}{\mathrm{m}^{k+1}(k + 1)!} \quad \text{a.s.} \tag{32}$$

Further,

$$\mathbb{E}\left[\left(\sum_{r\geq 1}\left(N_{1,\,k+1}^{(r)}\left(t-T_r^{(k)}\right)-V\left(t-T_r^{(k)}\right)\right)\mathbb{1}_{\{T_r^{(k)}\leq t\}}\right)^2\right]$$

$$= \sum_{r\geq 1}\mathbb{E}\left[\left(N_{1,\,k+1}^{(r)}\left(t-T_r^{(k)}\right)-V\left(t-T_r^{(k)}\right)\right)^2\mathbb{1}_{\{T_r^{(k)}\leq t\}}\right]$$

$$\leq \int_{[0,\,t]}\mathbb{E}[(N(t-y))^2]\,\mathrm{d}V_k(y)$$

$$\leq \mathbb{E}[(N(t))^2]V_k(t)$$

$$= \mathrm{O}(t^{k+2}), \quad t\to\infty,$$

having utilized monotonicity of $t\mapsto\mathbb{E}[(N(t))^2]$ for the last inequality and Lemmas 1 and 4 for the last equality. Now invoking the Markov inequality and the Borel–Cantelli lemma, we conclude that

$$\lim_{n\to\infty}\frac{\sum_{r\geq 1}\left(N_{1,\,k+1}^{(r)}\left(n^2-T_r^{(k)}\right)-V\left(n^2-T_r^{(k)}\right)\right)\mathbb{1}_{\{T_r^{(k)}\leq n^2\}}}{n^{2(k+1)}}=0 \quad\text{a.s.}$$

(n approaches ∞ along integers). This together with (32) yields

$$\lim_{n\to\infty}\frac{N_{k+1}(n^2)}{n^{2(k+1)}}=\frac{1}{\mathrm{m}^{k+1}(k+1)!} \quad\text{a.s.}$$

Thus it remains to show that we may pass to the limit continuously. To this end, note that for each $t\geq 0$ there exists $n\in\mathbb{N}$ such that $t\in[(n-1)^2, n^2)$, and use a.s. monotonicity of N_{k+1} to obtain

$$\frac{(n-1)^{2(k+1)}}{n^{2(k+1)}}\frac{N_{k+1}((n-1)^2)}{(n-1)^{2(k+1)}}\leq\frac{N_{k+1}(t)}{t^{k+1}}\leq\frac{N_{k+1}(n^2)}{n^{2(k+1)}}\frac{n^{2(k+1)}}{(n-1)^{2(k+1)}} \quad\text{a.s.}$$

Letting t tend to ∞, we arrive at

$$\lim_{t\to\infty}\frac{N_{k+1}(t)}{t^{k+1}}=\frac{1}{\mathrm{m}^{k+1}(k+1)!} \quad\text{a.s.,}$$

thereby completing the induction step. The proof of Theorem 5 is complete. $\qquad\square$

4.5. Proof of Theorem 6

In the case $\mathbb{E}[\eta]<\infty$ this result follows from Theorem 3.2 and Lemma 4.2 of [5]. Thus we concentrate on the case $\mathbb{E}[\eta^a]<\infty$ for $a\in(0,1)$ and $\mathbb{E}[\eta]=\infty$.

We are going to apply Theorem 7, stated in the Appendix, with $N_j^*=N_j$ for $j\in\mathbb{N}$. According to (14),

$$-c_1-c_2 t^{1-a}\leq V(t)-\mathrm{m}^{-1}t\leq c_U, \quad t\geq 0$$

for some positive constants c_1 and c_2 and $c_U=\mathrm{m}^{-2}\mathbb{E}[\xi^2]$, that is, condition (35) holds with $c=\mathrm{m}^{-1}$, $\omega=1$, $\varepsilon_1=a$, $\varepsilon_2=1$, $a_0+a_1=c_U$, $b_0=-c_1$, $b_1=-c_2$. By Assertion 3,

$$\mathbb{E}\left[\sup_{s\in[0,\,t]}(N(s)-V(s))^2\right]=\mathrm{O}(t), \quad t\to\infty,$$

that is, condition (36) holds with $\gamma = 1/2$. By Assertion 2,

$$\left(\frac{N(ut) - V(ut)}{\sqrt{m^{-3}s^2 t}} \right)_{u \geq 0} \Rightarrow (B(u))_{u \geq 0}, \quad t \to \infty$$

in the J_1-topology on D. This means that condition (37) holds with $\gamma = 1/2$, $b = m^{-3/2}s$ and $W = B$, a Brownian motion. Recall that the process B is locally Hölder-continuous with exponent β for any $\beta \in (0, 1/2)$. Thus, by Theorem 7, relation (8) is a specialization of (38) with $\gamma = 1/2$, $\omega = 1$, $R_j^{(1)} = B_{j-1}$, $j \in \mathbb{N}$, and $\rho_j = 1/(m^j j!)$, $j \in \mathbb{N}_0$.

