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Two problems on random analytic functions in Fock spaces

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Abstract. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function on the complex plane, and let $\mathcal{R}f(z) = \sum_{n=0}^{\infty} a_n X_n z^n$ be its randomization induced by a standard sequence $(X_n)_n$ of independent Bernoulli, Steinhaus, or Gaussian random variables. In this paper, we characterize those functions $f(z)$ such that $\mathcal{R}f(z)$ is almost surely in the Fock space $\mathcal{F}_\alpha^p$ for any $p, \alpha \in (0, \infty)$. Then such a characterization, together with embedding theorems which are of independent interests, is used to obtain a Littlewood-type theorem, also known as regularity improvement under randomization within the scale of Fock spaces. Other results obtained in this paper include: (a) a characterization of random analytic functions in the mixed-norm space $\mathcal{F}(\infty, q, \alpha)$, an endpoint version of Fock spaces, via entropy integrals; (b) a complete description of random lacunary elements in Fock spaces; and (c) a complete description of random multipliers between different Fock spaces.

1 Introduction

The study of random analytic functions in Hardy spaces, induced by a sequence of Bernoulli or Steinhaus random variables, was initiated by Littlewood [19, 20] and Paley and Zygmund [25] in 1930s. After that, the topic was extended to a standard Gaussian sequence [15] and many other situations. Of particular interest to us is a Banach space viewpoint, especially the study of random analytic functions in functions spaces on the unit disk, such as H^∞ [5, 15, 22, 23, 25, 27], the Bloch spaces [3, 10], BMOA spaces [28, 29, 37], Dirichlet spaces [7], and, recently, Bergman spaces [6].

It appears that, so far, activities along this line of research center around the unit disk only, and entire functions over the complex plane are largely untapped. The Fock spaces are paradigms of Banach spaces of entire functions [38], and in this paper, we initiate a study of random analytic functions in these spaces; in particular, we seek to answer the following two basic questions concerning them, as well as to consider several ramifications among which an endpoint Fock space $\mathcal{F}(\infty, q, \alpha)$ is particularly interesting.

Question A To characterize entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the randomization $\mathcal{R}f(z) = \sum_{n=0}^{\infty} \pm a_n z^n$ is almost surely in the Fock space $\mathcal{F}_\alpha^p$ for $p, \alpha \in (0, \infty)$.

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Question B To characterize indices $p, q, \alpha, \beta \in (0, \infty)$ such that

$$\mathcal{R} : \mathcal{F}^p_\alpha \hookrightarrow \mathcal{F}^q_\beta,$$

where $\mathcal{R} : E \hookrightarrow F$ denotes that $\mathcal{R}f \in F$ almost surely for any $f \in E$.

Motivations for these two questions will be given shortly. Here, we point out that, in contrast to the above Banach space viewpoint, the study of individual Gaussian analytic functions (GAF) on the complex plane has attracted a lot of attention in recent years [12, 24, 26, 30–33]. Hence, from the Banach space viewpoint, this work extends known results from the unit disk to the complex plane; from the GAF viewpoint, this work provides a new framework to study Gaussian entire functions.

Definition 1.1 A random variable X is called *Bernoulli* if $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$, *Steinhaus* if it is uniformly distributed on the unit circle, and by $N(0,1)$, we mean the law of a Gaussian variable with zero mean and unit variance. Moreover, let X be either Bernoulli, Steinhaus, or $N(0,1)$. Then a standard X sequence is a sequence of independent, identically distributed X variables, denoted by $(\varepsilon_n)_{n\geq 0}$, $(e^{2\pi i \alpha_n})_{n\geq 0}$, and $(\xi_n)_{n\geq 0}$, respectively. A *standard random sequence* $(X_n)_{n\geq 0}$ refers to either a standard Bernoulli, Steinhaus, or Gaussian $N(0,1)$ sequence.

To motivate the first question, we let $H(\mathbb{C})$ denote the space of all entire functions. We extend the definition of $\mathcal{R}f$ to a standard random sequence $(X_n)_{n\geq 0}$:

$$\mathcal{R}f(z) := \sum_{n=0}^{\infty} a_n X_n z^n \text{ for } f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C}).$$

Let $\mathcal{X} \subset H(\mathbb{C})$ be a p-Banach space of entire functions containing all polynomials such that the point evaluation functional δ_z is continuous on $\mathcal{X}$ for $z \in \mathbb{C}$. The Hewitt–Savage zero–one law [9, Theorem 2.5.4, p. 82] implies that

$$\mathbb{P}(\mathcal{R}f \in \mathcal{X}) \in \{0,1\} \text{ for any } f \in H(\mathbb{C}).$$

This prompts us to introduce a notion called *the symbol space* $\mathcal{X}_*$ by

$$\mathcal{X}_* := \{f \in H(\mathbb{C}) : \mathbb{P}(\mathcal{R}f \in \mathcal{X}) = 1\}.$$

In order to answer Question A, or, more generally, to characterize the symbol space $(\mathcal{F}^p_\alpha)_*$ for a standard random sequence, we introduce a class of mixed-norm spaces $\mathcal{F}(p, q, \alpha)$ with three parameters $p, q, \alpha \in (0, \infty)$ for entire functions in Section 2.1. This class, although not showing up in literature previously, is a natural generalization of its unit disk analog, which has a long history already (see [14] for more details). Then, motivated by the Bergman result in [6], it is natural to conjecture (and prove) the following:

Theorem 1.1 *Let $p, \alpha \in (0, \infty)$ and $(X_n)_{n\geq 0}$ be a standard random sequence. Then $(\mathcal{F}^p_\alpha)_* = \mathcal{F}(2, p, \alpha)$.*

In fact, we can do more by solving this problem for mixed-norm spaces $\mathcal{F}(p, q, \alpha)$; see Theorem 2.6 for a characterization of $(\mathcal{F}(p, q, \alpha))_*$ for all $p, q, \alpha \in (0, \infty)$, which reduces to Theorem 1.1 by choosing $p = q$. Compared with the proof in [6], the main challenge for Theorem 1.1, besides obviously different function-theoretic issues, is

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to circumvent the difficulty caused by coefficient multipliers which work effectively on the unit disk but not on the complex plane. Another novelty for Fock spaces in this paper is perhaps that, with the help of entropy-type integrals, we are able to characterize the space $(\mathcal{F}(\infty, q, \alpha))_*$ (Theorem 2.9), which does not admit a Bergman counterpart.

