Proceedings of the Edinburgh Mathematical Society (2023) **66**, 231–240
doi:10.1017/S0013091523000135

RIGIDITY OF CAPILLARY SURFACES IN COMPACT 3-MANIFOLDS WITH STRICTLY CONVEX BOUNDARY

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(Received 27 April 2022)

Abstract In this paper, we obtain one sharp estimate for the length $L(\partial\Sigma)$ of the boundary $\partial\Sigma$ of a capillary minimal surface Σ^2 in M^3, where M is a compact three-manifolds with strictly convex boundary, assuming Σ has index one. The estimate is in term of the genus of Σ, the number of connected components of $\partial\Sigma$ and the constant contact angle θ. Making an extra assumption on the geometry of M along ∂M, we characterize the global geometry of M, which is saturated only by the Euclidean three-balls. For capillary stable CMC surfaces, we also obtain similar results.

Keywords: capillary surfaces; minimal surfaces; constant mean curvature surfaces; compact three-manifolds; strictly convex boundary

2020 Mathematics subject classification: Primary 53A10; 53C42; 53C24

1. Introduction and statement of results

A very interesting and important variational problem in differential geometry is the free boundary problem for constant mean curvature (CMC) or minimal hypersurfaces. Given a compact Riemannian manifold (M^{n+1}, g) with nonempty boundary, the problem consists of finding critical points of the area functional among all compact hypersurfaces $\Sigma \subset M$ with $\partial\Sigma \subset \partial M$, which divides M into two subsets of prescribed volumes. Critical points for this problem are CMC or minimal hypersurfaces $\Sigma \subset M$ meeting ∂M orthogonally along $\partial\Sigma$, and they are known as CMC or minimal hypersurfaces with free boundary. In the last few years, this subject have been studied by many authors, for example, [1, 2, 6, 8–12].

A natural generalization of free boundary hypersurfaces are capillary hypersurfaces. These are critical points of a certain energy functional, which will be presented in § 2. As will be deduced later, they can be characterized as CMC or minimal hypersurfaces whose boundary meet the ambient boundary at a constant angle.

Like in the free boundary case, questions relating the topology and the geometry of hypersurfaces raise a lot of attention from geometers. The first result in this direction was obtained by Nitsche [7], who proved that any immersed capillary disc in the unit ball of $\mathbb{R}^3$ must be either a spherical cap or a flat disc. Later, Ros and Souam [11] extended this result to capillary discs in balls of three-dimensional space forms. Recently, Wang and Xia [13] analysed the problem in an arbitrary dimension and proved that any stable immersed capillary hypersurface in a ball in space forms is totally umbilical.

In this work, we impose curvature assumptions on the ambient three-manifold and look for restrictions in the topology of the possible immersed CMC or minimal capillary surfaces. Our goal in this work is to extend the result proved by Mendes [6], for free boundary minimal surfaces immersed in compact three-manifolds with strictly nonempty boundary, to capillary minimal surfaces. More precisely,

Theorem 1. *Let M^3 be a compact Riemannian three-manifold with non-empty boundary ∂M. Suppose that $\mathrm{Ric} \geq 0$ and $\mathrm{II} \geq 1$, where Ric is the Ricci tensor of M and II is the second fundamental form of ∂M. If Σ^2 is a properly embedded capillary minimal surface of index one in M, with constant contact angle $\theta \in (0, \pi)$, then the length $L(\partial\Sigma)$ of $\partial\Sigma$ satisfies*

$$L(\partial\Sigma) + \cos\theta \int_{\partial\Sigma} A(\nu, \nu)\,\mathrm{d}s \leq 2\pi(g + r)\sin\theta, \tag{1}$$

where A denotes the second fundamental form de Σ, ν is the outward unit conormal for $\partial\Sigma$ in Σ, g is the genus of Σ and r is the number of connected components of $\partial\Sigma$. Moreover, if equality holds, we have the following:

(i) Σ is isometric to a flat disk of radius $\sin\theta$;
(ii) Σ is totally geodesic in M;
(iii) the geodesic curvature of $\partial\Sigma$ in ∂M is $\overline{k} = \cot\theta$;
(iv) $\mathrm{II} = 1$; and
(v) all sectional curvatures of M vanish on Σ.