Now we prove the claim that the centering $V_j(ut)$ can be replaced by that given in (9). We first note that the equality in (9) follows with the help of mathematical induction in k from the representation

$$\mathbb{E}[(t - R_i)^k \mathbb{1}_{\{R_i \leq t\}}] = \int_{[0, t]} (t - y)^k \, d\mathbb{P}\{R_i \leq y\}$$

$$= k \int_0^t \int_{[0, s]} (s - y)^{k-1} \, d\mathbb{P}\{R_i \leq y\} \, ds, \quad i, k \in \mathbb{N}, \ t \geq 0,$$

where $R_i := \eta_1 + \cdots + \eta_i$. Here the first step of induction is justified by the equality

$$\int_{[0, t]} (t - y) \, d\mathbb{P}\{R_i \leq y\} = \int_0^t \mathbb{P}\{R_i \leq s\} \, ds, \quad i \in \mathbb{N}, \ t \geq 0.$$

Further, we show that whenever $\mathbb{E}[\xi^2] < \infty$, irrespective of the distribution of η, for all $T > 0$,

$$\lim_{t \to \infty} t^{-(j-1/2)} \sup_{u \in [0, T]} |V_j(ut) - (j! m^j)^{-1} \mathbb{E}[(ut - R_j)^j \mathbb{1}_{\{R_j \leq ut\}}]| = 0. \tag{33}$$

To this end, we recall that, according to formula (4.4) of [4] (we use the formula with $\eta = 0$),

$$U_j(t) - \frac{t^j}{j! m^j} \leq \sum_{i=0}^{j-1} \binom{j}{i} \frac{c_U^{j-i} t^i}{i! m^i}, \quad t \geq 0,$$

where $c_U = m^{-2} \mathbb{E}[\xi^2]$. Using this and Lemma 1, we obtain, for $u \in [0, T]$ and $t \geq 0$,

$$\left| V_j(ut) - \frac{\mathbb{E}[(ut - R_j)^j \mathbb{1}_{\{R_j \leq ut\}}]}{j! m^j} \right| = \int_{[0, ut]} \left(U_j(ut - y) - \frac{(ut - y)^j}{j! m^j} \right) d\mathbb{P}\{R_j \leq y\}$$

$$\leq \int_{[0, ut]} \sum_{i=0}^{j-1} \binom{j}{i} \frac{c_U^{j-i}(ut - y)^i}{i! m^i} \, d\mathbb{P}\{R_j \leq y\}$$

$$\leq \sum_{i=0}^{j-1} \binom{j}{i} \frac{c_U^{j-i}(Tt)^i}{i! m^i}$$

$$= o(t^{j-1/2}), \quad t \to \infty,$$

which proves (33).

It remains to show that if $\mathbb{E}[\eta^{1/2}] < \infty$, then the centering $V_j(ut)$ can be replaced by $(ut)^j/(j! m^j)$. To justify this, it suffices to check that

$$\lim_{t \to \infty} \frac{\sup_{u \in [0, T]} \left((ut)^j - j! \int_0^{ut} \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j \leq y\} \, dy \, dt_j \cdots dt_2 \right)}{t^{j-1/2}} = 0. \tag{34}$$

The numerator of the ratio under the limit on the left-hand side of (34) is equal to

$$\sup_{u\in[0,\,T]} \int_0^{tu} \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j > y\}\,\mathrm{d}y\,\mathrm{d}t_j \cdots \mathrm{d}t_2$$

$$= \int_0^{tT} \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j > y\}\,\mathrm{d}y\,\mathrm{d}t_j \cdots \mathrm{d}t_2.$$

Hence we are left with showing that

$$\lim_{t\to\infty} t^{-(j-1/2)} \int_0^t \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j > y\}\,\mathrm{d}y\,\mathrm{d}t_j \cdots \mathrm{d}t_2 = 0.$$

Assume that $\mathbb{E}[\eta] < \infty$, so $\mathbb{E}[R_j] < \infty$ and thus

$$\lim_{t\to\infty} \int_0^t \mathbb{P}\{R_j > y\}\,\mathrm{d}y = \mathbb{E}[R_j] < \infty.$$

Then, using L'Hôpital's rule $(j-1)$ times, we obtain

$$\lim_{t\to\infty} \frac{\int_0^t \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j > y\}\,\mathrm{d}y\,\mathrm{d}t_j \cdots \mathrm{d}t_2}{t^{j-1/2}} = \frac{2^{j-1}}{1\cdot 3\cdot\ldots\cdot(2j-1)} \lim_{t\to\infty} \frac{\int_0^t \mathbb{P}\{R_j > y\}\,\mathrm{d}y}{t^{1/2}} = 0.$$