The second question in this paper, i.e., to characterize $\mathcal{R} : \mathcal{F}^p_\alpha \hookrightarrow \mathcal{F}^q_\beta$, is motivated by the regularity improvement of series summation under randomization. It is well known that

$$1 \pm \frac{1}{2^p} \pm \frac{1}{3^p} \pm \cdots \pm \frac{1}{n^p} \pm \cdots$$

is convergent almost surely if and only if $p > \frac{1}{2}$. For similar phenomena in analytic functions, the classical theorem of Littlewood [19, 20] states that, for any function $f(z)$ in the Hardy space $H^2(\mathbb{D})$, the randomization $\mathcal{R}f(z)$ belongs to $H^q(\mathbb{D})$ almost surely for all $q > 0$, so a regularity improvement. More precisely, for $p, q \in (0, \infty)$,

$$\mathcal{R} : H^p(\mathbb{D}) \hookrightarrow H^q(\mathbb{D}) \tag{1.1}$$

if and only if $p \geq 2$ and $q > 0$. Then the Bergman case is solved in [6], and the Dirichlet case follows trivially from Bergman, due to the commutativity:

$$(\mathcal{R}f)' = \mathcal{R}(f').$$

Now, it is natural to consider this phenomenon for entire functions and we offer the following:

Theorem 1.2 *Let $p, q, \alpha, \beta \in (0, \infty)$ and $(X_n)_{n \geq 0}$ be a standard random sequence. Then $\mathcal{R} : \mathcal{F}^p_\alpha \hookrightarrow \mathcal{F}^q_\beta$ if and only if one of the following holds:*

(i) *$\alpha < \beta$; or*
(ii) *$\alpha = \beta$ and $q \geq \max\{2, p\}$.*

This implies that, in particular, there is no Littlewood-type improvement for any $\mathcal{F}^p_\alpha$; on the other hand, when $p < 2$, the loss is not as drastic as in the Littlewood theorem. With Theorem 1.1 in hand, the proof of Theorem 1.2 is quickly reduced to an embedding problem between Fock spaces and mixed-norm spaces. This problem, of independent interests, is solved in Section 3 (Theorems 3.3 and 3.4).

Section 4 studies random Hadamard lacunary series in Fock spaces induced by a more general sequence of random variables (Theorem 4.1).

Section 5 contains a complete description of random multipliers between Fock spaces (Theorem 5.1). To achieve this, we use Theorem 1.1 and characterize the space of multipliers between Fock spaces, which is of independent interests (Theorem 5.2).

1.1 Notations

The abbreviation "a.s." stands for "almost surely." We assume that all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation denoted by $\mathbb{E}(\cdot)$. Moreover, $A \lesssim B$ (or, $A \gtrsim B$) means that there exists a positive constant C dependent only on the indexes $p, q, \alpha, \beta, \ldots$ such that $A \leq CB$ (or, respectively, $A \geq \frac{B}{C}$), and $A \simeq B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

2 The mixed-norm space $\mathcal{F}(p,q,\alpha)$ and its symbol space

In this section, we introduce a class of mixed-norm spaces $\mathcal{F}(p,q,\alpha)$ of entire functions for $p,q \in (0,\infty]$ and $\alpha \in (0,\infty)$ and characterize the symbol space $(\mathcal{F}(p,q,\alpha))_*$, first for $p \in (0,\infty)$ and then for $p=\infty$, and $q,\alpha \in (0,\infty)$. This includes, in particular, a complete description of the symbol space $(\mathcal{F}^p_\alpha)_*$ for any $p,\alpha \in (0,\infty)$ as special cases since $\mathcal{F}^p_\alpha = \mathcal{F}(p,p,\alpha)$, hence answering Question A. This part follows the framework of [6], and hence certain details are skipped and we focus on where the proofs differ; the main challenge, as we mentioned before, is to circumvent the difficulty caused by coefficient multipliers. Then, with the help of entropy integrals, we characterize $(\mathcal{F}(\infty,q,\alpha))_*$, which does not admit a Bergman counterpart.

For the reader's convenience, we recall the definition of Fock spaces. For $p>0$ and $\alpha>0$, the Fock space $\mathcal{F}^p_\alpha$ consists of entire functions f on $\mathbb{C}$ such that

$$\|f\|_{p,\alpha} := \left(\frac{p\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha|z|^2}{2}}\,dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA is the Lebesgue measure on $\mathbb{C}$. For $p=\infty$ and $\alpha>0$, the Fock space $\mathcal{F}^\infty_\alpha$ is defined as

$$\mathcal{F}^\infty_\alpha := \left\{ f \in H(\mathbb{C}) : \|f\|_{\infty,\alpha} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha|z|^2}{2}} < \infty \right\}.$$

2.1 The mixed-norm space $\mathcal{F}(p,q,\alpha)$

For $p \in (0,\infty]$ and $q,\alpha \in (0,\infty)$, we introduce the following mixed-norm space:

$$\mathcal{F}(p,q,\alpha) := \left\{ f \in H(\mathbb{C}) : \|f\|_{p,q,\alpha} := \left(\int_0^\infty M_p^q(f,r)\,d\lambda_{q\alpha}(r)\right)^{\frac{1}{q}} < \infty \right\},$$

where $d\lambda_{q\alpha}(r) := q\alpha r e^{-\frac{q\alpha r^2}{2}}dr$ and, as usual,

$$M_p(f,r) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}} \quad \text{for } p<\infty$$

and

$$M_\infty(f,r) := \sup_{\theta\in[0,2\pi]} |f(re^{i\theta})|.$$

Moreover, for $q=\infty$, the space $\mathcal{F}(p,\infty,\alpha)$ consists of entire functions f such that

$$\|f\|_{p,\infty,\alpha} := \sup_{r>0} M_p(f,r)e^{-\frac{\alpha r^2}{2}} < \infty.$$

Note that $\mathcal{F}^p_\alpha = \mathcal{F}(p,p,\alpha)$ for $p \in (0,\infty]$ and $\alpha \in (0,\infty)$.

The corresponding space $H(p,q,\alpha)$ on the unit disk has been intensively studied for a long time [14, Chapters 7 and 8]. Here, we establish a few properties of $\mathcal{F}(p,q,\alpha)$ in order to answer Question A.

The proof of the following lemma is standard, and details are skipped.

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Lemma 2.1 *Let $p, q \in (0, \infty]$ and $\alpha \in (0, \infty)$. If $p \geq 1$ and $q \geq 1$, then $\mathcal{F}(p, q, \alpha)$ is a Banach space, else $\mathcal{F}(p, q, \alpha)$ is an s-Banach space with $s = \min\{p, q\}$.*

The next lemma is an extension of [38, Proposition 2.9].

Lemma 2.2 *Let $p, q \in (0, \infty]$, $\alpha \in (0, \infty)$, and $f \in \mathcal{F}(p, q, \alpha)$, $f_r(z) := f(rz), r \in (0, 1)$.*

(a) *One has $\|f_r\|_{p,q,\alpha} \to \|f\|_{p,q,\alpha}$ as $r \to 1^-$. Moreover, if $q < \infty$, then $f_r \to f$ in $\mathcal{F}(p, q, \alpha)$ as $r \to 1^-$.*
(b) *For $r \in (0, 1)$, the Taylor polynomials of f_r converge to f_r in $\mathcal{F}(p, q, \alpha)$.*

Proof (a) Using $\|f_r\|_{p,q,\alpha} \leq \|f\|_{p,q,\alpha}$ for $r < 1$, if $q < \infty$, then by the dominated convergence theorem,

$$\|f_r\|_{p,q,\alpha} = \left(\int_0^\infty M_p^q(f_r, t) d\lambda_{q\alpha}(t)\right)^{\frac{1}{q}} \to \left(\int_0^\infty M_p^q(f, t) d\lambda_{q\alpha}(t)\right)^{\frac{1}{q}} = \|f\|_{p,q,\alpha},$$

as $r \to 1^-$. From this and [11, Lemma 3.17], it follows that $\|f_r - f\|_{p,q,\alpha} \to 0$, i.e., $f_r \to f$ in $\mathcal{F}(p, q, \alpha)$ as $r \to 1^-$. For $q = \infty$, we find a sequence $(t_n)_n$ so that

$$\lim_{n\to\infty} M_p(f, t_n) e^{-\frac{\alpha t_n^2}{2}} = \sup_{t>0} M_p(f, t) e^{-\frac{\alpha t^2}{2}} = \|f\|_{p,\infty,\alpha}.$$

Then, for $n \in \mathbb{N}$,

$$\lim_{r\to 1^-} \|f_r\|_{p,\infty,\alpha} \geq \lim_{r\to 1^-} M_p(f_r, t_n) e^{-\frac{\alpha t_n^2}{2}} = M_p(f, t_n) e^{-\frac{\alpha t_n^2}{2}} \to \|f\|_{p,\infty,\alpha},$$

as $n \to \infty$.