Making an extra assumption on the geometry of M along ∂M and by using Theorem 1, we characterize the global geometry of M when equality in Equation (1) holds.

Corollary 1. *Let M^3 be a compact Riemannian three-manifold with non-empty boundary ∂M. Suppose that $\mathrm{Ric} \geq 0$ and $\mathrm{II} \geq 1$ and $K_M(T_p\partial M) \geq 0$ for all $p \in \partial M$, where K_M is the sectional curvature of M. If Σ^2 is a properly embedded capillary minimal surface of index one in M, with constant contact angle $\theta \in (0, \pi)$, then*

$$L(\partial\Sigma) + \cos\theta \int_{\partial\Sigma} A(\nu, \nu)\,\mathrm{d}s \leq 2\pi(g + r)\sin\theta.$$

Furthermore, if equality holds, M^3 is isometric to the Euclidean unit three-ball and Σ^2 is isometric to the Euclidean disk of radius $\sin\theta$.

Wang and Xia [13] proved that any stable immersed capillary hypersurface in a ball in space forms is totally umbilical. In our next result, we consider immersed stable capillary

https://doi.org/10.1017/S0013091523000135 Published online by Cambridge University Press

CMC surfaces in three-dimensional compact Riemannian manifold with non-negative Ricci curvature and strictly convex boundary and prove the following:

Theorem 2. *Let M^3 be a compact Riemannian three-manifold with nonempty boundary ∂M. Suppose that* $\mathrm{Ric} \geq 0$ *and* $\mathrm{II} \geq 1$. *If Σ^2 is a properly embedded capillary stable CMC surface in M, with constant contact angle $\theta \in (0, \pi)$, then the length $L(\partial\Sigma)$ of $\partial\Sigma$ satisfies*

$$L(\partial\Sigma) + \cos\theta \int_{\partial\Sigma} A(\nu, \nu)\, \mathrm{d}s \leq 2\pi(g + r)\sin\theta,$$

where A denotes the second fundamental form de Σ, ν is outward unit conormal for $\partial\Sigma$ in Σ, g is the genus of Σ and r is the number of connected components of $\partial\Sigma$. Moreover, if equality holds, we have the following:

(i) *Σ is isometric to a flat disk of radius* $\sin\theta$*;*
(ii) *Σ is totally geodesic in M;*
(iii) *the geodesic curvature of $\partial\Sigma$ in ∂M is* $\overline{k} = \cot\theta$*;*
(iv) $\mathrm{II} = 1$*; and*
(v) *all sectional curvatures of M vanish on Σ.*

Observation 1. Note that in our results by taking $\theta = \pi/2$, we get the theorems proved by Mendes [6], for free boundary minimal surfaces.

Corollary 1 is also true if we change the hypothesis 'minimal of index one' by 'stable CMC and minimal'.

2. Preliminaries and basic results

The purpose of this section is to formally introduce the concept of capillary CMC and minimal hypersurfaces. Let (M^{n+1}, g) be a Riemannian manifold with non-empty boundary ∂M. Let Σ^n be a smooth compact manifold with non-empty boundary, and let $\varphi : \Sigma \to M$ be a smooth immersion of Σ into M. We say that φ is a proper immersion if $\varphi(\Sigma) \cap \partial M = \varphi(\partial\Sigma)$.

We assume that φ is orientable. Fix a unit normal vector field N for Σ along φ and denote by ν the outward unit conormal for $\partial\Sigma$ in Σ. Moreover, let $\overline{N}$ be the outward pointing unit normal for ∂M and let $\overline{\nu}$ be the unit normal for $\partial\Sigma$ in ∂M such that the bases $\{N, \nu\}$ and $\{\overline{N}, \overline{\nu}\}$ determine the same orientation in $(T\partial\Sigma)^{\perp}$.