Assume that $\mathbb{E}[\eta] = \infty$. Since $\mathbb{E}[\eta^{1/2}] < \infty$ is equivalent to $\mathbb{E}[R_j^{1/2}] < \infty$, we infer

$$\lim_{t\to\infty} t^{1/2}\mathbb{P}\{R_j > t\} = 0.$$

With this at hand, using L'Hôpital's rule j times, we infer

$$\lim_{t\to\infty} \frac{\int_0^t \int_0^{t_2} \cdots \int_0^{t_j} \mathbb{P}\{R_j > y\}\,\mathrm{d}y\,\mathrm{d}t_j \cdots \mathrm{d}t_2}{t^{j-1/2}} = \frac{2^j}{1\cdot 3\cdot\ldots\cdot(2j-1)} \lim_{t\to\infty} t^{1/2}\mathbb{P}\{R_j > t\} = 0.$$

The proof of Theorem 6 is complete. $\qquad\qquad\square$

Appendix A. Auxiliary results

In this section we state several results borrowed from other sources. The first of these can be found in the proof of Lemma 7.3 of [1].

Lemma 5. *Let $f\colon [0,\infty) \to [0,\infty)$ be a locally bounded function. Then, for any $l\in\mathbb{N}$,*

$$\mathbb{E}\left[\left(\sum_{k\geq 0} f(t-S_k)\mathbb{1}_{\{S_k\leq t\}}\right)^l\right] \leq \left(\sum_{j=0}^{[t]} \sup_{y\in[j,\,j+1)} f(y)\right)^l \mathbb{E}[(\nu(1))^l], \quad t\geq 0,$$

where $\nu(t) = \inf\{k\in\mathbb{N}\colon S_k > t\}$.

For $j\in\mathbb{N}$ and $t\geq 0$, let $N_j^*(t)$ denote the number of the jth generation individuals with birth times $\leq t$ in a general branching process generated by an arbitrary locally finite point process T^*, and put $V_j^*(t) := \mathbb{E}[N_j^*(t)]$. In particular, $N_j^* = N_j$ for $j\in\mathbb{N}$ when $T^* = T$. For notational simplicity, put $N^* := N_1^*$ and $V^* := V_1^*$.

Let $W := (W(s))_{s \geq 0}$ denote a centered Gaussian process which is a.s. locally Hölder-continuous and satisfies $W(0) = 0$. For each $u > 0$, put

$$R_1^{(u)}(s) := W(s), \quad R_j^{(u)}(s) := \int_{[0,\,s]} (s-y)^{u(j-1)} \, \mathrm{d}W(y), \quad s \geq 0, \ j \geq 2.$$

The next result follows from Theorem 3.2 of [5] and its proof.

Theorem 7. *Assume the following conditions hold:*

(i)
$$b_0 + b_1 t^{\omega - \varepsilon_1} \leq V^*(t) - ct^\omega \leq a_0 + a_1 t^{\omega - \varepsilon_2} \tag{35}$$

for all $t \geq 0$ and some constants $c, \omega, a_0, a_1 > 0$, $0 < \varepsilon_1, \varepsilon_2 \leq \omega$ and $b_0, b_1 \in \mathbb{R}$,

(ii)
$$\mathbb{E}\left[\sup_{s \in [0,\,t]} (N^*(s) - V^*(s))^2 \right] = \mathrm{O}(t^{2\gamma}), \quad t \to \infty \tag{36}$$

for some $\gamma \in (0, \omega)$.

(iii)
$$\left(\frac{N^*(ut) - V^*(ut)}{bt^\gamma} \right)_{u \geq 0} \Rightarrow (W(u))_{u \geq 0}, \quad t \to \infty \tag{37}$$

in the J_1-topology on D for some $b > 0$ and the same γ as in (36).

Then

$$\left(\left(\frac{N_j^*(ut) - V_j^*(ut)}{b\rho_{j-1}t^{\gamma + \omega(j-1)}} \right)_{u \geq 0} \right)_{j \in \mathbb{N}} \Rightarrow \left((R_j^{(\omega)}(u))_{u \geq 0} \right)_{j \in \mathbb{N}} \tag{38}$$

in the J_1-topology on $D^{\mathbb{N}}$, where

$$\rho_j := \frac{(c\Gamma(\omega + 1))^j}{\Gamma(\omega j + 1)}, \quad j \in \mathbb{N}_0,$$

with $\Gamma(\cdot)$ denoting the gamma function.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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