(b) We claim that $M_p(f, t) \leq \|f\|_{p,q,\alpha} e^{\frac{\alpha t^2}{2}}$; hence, $M_p(f_r, t) \leq \|f\|_{p,q,\alpha} e^{\frac{\alpha r^2 t^2}{2}}$ for $t > 0$. This implies $f_r \in \mathcal{F}(p, \infty, r^2\alpha)$. Fix a number $\beta \in (r^2\alpha, \alpha)$. By Theorem 3.3, we have

$$\mathcal{F}(p, \infty, r^2\alpha) \subset \mathcal{F}_\beta^2 \subset \mathcal{F}(p, q, \alpha).$$

Thus, f_r belongs to the Hilbert Fock space $\mathcal{F}_\beta^2$, which implies that the Taylor polynomials $(p_n)_n$ of f_r converge to f_r in $\mathcal{F}_\beta^2$, and, hence, also in $\mathcal{F}(p, q, \alpha)$. It remains to prove the claim, which is trivial when $q = \infty$. If $q < \infty$, then, for $t \geq 1$, we have

$$\|f\|_{p,q,\alpha}^q \geq \int_t^\infty M_p^q(f, x) d\lambda_{q\alpha}(x) \geq M_p^q(f, t) \int_t^\infty d\lambda_{q\alpha}(x) = M_p^q(f, t) e^{-\frac{q\alpha t^2}{2}}. \quad \blacksquare$$

2.2 The symbol space $(\mathcal{F}(p, q, \alpha))_*$

As the first step to characterize the symbol space $(\mathcal{F}(p, q, \alpha))_*$, we extend [6, Theorem 8] from the unit disk to the complex plane.

Proposition 2.3 *Let $p, q \in (0, \infty]$, $\alpha \in (0, \infty)$, and $(X_n)_{n\geq 0}$ be a standard random sequence. Let $f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{C})$. Then the following statements are equivalent:*

(i) *$\mathcal{R}f \in \mathcal{F}(p, q, \alpha)$ a.s.;*
(ii) *$\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^s) < \infty$ for some $s > 0$; and*
(iii) *$\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^s) < \infty$ for any $s > 0$.*

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Moreover, the quantities $\left(\mathbb{E}(\|\mathcal{R}f\|^s_{p,q,\alpha})\right)^{\frac{1}{s}}$ *are equivalent for all* $s > 0$ *with some constant depending only on s.*

To prove Proposition 2.3, we need the following two auxiliary lemmas. The first lemma is the Khintchine–Kahane inequality for p-Banach spaces (see [6, Lemma 11]).

Lemma 2.4 *Let* $(e_n)_{n\geq 0}$ *be a sequence of elements in a p-Banach space* $\mathcal{X}$*, and let* $(X_n)_{n\geq 0}$ *be a standard random sequence. Let* $S := \sum_{n=0}^{\infty} X_n e_n$ *be an a.s. convergent series in* $\mathcal{X}$*. Then* $S \in L^q(\Omega, \mathcal{X})$ *for all* $q \in (0, \infty)$*, and moreover,*

$$\|S\|_{L^{q_1}(\Omega,\mathcal{X})} \simeq \|S\|_{L^{q_2}(\Omega,\mathcal{X})}$$

for any $q_1, q_2 \in (0, \infty)$*, where* $\|S\|^q_{L^q(\Omega,\mathcal{X})} = \mathbb{E}(\|S\|^q_{\mathcal{X}})$*.*

We also use this lemma several times with $\mathcal{X} = \mathbb{C}$ in this section. The second lemma is the extension of [6, Lemma 10] from the spaces $H(p, q, \alpha)$ on the unit disk to the spaces $\mathcal{F}(p, q, \alpha)$ on the complex plane.

Lemma 2.5 *Let* $p, q \in (0, \infty]$*,* $\alpha \in (0, \infty)$*, and* $(X_n)_{n\geq 0}$ *be a sequence of independent and symmetric random variables. If* $f(z) = \sum_{n=0}^{\infty} a_n X_n z^n$ *belongs to* $\mathcal{F}(p, q, \alpha)$ *a.s., then the Taylor series* $S_n(z) = \sum_{k=0}^{n} a_k X_k z^k$ *converges a.s. to* $f(z)$ *in* $\mathcal{F}(p, q, \alpha)$ *with* $q < \infty$*, whereas it is a.s. bounded in* $\mathcal{F}(p, \infty, \alpha)$*.*

Proof We follow the strategy in [6, Lemma 10]. The proof outlined below is based on Lemma 2.2 and the Marcinkiewicz–Zygmund–Kahane (MZK) theorem for s-Banach spaces. When $q < \infty$, we fix an increasing sequence of positive number $r_m \to 1^-$ as $m \to \infty$. Then, for $m \geq 1$, $f_{r_m}(z) := \sum_{n=0}^{\infty} r_m^n a_n X_n z^n$ a.s. in $\mathcal{F}(p, q, \alpha)$. Moreover, by Lemma 2.2, $f_{r_m} \to f$ a.s. in $\mathcal{F}(p, q, \alpha)$ as $m \to \infty$. From this and the MZK theorem [6, p. 11], we arrive at the assertion.

When $q = \infty$, by Lemma 2.2, $\|f_{r_m}\|_{p,\infty,\alpha} \to \|f\|_{p,\infty,\alpha}$ as $m \to \infty$ and, for fixed $m \in \mathbb{N}$, the Taylor polynomials of f_{r_m} converge to f_{r_m} in $\mathcal{F}(p, \infty, \alpha)$. This allows us to replace the A-convergent version of the MZK theorem with an s-Banach space version of the A-bounded MZK theorem [17, Theorem II.4] to conclude that the Taylor series $(S_n)_n$ is a.s. bounded in $\mathcal{F}(p, \infty, \alpha)$. Note that we need the A-bounded MZK theorem in this case because we do not have $f_{r_m} \to f$ in $\mathcal{F}(p, \infty, \alpha)$. ■

Remark 2.1 This result is interesting since there exists a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}^p_\alpha$, with $p < 1$ in [18] and with $p = 1$ in [21], whose Taylor series $S_n(z) = \sum_{k=0}^{n} a_k z^k$ does not converge to f; hence, $\sup_{n\geq 0} \|S_n\|_{p,\alpha} = \infty$.