Denote by A and H the second fundamental form and the mean curvature of the immersion φ, respectively. Precisely, $A(X, Y) = -g(D_X N, Y)$ and $H = \mathrm{tr}(A)$, where D is the Levi–Civita connection of M. Moreover, let $\mathrm{II}(v, w) = g(D_v\overline{N}, w)$ be the second fundamental form of ∂M.

A smooth function $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ is called a proper variation of φ if the maps $\varphi_t : \Sigma \to M$, defined by $\varphi_t(x) = \Phi(x, t)$, are proper immersions for all $t \in (-\varepsilon, \varepsilon)$ and if $\varphi_0 = \varphi$.

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Let us fix a proper variation Φ of φ. The variational vector field associated to Φ is the vector field $\xi : \Sigma \to TM$ along φ defined by

$$\xi(x) = \frac{\partial \Phi}{\partial t}(x, 0), \quad x \in \Sigma.$$

For this variation, the area functional $\mathcal{A} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and the volume functional $\mathcal{V} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ are defined by

$$\mathcal{A}(t) = \int_{\Sigma} \mathrm{d}A_{\varphi_t^* g}$$
$$\mathcal{V}(t) = \int_{\Sigma \times [0,t]} \Phi^*(\mathrm{d}V),$$

where $\mathrm{d}A_{\varphi_t^* g}$ denotes the area element of $(\Sigma, \varphi_t^* g)$ and $\mathrm{d}V$ is the volume element of M. We say that the variation Φ is volume preserving if $\mathcal{V}(t) = 0$ for every $t \in (-\varepsilon, \varepsilon)$. Another area functional called wetting area functional $\mathcal{W} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by

$$\mathcal{W}(t) = \int_{\partial\Sigma \times [0,t]} \Phi^*(\mathrm{d}A_{\partial M}),$$

where $\mathrm{d}A_{\partial M}$ denotes the area element of ∂M. Fix a real number $\theta \in (0, \pi)$, the energy functional $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by

$$E(t) = \mathcal{A}(t) - \cos\theta \cdot \mathcal{W}(t).$$

The first variation formulae of $\mathcal{V}(t)$ and $E(t)$ for a variation with a variation vector field $\xi(x)$ are given by

$$\mathcal{V}'(0) = \int_{\Sigma} g(\xi, N)\,\mathrm{d}A;$$
$$E'(0) = -\int_{\Sigma} Hg(\xi, N)\,\mathrm{d}A + \int_{\partial\Sigma} g(\xi, \nu - \cos\theta\overline{\nu})\,\mathrm{d}s,$$

where $\mathrm{d}A$ and $\mathrm{d}s$ are the area element of Σ and $\partial\Sigma$, respectively.

We say that the immersion φ is a capillary CMC immersion if $E'(0) = 0$ for every volume preserving variation of φ. If $E'(0) = 0$ for every variation of φ, we call φ a capillary minimal immersion.

Notice that Σ is a capillary CMC hypersurface if and only if Σ has constant mean curvature and $g(N, \overline{N}) = \cos\theta$ along $\partial\Sigma$; this last condition means that $\partial\Sigma$ meets ∂M at an angle of θ. Similarly, Σ is a capillary minimal hypersurface when Σ is a minimal hypersurface and $\partial\Sigma$ meets ∂M at an angle of θ. When $\theta = \pi/2$, we use the term free boundary CMC (or minimal) hypersurface.

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For a capillary CMC or minimal hypersuface Σ with contact angle $\theta \in (0,\pi)$, the orthonormal bases $\{N,\nu\}$ and $\{\overline{N},\overline{\nu}\}$ are related by the following equations:

$$\begin{aligned}\overline{N} &= \cos\theta \cdot N + \sin\theta \cdot \nu;\\ \overline{\nu} &= -\sin\theta \cdot N + \cos\theta \cdot \nu.\end{aligned}$$

Let $f : \Sigma \to \mathbb{R}$ be a smooth function which satisfies $\int_\Sigma f\,\mathrm{d}A = 0$ and φ a capillary CMC immersion; for a volume preserving proper variation of φ such that $f = g(\xi, N)$, the second variational formula of E is given by

$$E''(0) = -\int_\Sigma \left[\Delta f + \left(\mathrm{Ric}(N) + \|A\|^2\right) f\right] f\,\mathrm{d}A + \int_{\partial\Sigma} \left(\frac{\partial f}{\partial \nu} - qf\right) f\,\mathrm{d}s,$$

where Δ is the Laplace operator on Σ with respect to the induced metric from M and

$$q = \frac{\mathrm{II}(\overline{\nu},\overline{\nu})}{\sin\theta} + \cot\theta A(\nu,\nu).$$

Definition 1. *The capillary CMC immersion $\varphi : \Sigma \to M$ (or just Σ) is called stable if $E''(0) \geq 0$ for any volume preserving variation of φ. If φ is a capillary minimal immersion, we call it stable whenever $E''(0) \geq 0$ for every variation of φ.*

Alternatively, let $\mathcal{F} = \{f \in H^1(\Sigma) : \int_\Sigma f\,\mathrm{d}A = 0\}$, where $H^1(\Sigma)$ is the first Sobolev space of Σ. The index form $Q : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ of Σ is given by

$$Q(f,h) = \int_\Sigma \left[g(\nabla f, \nabla h) - (\mathrm{Ric}(N) + \|A\|^2) fh\right]\,\mathrm{d}A - \int_{\partial\Sigma} qfh\,\mathrm{d}s,$$

where ∇ is the gradient on Σ with respect to the induced metric from M. Then φ is a capillary CMC stable immersion if and only if $Q(f,f) \geq 0$ for every $f \in \mathcal{F}$. If φ is a capillary minimal immersion, then it is stable precisely when $Q(f,f) \geq 0$ for every $f \in H^1(\Sigma)$.

Ros and Souam [11] showed that totally geodesic balls and spherical caps immersed in the Euclidean ball are capillary CMC stable. Conversely, the uniqueness problem was first studied by Ros and Vergasta [12] for minimal or CMC hypersurfaces in free boundary case, that is, $\theta = \pi/2$, and later Ros and Souam [11] for general capillary ones. In [13], Wang and Xia proved that any immersed stable capillary hypersurfaces in a ball in space forms are totally umbilical.

On the other hand, considering the totally geodesic balls immersed in the Euclidean ball with contact angle θ as capillary minimal hypersurfaces, we have that $1 \in H^1(\Sigma)$ is an admissible function for testing stability. Then,

$$Q(1,1) = -\frac{\mathrm{Area}(\partial\Sigma)}{\sin\theta} < 0.$$

Therefore, totally geodesic balls with contact angle θ are capillary unstable minimal hypersurfaces. The stability index of a capillary CMC (respectively, minimal) hypersurface Σ is the dimension of the largest vector subspace of $\mathcal{F}$ (respectively, $H^1(\Sigma)$) restricted to which the bilinear form Q is negative definite. The index of Σ is denoted by $\mathrm{ind}(\Sigma)$. Thus, stable hyperfurfaces are those which have index equal to zero.

3. Proof of the results

Proof of Theorem 1. Let $\phi_1 : \Sigma \to \mathbb{R}$ be the first eigenfunction of Q. We know that ϕ_1 does not change sign. Then, without loss of generality, we can assume $\phi_1 \geq 0$. Since $\mathrm{ind}(\Sigma) = 1$, for all $f \in C^\infty(\Sigma)$ with $\int_\Sigma f \cdot \phi_1 \, \mathrm{d}A = 0$, we have $Q(f,f) \geq 0$, that is,

$$\int_\Sigma \left[\|\nabla f\|^2 - (\mathrm{Ric}(N) + \|A\|^2) f^2\right] \mathrm{d}A \geq \int_{\partial\Sigma} \left(\frac{\mathrm{II}(\overline{\nu},\overline{\nu})}{\sin\theta} + \cot\theta \cdot A(\nu,\nu)\right) f^2 \, \mathrm{d}s.$$