The proof of Proposition 2.3 (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is trivial, and (ii) $\Longrightarrow$ (iii) follows from Lemma 2.4. It remains to prove (i) $\Longrightarrow$ (ii). If $\mathcal{R}f \in \mathcal{F}(p, q, \alpha)$ a.s., then by Lemma 2.5, the Taylor series $S_n(z) = \sum_{k=0}^{n} a_k X_k z^k$ is a.s. bounded in $\mathcal{F}(p, q, \alpha)$, i.e., $\mathbb{P}(M < \infty) = 1$, where $M := \sup_{n\geq 0} \|S_n\|^s_{p,q,\alpha}$ and $s := \min\{p, q, 1\}$. Thus, by [6, Lemma 9],

$$\mathbb{E}\left(\exp\left(\lambda\|\mathcal{R}f\|^s_{p,q,\alpha}\right)\right) \leq \mathbb{E}(\exp(\lambda M)) < \infty$$

for a small enough constant $\lambda > 0$, from which and Jensen's inequality, (ii) follows. Moreover, Lemma 2.4 implies that the quantities $(\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^{s}))^{\frac{1}{s}}$ are equivalent for all $s > 0$ by a constant depending only on s.

Now, we come to the first of the two symbol space theorems in this section.

Theorem 2.6 *Let $p, q, \alpha \in (0, \infty)$ and $(X_n)_{n\geq 0}$ be a standard random sequence. Then $(\mathcal{F}(p, q, \alpha))_* = \mathcal{F}(2, q, \alpha)$.*

Remark 2.2 A reader acquainted with [6] should be able to predict the above theorem quickly since it was observed (before Theorem 7) in [6] that the randomization $\mathcal{R}(\cdot)$ is an operation of "circular orthogonalization"; namely, it changes the circular p-norm to an orthogonal 2-norm, and it does nothing to the radial parameters q in the definition of mixed-norm spaces. This echoes well with the classical Littlewood theorem $(H^p)_* = H^2$ for all $0 < p < \infty$. The root for this phenomenon should, perhaps, be credited to Khintchine's inequality.

Proof Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$. By Proposition 2.3, $\mathcal{R}f \in \mathcal{F}(p, q, \alpha)$ a.s. if and only if $\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^{q}) < \infty$. Moreover,

$$\begin{aligned}\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^{q}) &= \int_\Omega \int_0^\infty \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} d\lambda_{q\alpha}(r) d\mathbb{P}\\ &= \int_0^\infty \left[\int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} d\mathbb{P}\right] d\lambda_{q\alpha}(r).\end{aligned}$$

Let $p \leq q$. Using the Minkowski inequality, we get

$$\int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} d\mathbb{P} \leq \left[\frac{1}{2\pi}\int_0^{2\pi} \left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^q d\mathbb{P}\right)^{\frac{p}{q}} d\theta\right]^{\frac{q}{p}}.$$

Moreover, by Lemma 2.4,

$$\left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^q d\mathbb{P}\right)^{\frac{1}{q}} \simeq \left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^2 d\mathbb{P}\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n}\right)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned}\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^{q}) &\lesssim \int_0^\infty \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n}\right)^{\frac{q}{2}} d\lambda_{q\alpha}(r)\\ &= \int_0^\infty \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^{\frac{q}{2}} d\lambda_{q\alpha}(r) = \|f\|_{2,q,\alpha}^{q}.\end{aligned}$$

Thus, $\mathcal{F}(2, q, \alpha) \subset (\mathcal{F}(p, q, \alpha))_*$. On the other hand, by Lemma 2.4,

$$\begin{aligned}\int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} d\mathbb{P} &\geq \left[\int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right) d\mathbb{P}\right]^{\frac{q}{p}}\\ &\simeq \left[\frac{1}{2\pi}\int_0^{2\pi} \left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^2 d\mathbb{P}\right)^{\frac{p}{2}} d\theta\right]^{\frac{q}{p}},\end{aligned}$$

which is equal to $\left(\sum_{n=0}^{\infty}|a_n|^2 r^{2n}\right)^{\frac{q}{2}}$. Therefore,

$$\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^q) \gtrsim \int_0^\infty \left(\sum_{n=0}^{\infty}|a_n|^2 r^{2n}\right)^{\frac{q}{2}} d\lambda_{q\alpha}(r) = \|f\|_{2,q,\alpha}^q.$$

Thus, $(\mathcal{F}(p,q,\alpha))_* \subset \mathcal{F}(2,q,\alpha)$. Similar methods apply to the case $q < p$ in order to prove the inclusion $\mathcal{F}(2,q,\alpha) \subset (\mathcal{F}(p,q,\alpha))_*$ and the inverse inclusion. Details are skipped. ■

It is interesting to observe that one can reformulate the a.s. membership problem $\mathcal{R} : \mathcal{F}(2,p,\alpha) \hookrightarrow \mathcal{F}_\alpha^p$ as the boundedness of a (deterministic) operator.

Corollary 2.7 *Let $p, \alpha \in (0,\infty)$ and $(X_n)_{n\geq 0}$ be a standard random sequence. Then the map $\mathcal{R} : \mathcal{F}(2,p,\alpha) \to L^s(\Omega, \mathcal{F}_\alpha^p)$ is continuous for $s \in (0,\infty)$.*

Proof By Lemma 2.4, it is sufficient to consider $s = p$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}(2,p,\alpha)$, we get

$$\begin{aligned}
\|\mathcal{R}f\|_{L^p(\Omega,\mathcal{F}_\alpha^p)} &= \left(\int_\Omega \left(\frac{p\alpha}{2\pi}\int_{\mathbb{C}} |\mathcal{R}f(z)|^p e^{-\frac{p\alpha|z|^2}{2}} dA(z)\right) d\mathbb{P}\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \left(\int_0^\infty \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(re^{i\theta})|^p d\theta\right) d\lambda_{p\alpha}(r)\right) d\mathbb{P}\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(\frac{1}{2\pi}\int_0^{2\pi} \left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^p d\mathbb{P}\right) d\theta\right) d\lambda_{p\alpha}(r)\right)^{\frac{1}{p}} \\
&\simeq \left(\int_0^\infty \left(\frac{1}{2\pi}\int_0^{2\pi} \left(\int_\Omega |\mathcal{R}f(re^{i\theta})|^2 d\mathbb{P}\right)^{\frac{p}{2}} d\theta\right) d\lambda_{p\alpha}(r)\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(\sum_{n=0}^{\infty}|a_n|^2 r^{2n}\right)^{\frac{p}{2}} d\lambda_{p\alpha}(r)\right)^{\frac{1}{p}} = \|f\|_{2,p,\alpha}.
\end{aligned}$$

■

Remark 2.3 From the proof of Theorem 2.6 and Lemma 2.4, we see that, for $s > 0$, the quantities $\mathbb{E}(\|\mathcal{R}f\|_{p,q,\alpha}^s) < \infty$ for $\mathcal{R}^{(B)}f$, $\mathcal{R}^{(S)}f$, and $\mathcal{R}^{(G)}f$ are equivalent.