By [3, Theorem 7.2], there exists a proper conformal branched cover $F = (f_1, f_2) \colon \Sigma \to \overline{\mathbb{D}}^2$ satisfying $\deg(F) \leq g + r$, where $\overline{\mathbb{D}}^2 = \{z \in \mathbb{R}^2 : \|z\| \leq 1\}$ is the Euclidean unit disk. By [6, Lemma 2.1], we can assume $\int_\Sigma f_i \cdot \phi_1 \, \mathrm{d}A = 0$. Then, using f_i as a test function, we obtain

$$\int_{\partial\Sigma} \left(\frac{\mathrm{II}(\overline{\nu},\overline{\nu})}{\sin\theta} + \cot\theta \cdot A(\nu,\nu)\right) f_i^2 \, \mathrm{d}s \leq \int_\Sigma \left[\|\nabla f_i\|^2 - (\mathrm{Ric}(N) + \|A\|^2) f_i^2\right] \mathrm{d}A.$$

Note that, because F is conformal,

$$\sum_{i=1}^{2} \int_\Sigma \|\nabla f_i\|^2 \, \mathrm{d}A = \int_\Sigma \|\nabla F\|^2 \, \mathrm{d}A = 2 \cdot \mathrm{Area}(F(\Sigma)) = 2 \cdot \mathrm{Area}(\overline{\mathbb{D}}^2)\deg(F) \leq 2\pi(g + r).$$

Hence, since $F(\partial\Sigma) \subset \mathbb{S}^1$ (since F is proper), $\mathrm{Ric} \geq 0$ and $\mathrm{II} \geq 1$,

$$\int_{\partial\Sigma} \left(\frac{1}{\sin\theta} + \cot\theta \cdot A(\nu,\nu)\right) \mathrm{d}s \leq 2\pi \deg(F) \leq 2\pi(g + r),$$

which implies

$$L(\partial\Sigma) + \cos\theta \int_{\partial\Sigma} A(\nu,\nu) \, \mathrm{d}s \leq 2\pi(g + r)\sin\theta.$$

Proceeding, we notice that if equality occurs, then every inequality that appears in the previous argument will be an equality. In particular, $A \equiv 0$ (Σ is totally geodesic), $\mathrm{Ric}(N) = 0$ and $\mathrm{II}(\overline{\nu},\overline{\nu}) = 1$. Using the Gauss equation $R + H^2 - \|A\|^2 = 2(\mathrm{Ric}(N) + K)$, where K is the Gaussian curvature of Σ and R is the scalar curvature of M, we have $2K = R \geq 0$.

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Consider T the unit tangent to $\partial\Sigma$. Since $\nu = \sin\theta\,\overline{N} + \cos\theta\,\overline{\nu}$ along $\partial\Sigma$, the geodesic curvature of $\partial\Sigma$ in Σ is given by

$$\begin{aligned} k = -g(D_T T, \nu) &= g(D_T \nu, T) \\ &= g(D_T(\sin\theta\overline{N} + \cos\theta\overline{\nu}), T) \\ &= \sin\theta \cdot \mathrm{II}(T,T) + \cos\theta g(D_T\overline{\nu}, T). \end{aligned}$$

On the other hand,

$$\begin{aligned} g(D_T\overline{\nu}, T) &= g(D_T(-\sin\theta\, N + \cos\theta\,\nu), T) \\ &= \sin\theta \cdot A(T,T) + \cos\theta \cdot k = \cos\theta \cdot k. \end{aligned}$$

From which we conclude that

$$k = \sin\theta \cdot \mathrm{II}(T,T) + \cos^2\theta \cdot k,$$

that is,

$$k = \frac{\mathrm{II}(T,T)}{\sin\theta} \geq \frac{1}{\sin\theta}.$$

Moreover, equality occurs if and only if $\mathrm{II}(T,T) = 1$. By Gauss–Bonnet theorem,

$$\begin{aligned} 2\pi(2-2g-r) = 2\pi\chi(\Sigma) &= \int_\Sigma K\,\mathrm{d}A + \int_{\partial\Sigma} k\,\mathrm{d}s \\ &\geq \frac{L(\partial\Sigma)}{\sin\theta} = 2\pi(g+r), \end{aligned}$$