Now, we move on toward the second theorem in this section, which treats the case $p = \infty$ and $q < \infty$. More notations and auxiliary results are needed for this purpose. Let $(a_n)_n$ be a sequence of complex numbers. For simplicity, we define, if finite, the following quantities:

$$\rho(t) := \left(\sum_{n=0}^{\infty}|a_n|^2|e^{2\pi nti} - 1|^2\right)^{\frac{1}{2}} \quad \text{and} \quad I := \int_0^1 \frac{\overline{\rho}(t)}{t\sqrt{\log e/t}} dt,$$

where $\overline{\rho}(s) := \sup\{y : m(\{t : \rho(t) < y\}) < s\}$ is the nondecreasing rearrangement of $\rho(t)$. For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we define

$$\rho_r(t) := \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n} |e^{2\pi n t i} - 1|^2\right)^{\frac{1}{2}} \quad \text{and} \quad I_f(r) := \int_0^1 \frac{\overline{\rho_r}(t)}{t\sqrt{\log e/t}} dt,$$

where, as above, $\overline{\rho_r}$ is the nondecreasing rearrangement of ρ_r.

The $p = 1$ case of the following auxiliary result can be found in [23, Theorem 1.4, p. 11]. The general case follows from the Fernique theorem ([8, Theorem 2.7, p. 37] or [16, Corollary 3.2, p. 59]), since $X := \sum_{n=0}^{\infty} a_n e^{in\theta} X_n$ defines a Gaussian vector in H^∞ when $I < \infty$ by the Marcus–Pisier theorem, so the qth-moment of $\|X\|_\infty$ is equivalent to $\mathbb{E}(\|X\|_\infty)$.

Proposition 2.8 *Let $q \in (0, \infty)$ and $(X_n)_{n\geq 0}$ be a sequence of independent, symmetric random variables. Then there is a constant K such that*

$$\frac{1}{K}\left(\inf_n \mathbb{E}|X_n|\right)\left(\sqrt{\sum_{n=0}^{\infty}|a_n|^2} + I\right) \leq \left[\mathbb{E}\left(\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty} a_n e^{in\theta} X_n\right|^q\right)\right]^{1/q}$$
$$\leq K\sqrt{\sup_n \mathbb{E}|X_n|^2}\left(\sqrt{\sum_{n=0}^{\infty}|a_n|^2} + I\right).$$

Now, we present the second theorem in this section.

Theorem 2.9 *Let $q, \alpha \in (0, \infty)$ and $(X_n)_{n\geq 0}$ be a standard random sequence. Then*

$$(\mathcal{F}(\infty, q, \alpha))_* = \left\{f \in H(\mathbb{C}) : I_f(r) \in L^q(\mathbb{R}^+, d\lambda_{q\alpha})\right\}.$$

Proof Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. For $r \in (0, \infty)$, applying Proposition 2.8 to the sequence $(a_n r^n)_{n\geq 0}$ instead of $(a_n)_{n\geq 0}$, we get

$$\left(M_2(f, r) + I_f(r)\right)^q \simeq \mathbb{E}\left(M_\infty^q(\mathcal{R}f, r)\right). \tag{2.1}$$

First, suppose that $f \in (\mathcal{F}(\infty, q, \alpha))_*$, i.e., $\mathcal{R}f$ belongs to $\mathcal{F}(\infty, q, \alpha)$ a.s. Then, by Proposition 2.3, $\mathbb{E}(\|\mathcal{R}f\|_{\infty,q,\alpha}^q) < \infty$. Using this and (2.1), we get

$$\|I_f\|_{L^q(\mathbb{R}^+, d\lambda_{q\alpha})}^q \lesssim \int_0^\infty \mathbb{E}\left(M_\infty^q(\mathcal{R}f, r)\right) d\lambda_{q\alpha}(r)$$
$$= \mathbb{E}\left(\int_0^\infty M_\infty^q(\mathcal{R}f, r) d\lambda_{q\alpha}(r)\right) = \mathbb{E}(\|\mathcal{R}f\|_{\infty,q,\alpha}^q) < \infty,$$

which implies that $I_f(r) \in L^q(\mathbb{R}^+, d\lambda_{q\alpha})$.

Conversely, suppose that $I_f(r) \in L^q(\mathbb{R}^+, d\lambda_{q\alpha})$. We claim that $f \in \mathcal{F}(2, q, \alpha)$. Using this and (2.1), we obtain

$$\mathbb{E}\left(\|\mathcal{R}f\|_{\infty,q,\alpha}^q\right) = \int_0^\infty \mathbb{E}\left(M_\infty^q(\mathcal{R}f, r)\right) d\lambda_{q\alpha(r)}$$
$$\lesssim \int_0^\infty \left(M_2(f, r) + I_f(r)\right)^q d\lambda_{q\alpha(r)}$$
$$\lesssim \left(\|f\|_{2,q,\alpha}^q + \|I_f\|_{L^q(\mathbb{R}^+, d\lambda_{q\alpha})}^q\right) < \infty.$$

From this and Proposition 2.3, it follows that $\mathcal{R}f$ belongs to $\mathcal{F}(\infty, q, \alpha)$ a.s. Now, we prove the claim. For $n \in \mathbb{N}$ and $r > 0$, we get

$$\rho_r(t) \geq 2|a_n|r^n|\sin(n\pi t)|,$$

and hence,

$$\overline{\rho_r}(t) \geq \frac{4}{\pi}|a_n|r^n \arcsin t, \;\; t \in (0,1).$$

Thus,

$$I_f(r) \geq \frac{4}{\pi}|a_n|r^n \int_0^1 \frac{\arcsin t}{t\sqrt{\log e/t}} dt \gtrsim |a_n|r^n, \tag{2.2}$$

for $n \in \mathbb{N}$ and $r > 0$. Moreover, using [23, Theorem 1.2, p. 126], we get

$$I_f(r) = \int_0^1 \frac{\overline{\rho_r}(t)}{t\sqrt{\log e/t}} dt \geq \int_{\frac{1}{2}}^1 \frac{\overline{\rho_r}(t)}{t\sqrt{\log e/t}} dt \geq \overline{\rho_r}\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{dt}{t\sqrt{\log e/t}}$$
$$\gtrsim \overline{\rho_r}\left(\frac{1}{2}\right) \gtrsim \left(\sum_{k\geq 2} (a_k^*)^2\right)^{\frac{1}{2}} = \left(\sum_{n\geq 1} |a_n|^2 r^{2n} - \sup_{n\geq 1} |a_n|^2 r^{2n}\right)^{\frac{1}{2}},$$

where $(a_k)^*$ is the nonincreasing rearrangement of the sequence $(|a_n|r^n)_n$. From this and (2.2), it follows that $M_2(f,r) \lesssim I_f(r)$ for all $r > 0$, which implies that $f \in \mathcal{F}(2, q, \alpha)$. ∎

3 Embedding theorems

The purpose of this section is to answer Question B in the introduction; namely, we characterize $\mathcal{R} : \mathcal{F}^p_\alpha \hookrightarrow \mathcal{F}^q_\beta$. To achieve this, we solve completely the embedding problem from the Fock space $\mathcal{F}^p_\alpha$ to the mixed-norm space $\mathcal{F}(s, q, \beta)$. We shall do a little more and cover the case $p = \infty$. On the other hand, it would be desirable to solve the full embedding problem for mixed-norm spaces at the generality of Arévalo's solution for the disk case [4], namely, to characterize the embedding $\mathcal{F}(t, p, \alpha) \subset \mathcal{F}(s, q, \beta)$ with six parameters. (Despite our repeated efforts, in this paper, we can handle five parameters only.) An obstacle is our limited knowledge on the mean growth function $M_p(r, f)$ as $r \to \infty$, which calls for more study. In contrast, the same function is well understood in the unit disk (with $r \to 1^-$).

Theorem 3.1 *Let $p \in (0, \infty]$, $q, s, \alpha, \beta \in (0, \infty)$, and $(X_n)_{n\geq 0}$ be a standard random sequence. Then $\mathcal{F}^p_\alpha \subset (\mathcal{F}(s, q, \beta))_*$ if and only if one of the following conditions holds:*

(i) *$\alpha < \beta$; or*
(ii) *$\alpha = \beta$ and $q \geq \max\{2, p\}$.*

Corollary 3.2 *Let $p \in (0, \infty]$, $q, \alpha, \beta \in (0, \infty)$, and $(X_n)_{n\geq 0}$ be a standard random sequence. Then $\mathcal{R} : \mathcal{F}^p_\alpha \hookrightarrow \mathcal{F}^q_\beta$ iff either (i) or (ii) in Theorem 3.1 holds.*

The proof of Theorem 3.1 follows from the following characterizations of the inclusion relationship between $\mathcal{F}^q_\beta$ and $\mathcal{F}(s, p, \alpha)$. In the case $\alpha \neq \beta$, we get the following theorem.

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Theorem 3.3 *Let $\alpha, \beta \in (0,\infty)$. The following statements are true:*

(a) *If $\alpha < \beta$, then $\mathcal{F}^p_\alpha \subset \mathcal{F}(s,q,\beta)$ and the inclusion is proper for $p,q,s \in (0,\infty]$. In particular, $\mathcal{F}^p_\alpha \subset \mathcal{F}^q_\beta$ for $p,q \in (0,\infty]$.*
(b) *If $\beta < \alpha$, then $\mathcal{F}(s,q,\beta) \subset \mathcal{F}^p_\alpha$ and the inclusion is proper for $p,q,s \in (0,\infty]$.*

In the case $\alpha = \beta$, by [38, Corollary 2.8 and Theorem 2.10], the family of Fock spaces $\mathcal{F}^p_\alpha$ is nested, i.e.,

$$\mathcal{F}^p_\alpha \subsetneqq \mathcal{F}^q_\alpha \text{ for } 0 < p < q \le \infty. \tag{3.1}$$

In view of this, we only need to consider the case $q \neq s$.

Theorem 3.4 *Let $p,q,s \in (0,\infty]$ and $\alpha \in (0,\infty)$.*

(a) *If $s < q$ and $p \le q$, then $\mathcal{F}^p_\alpha \subset \mathcal{F}(s,q,\alpha)$ and the inclusion is proper.*
(b) *If $s < q < p$, then $\mathcal{F}^p_\alpha \not\subset \mathcal{F}(s,q,\alpha)$ and $\mathcal{F}(s,q,\alpha) \not\subset \mathcal{F}^p_\alpha$.*
(c) *If $q < s$ and $q \le p$, then $\mathcal{F}(s,q,\alpha) \subset \mathcal{F}^p_\alpha$ and the inclusion is proper.*
(d) *If $p < q < s$, then $\mathcal{F}^p_\alpha \not\subset \mathcal{F}(s,q,\alpha)$ and $\mathcal{F}(s,q,\alpha) \not\subset \mathcal{F}^p_\alpha$.*

The rest of this section is devoted to the proofs of Theorems 3.3 and 3.4. For the reader's convenience, we recall two facts first. We consider the reproducing kernel $K_{\alpha,w}(z) := e^{\alpha\overline{w}z}$, $w \in \mathbb{C}$, of the Hilbert Fock space $\mathcal{F}^2_\alpha$. Then

$$\|K_{\alpha,w}\|_{p,\alpha} = e^{\frac{\alpha|w|^2}{2}} \text{ for } p \in (0,\infty]. \tag{3.2}$$

By [38, Corollary 2.8],

$$|f(z)| \le e^{\frac{\alpha|z|^2}{2}} \|f\|_{p,\alpha} \text{ for } f \in \mathcal{F}^p_\alpha \text{ and } 0 < p \le \infty. \tag{3.3}$$

Proposition 3.5 *Let $p,q \in (0,\infty]$ and $\alpha \in (0,\infty)$. The following holds:*

(a) *$\mathcal{F}^q_\alpha \subsetneqq \mathcal{F}(p,q,\alpha)$ for $0 < p < q \le \infty$; and*
(b) *$\mathcal{F}(p,q,\alpha) \subsetneqq \mathcal{F}^q_\alpha$ for $0 < q < p \le \infty$.*

Proof We first observe that $\mathcal{F}(p_2,q,\alpha) \subset \mathcal{F}(p_1,q,\alpha)$ for $0 < p_1 < p_2 \le \infty$, which is a consequence of $M_{p_1}(f,r) \le M_{p_2}(f,r)$ for an entire function f and $r > 0$.

(a) Suppose that $0 < p < q \le \infty$. Then $\mathcal{F}^q_\alpha = \mathcal{F}(q,q,\alpha) \subset \mathcal{F}(p,q,\alpha)$. We need to prove that $\mathcal{F}^q_\alpha \neq \mathcal{F}(p,q,\alpha)$. Otherwise, by the open mapping theorem, there is a constant $C > 0$ such that

$$C^{-1}\|f\|_{q,\alpha} \le \|f\|_{p,q,\alpha} \le C\|f\|_{q,\alpha} \text{ for } f \in \mathcal{F}^q_\alpha. \tag{3.4}$$

We estimate the norm $\|K_{\alpha,w}\|_{p,q,\alpha}$.

$$\begin{aligned} M^p_p(K_{\alpha,w},r) &= \frac{1}{2\pi}\int_0^{2\pi} \left|e^{\alpha\overline{w}re^{i\theta}}\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{p\alpha|w|r\cos\theta}\,d\theta \\ &= I_0(p\alpha|w|r) \sim \frac{e^{p\alpha|w|r}}{\sqrt{2\pi p\alpha|w|r}} \text{ as } |w|r \to \infty, \end{aligned}$$

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where I_0 is the modified Bessel function and the last estimate is deduced from [2, Section 9.7]. Thus,

$$M_p(K_{\alpha,w},r) \simeq \frac{e^{\alpha|w|r}}{(|w|r)^{\frac{1}{2p}}} \text{ for } r \geq 1 \text{ and } |w| \geq 1. \tag{3.5}$$

If $q = \infty$, then by (3.5), for $|w| \geq 1$, we get

$$\begin{aligned} \|K_{\alpha,w}\|_{p,\infty,\alpha} &\lesssim \sup_{r<1} M_p(K_{\alpha,w},r)e^{-\frac{\alpha r^2}{2}} + |w|^{-\frac{1}{2p}} \sup_{r\geq 1} e^{\alpha|w|r-\frac{\alpha r^2}{2}} r^{-\frac{1}{2p}} \\ &\lesssim e^{\alpha|w|} + e^{\frac{\alpha|w|^2}{2}}|w|^{-\frac{1}{2p}} \sup_{r\geq 1} e^{-\frac{\alpha(r-|w|)^2}{2}} \lesssim e^{\frac{\alpha|w|^2}{2}}|w|^{-\frac{1}{2p}}. \end{aligned} \tag{3.6}$$