that is,

$$2\pi(2-2g-r) \geq 2\pi(g+r),$$

which implies $g=0$ and $r=1$. Then all inequalities above must be equalities. So $K=0$, $L(\partial\Sigma) = 2\pi\,\sin\theta$, $k = 1/\sin\theta$ and $\mathrm{II}(T,T) = 1$. Also, observe that the geodesic curvature $\overline{k}$ of $\partial\Sigma$ in ∂M satisfies

$$\begin{aligned} \overline{k} &= -g(D_T T, \overline{\nu}) = -g(D_T T, -\sin\theta \cdot N + \cos\theta \cdot \nu) \\ &= \sin\theta\, g(D_T T, N) - \cos\theta\, g(D_T T, \nu) \\ &= \sin\theta\, A(T,T) + \cos\theta\, k = \cos\theta\, k \\ &= \cot\theta. \end{aligned}$$

Now, let $x \in \Sigma$ and $\{e_1, e_2, e_3 = N\} \subset T_xM$ be such that $\{e_1, e_2\}$ is an orthonormal basis of $T_x\Sigma$ and denote by K_M the sectional curvature of M. Since

$$\mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) + \mathrm{Ric}(e_3,e_3) = R = 0$$

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on Σ and $\mathrm{Ric} \geq 0$ everywhere, we have $\mathrm{Ric}(e_i, e_i) = 0$ on Σ for $i = 1, 2, 3$, which implies $K_M(e_i, e_j) = 0$ for $i \neq j$.

Let $a, b \in \mathbb{R}$ be such that $a^2 + b^2 = 1$, since $\mathrm{II}(\overline{\nu}, \overline{\nu}) = 1$ and $\mathrm{II}(T, T) = 1$, then

$$\mathrm{II}(a\overline{\nu} + bT, a\overline{\nu} + bT) \geq 1 \;\Rightarrow\; 2ab \cdot \mathrm{II}(\overline{\nu}, T) \geq 0;$$
$$\mathrm{II}(a\overline{\nu} - bT, a\overline{\nu} - bT) \geq 1 \;\Rightarrow\; -2ab \cdot \mathrm{II}(\overline{\nu}, T) \geq 0,$$

and we infer that $\mathrm{II}(\overline{\nu}, T) = 0$ and $\mathrm{II} = 1$. □

If we make an extra assumption on the geometry of M along ∂M, we can characterize the global geometry of M when equality in (1) holds.

Corollary 2. *Let M^3 be a compact Riemannian three-manifold with non-empty boundary ∂M. Suppose that $\mathrm{Ric} \geq 0$ and $\mathrm{II} \geq 1$ and $K_M(T_p\partial M) \geq 0$ for all $p \in \partial M$, where K_M is the sectional curvature of M. If Σ^2 is a properly embedded capillary minimal surface of index one in M, with constant contact angle $\theta \in (0, \pi)$, then*

$$L(\partial\Sigma) + \cos\theta \int_{\partial\Sigma} A(\nu, \nu)\,\mathrm{d}s \leq 2\pi(g + r)\sin\theta.$$

Furthermore, if equality holds, M^3 is isometric to the Euclidean unit three-ball and Σ^2 is isometric to the Euclidean disk of radius $\sin\theta$.

Proof. According to Theorem 1, inequality is valid. Furthermore, if equality occurs, Σ^2 is totally geodesic and the geodesic curvature of $\partial\Sigma$ in ∂M is $\overline{k} = \cot\theta$. In addition, we get $L(\partial\Sigma) = 2\pi\sin\theta$. We can assume, by a possible change of orientation, that $\overline{k} = \cot\theta \geq 0$.

Now, denote by $K_{\partial M}$ the Gaussian curvature of ∂M. Also, denote by k_1 and k_2 the principal curvatures of ∂M in M. By Gauss equation

$$K_{\partial M} = K_M(T_p\,\partial M) + k_1 k_2 \geq 1.$$

Since $\partial\Sigma$ is a simple curve of ∂M (because Σ is embedded into M), it follows from [4, Theorem 4] that $\partial\Sigma$ bounds a domain in ∂M which is isometric to a geodesic ball in $\mathbb{S}^2$. We cut ∂M along $\partial\Sigma$ to obtain two compact surfaces with the geodesic $\partial\Sigma$ as their common boundary. Applying [4, Theorem 4] to either of these two compact surfaces with boundary, we conclude that ∂M is isometric to the standard unit two-sphere.