From this and (3.2), it follows that

$$\frac{\|K_{\alpha,w}\|_{p,\infty,\alpha}}{\|K_{\alpha,w}\|_{\infty,\alpha}} \lesssim |w|^{-\frac{1}{2p}} \to 0 \text{ as } |w| \to \infty.$$

If $2p < q < \infty$, then by (3.5), for $|w| \geq 1$, we get

$$\begin{aligned} &\|K_{\alpha,w}\|^q_{p,q,\alpha} \\ &\simeq \int_0^1 M_p^q(K_{\alpha,w},r)e^{-\frac{q\alpha r^2}{2}} r\,dr + |w|^{-\frac{q}{2p}} \int_1^\infty e^{q\alpha|w|r-\frac{q\alpha r^2}{2}} r^{1-\frac{q}{2p}}\,dr \\ &\lesssim e^{q\alpha|w|} + e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}} \int_1^\infty e^{-\frac{q\alpha(r-|w|)^2}{2}}\,dr \quad \left(\text{since } 1-\frac{q}{2p} < 0\right) \\ &\lesssim e^{q\alpha|w|} + e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}} \int_{-\infty}^\infty e^{-\frac{p\alpha(r-|w|)^2}{2}}\,dr \lesssim e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}}. \end{aligned} \tag{3.7}$$

Similarly as above, we get

$$\frac{\|K_{\alpha,w}\|_{p,q,\alpha}}{\|K_{\alpha,w}\|_{q,\alpha}} \lesssim |w|^{-\frac{1}{2p}} \to 0 \text{ as } |w| \to \infty.$$

If $p < q \leq 2p$, then by (3.5), for $|w| \geq 1$, we get

$$\begin{aligned} \|K_{\alpha,w}\|^q_{p,q,\alpha} &\lesssim e^{q\alpha|w|} + e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}} \int_1^\infty e^{-\frac{q\alpha(r-|w|)^2}{2}} r^{1-\frac{q}{2p}}\,dr \\ &\lesssim e^{q\alpha|w|} + e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}} \int_{-\infty}^\infty e^{-\frac{q\alpha t^2}{2}}(|t|+|w|)^{1-\frac{q}{2p}}\,dt \quad \left(\text{since } 1-\frac{q}{2p} \geq 0\right) \\ &\lesssim e^{q\alpha|w|} + e^{\frac{q\alpha|w|^2}{2}}|w|^{-\frac{q}{2p}} \left(|w|^{1-\frac{q}{2p}} \int_{-\infty}^\infty e^{-\frac{q\alpha t^2}{2}}\,dt + \int_{-\infty}^\infty e^{-\frac{q\alpha t^2}{2}}|t|^{1-\frac{q}{2p}}\,dt\right) \\ &\lesssim e^{\frac{q\alpha|w|^2}{2}}|w|^{1-\frac{q}{p}}, \end{aligned} \tag{3.8}$$

which, together with (3.2), implies that

$$\frac{\|K_{\alpha,w}\|_{p,q,\alpha}}{\|K_{\alpha,w}\|_{q,\alpha}} \lesssim |w|^{\frac{1}{q}-\frac{1}{p}} \to 0 \text{ as } |w| \to \infty \quad (\text{since } q > p).$$

Thus, we get a contradiction to (3.4).

(b) Let $0 < q < p \leq \infty$. Then $\mathcal{F}(p,q,\alpha) \subset \mathcal{F}(q,q,\alpha) = \mathcal{F}^q_\alpha$. It remains to show that the inclusion from $\mathcal{F}(p,q,\alpha)$ into $\mathcal{F}^q_\alpha$ is proper. Otherwise, as above, we assume that

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(3.4) holds by contradiction. If $q < p < \infty$, then using (3.5), for $|w| \geq 1$, we get

$$\begin{aligned}\|K_{\alpha,w}\|_{p,q,\alpha} &\gtrsim \left(|w|^{-\frac{q}{2p}} \int_{|w|}^{\infty} e^{q\alpha|w|r - \frac{q\alpha r^2}{2}} r^{1-\frac{q}{2p}} dr\right)^{\frac{1}{q}} \\ &\geq e^{\frac{\alpha|w|^2}{2}} |w|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{|w|}^{\infty} e^{-\frac{q\alpha(r-|w|)^2}{2}} dr\right)^{\frac{1}{q}} \quad \left(\text{since } 1 - \frac{q}{2p} > 0\right) \\ &\gtrsim e^{\frac{\alpha|w|^2}{2}} |w|^{\frac{1}{q}-\frac{1}{p}}. \end{aligned} \tag{3.9}$$

From this and (3.2), it follows that

$$\frac{\|K_{\alpha,w}\|_{p,q,\alpha}}{\|K_{\alpha,w}\|_{q,\alpha}} \gtrsim |w|^{\frac{1}{q}-\frac{1}{p}} \to \infty \text{ as } |w| \to \infty.$$

If $q < p = \infty$, then $M_\infty(K_{\alpha,w}, r) = e^{\alpha|w|r}$, and hence,

$$\begin{aligned}\|K_{\alpha,w}\|_{\infty,q,\alpha} &\simeq \left(\int_0^{\infty} e^{q\alpha|w|r - \frac{q\alpha r^2}{2}} r\, dr\right)^{\frac{1}{q}} = e^{\frac{\alpha|w|^2}{2}} \left(\int_0^{\infty} e^{-\frac{q\alpha(r-|w|)^2}{2}} r\, dr\right)^{\frac{1}{q}} \\ &\geq e^{\frac{\alpha|w|^2}{2}} \left(\int_{-|w|}^{\infty} e^{-\frac{q\alpha t^2}{2}} (t + |w|) dt\right)^{\frac{1}{q}} \\ &\gtrsim e^{\frac{\alpha|w|^2}{2}} \left(|w| \int_{-|w|}^{\infty} e^{-\frac{q\alpha t^2}{2}} dt + \int_{-|w|}^{\infty} e^{-\frac{q\alpha t^2}{2}} t dt\right)^{\frac{1}{q}} \gtrsim e^{\frac{\alpha|w|^2}{2}} |w|^{\frac{1}{q}}.\end{aligned}$$

Similarly as above, we obtain

$$\frac{\|K_{\alpha,w}\|_{p,q,\alpha}}{\|K_{\alpha,w}\|_{q,\alpha}} \gtrsim |w|^{\frac{1}{q}} \to \infty \text{ as } |w| \to \infty.$$

Thus, we obtain a contradiction to (3.4). ∎

Now, we come to the proof of Theorem 3.3.