Thus, by Xia theorem ([14, Theorem 1]) M^3 is isometric to the Euclidean unit three-ball. Finally, by using that Σ^2 is totally geodesic, we can conclude that Σ is isometric to the Euclidean disk of radius $\sin\theta$. □

Below, we get a sharp upper bound for the area of Σ, when M^3 is a strictly convex body in $\mathbb{R}^3$.

Corollary 3. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is strictly convex, say $\mathrm{II} \geq 1$, where II is the second fundamental form of $\partial\Omega$ in $\mathbb{R}^3$. If Σ^2 is a* *properly embedded capillary minimal disk of index one in Ω, with constant contact angle $\theta \in (0, \pi)$, then the area of Σ satisfies*

$$A(\Sigma) \leq \frac{\left(2\pi \sin\theta - \cos\theta \int_{\partial\Sigma} A(\nu,\nu)\,\mathrm{d}s\right)^2}{4\pi}.$$

Moreover, if equality holds, Ω is the Euclidean unit three-ball and Σ^2 is the Euclidean disk of radius θ.

Proof. The isoperimetric inequality for minimal surfaces (see [5, Theorem 1]) says that

$$4\pi A(\Sigma) \leq L^2(\partial\Sigma).$$

Then, by Theorem 1,

$$A(\Sigma) \leq \frac{L^2(\partial\Sigma)}{4\pi} \leq \frac{\left(2\pi \sin\theta - \cos\theta \int_{\partial\Sigma} A(\nu,\nu)\,\mathrm{d}s\right)^2}{4\pi}.$$

Notice that if equality occurs, by Corollary 2, we infer that Ω is the Euclidean unit three-ball and Σ^2 is the Euclidean disk of radius $\sin\theta$. □

Proof of Theorem 2. Let $F = (f_1, f_2) : \Sigma \to \overline{\mathbb{D}}^2$ be a proper conformal branched cover as in the proof of Theorem 1. Taking $\phi_1 = 1$ in [6, Lemma 2.1], we can assume

$$\int_\Sigma f_i \,\mathrm{d}A = 0$$

for $i = 1, 2$. Because Σ is stable

$$\int_\Sigma \left[\|\nabla f_i\|^2 - (\mathrm{Ric}(N) + \|A\|^2) f_i^2\right] \mathrm{d}A \geq \int_{\partial\Sigma} \left(\frac{\mathrm{II}(\overline{\nu},\overline{\nu})}{\sin\theta} + \cot\theta \cdot A(\nu,\nu)\right) f_i^2 \,\mathrm{d}s.$$

Summing over i and since $f_1^2 + f_2^2 = 1$ on $\partial\Sigma$, we get

$$\int_\Sigma \left[\|\nabla F\|^2 - (\mathrm{Ric}(N) + \|A\|^2)(f_1^2 + f_2^2)\right] \mathrm{d}A \geq \int_{\partial\Sigma} \left(\frac{\mathrm{II}(\overline{\nu},\overline{\nu})}{\sin\theta} + \cot\theta \cdot A(\nu,\nu)\right) \mathrm{d}s.$$

Thereby,

$$\frac{L(\partial\Sigma)}{\sin\theta} + \cot\theta \int_{\partial\Sigma} A(\nu,\nu)\,\mathrm{d}s \leq 2\pi(g + r).$$

Furthermore, if equality holds, $A \equiv 0$ (Σ is totally geodesic), $\mathrm{Ric}(N) = 0$ and $\mathrm{II}(\overline{\nu},\overline{\nu}) = 1$. Working exactly as in the proof of Theorem 1, we have the result. □

Observation 2. Corollaries 2 and 3 are also true if we change the hypothesis 'minimal of index one' by 'stable CMC and minimal'.

https://doi.org/10.1017/S0013091523000135 Published online by Cambridge University Press

Funding Statement. P.A.S. was partially supported by CNPq (grant 402668/2016-2).

Competing Interests. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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