The proof of Theorem 3.3 (a) Let $\alpha < \beta$. For $f \in \mathcal{F}_\alpha^p$, by (3.3),

$$|f(z)| \leq \|f\|_{p,\alpha} e^{\frac{\alpha|z|^2}{2}} \text{ for all } z \in \mathbb{C}.$$

Hence,

$$M_s(f, r) \leq \|f\|_{p,\alpha} e^{\frac{\alpha r^2}{2}} \text{ for all } r > 0.$$

If $q < \infty$, then

$$\begin{aligned}\|f\|_{s,q,\beta} &\simeq \left(\int_0^{\infty} M_s^q(f, r) e^{-\frac{q\beta r^2}{2}} r\, dr\right)^{\frac{1}{q}} \\ &\leq \|f\|_{p,\alpha} \left(\int_{\mathbb{C}} e^{-\frac{q(\beta-\alpha)r^2}{2}} r\, dr\right)^{\frac{1}{q}} \lesssim \|f\|_{p,\alpha}.\end{aligned}$$

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If $q = \infty$, then

$$\|f\|_{s,\infty,\beta} = \sup_{r>0} M_s(f,r)e^{-\frac{\beta r^2}{2}} \le \|f\|_{p,\alpha} \sup_{r>0} e^{-\frac{(\beta-\alpha)r^2}{2}} = \|f\|_{p,\alpha}.$$

Therefore, $\mathcal{F}^p_\alpha \subset \mathcal{F}(s,q,\beta)$. As above, to prove properness of the inclusion, we assume by contraction and by the open mapping theorem that there is a constant $C > 0$ such that

$$C^{-1}\|f\|_{s,q,\beta} \le \|f\|_{p,\alpha} \le C\|f\|_{s,q,\beta} \text{ for } f \in \mathcal{F}^p_\alpha. \tag{3.10}$$

Moreover, for $p, q \in (0,\infty]$, we have

$$\|z^n\|_{p,\alpha} \sim \left(\frac{n}{e\alpha}\right)^{\frac{n}{2}} n^{\frac{1}{2p}} \text{ and } \|z^n\|_{s,q,\beta} = \|z^n\|_{q,\beta} \sim \left(\frac{n}{e\beta}\right)^{\frac{n}{2}} n^{\frac{1}{2q}} \text{ as } n \to \infty, \tag{3.11}$$

where, here and below, we write $\frac{1}{p} := 0$ if $p = \infty$ for simplicity. Thus,

$$\frac{\|z^n\|_{s,q,\beta}}{\|z^n\|_{p,\alpha}} \sim \left(\frac{\alpha}{\beta}\right)^{\frac{n}{2}} n^{\frac{1}{2q}-\frac{1}{2p}} \to 0 \text{ as } n \to \infty,$$

which is a contradiction to (3.10).

(b) The case $\beta < \alpha$ is divided into three subcases.

Subcase 1. Suppose that $q \le s$. By Proposition 3.5 and part (a), we have

$$\mathcal{F}(s,q,\beta) \subset \mathcal{F}^q_\beta \subset \mathcal{F}^p_\alpha \text{ for } p \in (0,\infty].$$

Subcase 2. Suppose that $p \le s < q$. If $q < \infty$, then

$$\begin{aligned}
\|f\|^p_{p,\alpha} &\simeq \int_0^\infty M^p_p(f,r)e^{-\frac{p\alpha r^2}{2}} rdr \\
&\le \int_0^\infty \left(M^p_s(f,r)e^{-\frac{p\beta r^2}{2}} r^{\frac{p}{q}}\right)\left(e^{\frac{p(\beta-\alpha)r^2}{2}} r^{1-\frac{p}{q}}\right) dr \\
&\le \left(\int_0^\infty M^q_s(f,r)e^{-\frac{q\beta r^2}{2}} rdr\right)^{\frac{p}{q}} \left(\int_0^\infty e^{\frac{pq(\beta-\alpha)r^2}{2(q-p)}} rdr\right)^{\frac{q-p}{q}} \\
&\simeq \|f\|^p_{s,q,\beta}.
\end{aligned}$$

If $q = \infty$, then

$$\begin{aligned}
\|f\|^p_{p,\alpha} &\simeq \int_0^\infty M^p_p(f,r)e^{-\frac{p\alpha r^2}{2}} rdr \\
&\le \int_0^\infty \left(M^p_s(f,r)e^{-\frac{p\beta r^2}{2}}\right)\left(e^{\frac{p(\beta-\alpha)r^2}{2}} r\right) dr \\
&\le \|f\|^p_{s,\infty,\beta} \int_0^\infty e^{\frac{p(\beta-\alpha)r^2}{2}} rdr \simeq \|f\|^p_{s,\infty,\beta}.
\end{aligned}$$

Subcase 3. Let $q > s$ and $p > s$. By Subcase 2 and (3.1), $\mathcal{F}(s,q,\beta) \subset \mathcal{F}^s_\alpha \subset \mathcal{F}^p_\alpha$. The remaining arguments for the properness of the inclusion are similar to the above, hence skipped. ∎

Now, we come to the proof of Theorem 3.4.

The proof of Theorem 3.4 (a) Let $s < q$ and $p \le q$. Then, by Proposition 3.5 and (3.1),

$$\mathcal{F}^p_\alpha \subset \mathcal{F}^q_\alpha \subsetneqq \mathcal{F}(s,q,\alpha).$$

(b) Let $s < q < p$. First, we assume that $\mathcal{F}^p_\alpha \subset \mathcal{F}(s,q,\alpha)$. As above, there is a constant $C > 0$ such that

$$\|f\|_{s,q,\alpha} \le C\|f\|_{p,\alpha} \text{ for } f \in \mathcal{F}^p_\alpha. \tag{3.12}$$

On the other hand, by (3.11), we get

$$\frac{\|z^n\|_{s,q,\alpha}}{\|z^n\|_{p,\alpha}} \sim n^{\frac{1}{2q}-\frac{1}{2p}} \to \infty \text{ as } n \to \infty.$$

Thus, we get a contradiction to (3.12), which implies that $\mathcal{F}^p_\alpha \not\subset \mathcal{F}(s,q,\alpha)$. Next, we assume that $\mathcal{F}(s,q,\alpha) \subset \mathcal{F}^p_\alpha$. Then, there is a constant $C > 0$ such that

$$\|f\|_{p,\alpha} \le C\|f\|_{s,q,\alpha} \text{ for } f \in \mathcal{F}(s,q,\alpha). \tag{3.13}$$

On the other hand, using (3.2) and (3.6)–(3.8), we get

$$\frac{\|K_{\alpha,w}\|_{s,q,\alpha}}{\|K_{\alpha,w}\|_{p,\alpha}} \to 0 \text{ as } |w| \to \infty.$$

Thus, we get a contradiction to (3.13), which implies that $\mathcal{F}(s,q,\alpha) \not\subset \mathcal{F}^p_\alpha$.

(c) Let $q < s$ and $q \le p$. Then, by Proposition 3.5 and (3.1), we get

$$\mathcal{F}(s,q,\alpha) \subsetneqq \mathcal{F}^q_\alpha \subset \mathcal{F}^p_\alpha.$$

(d) Let $p < q < s$. Similar to the proof of part (b), by contradiction, we assume that $\mathcal{F}^p_\alpha \subset \mathcal{F}(s,q,\alpha)$, and hence, (3.12) holds. On the other hand, using (3.2) and (3.9), we get

$$\frac{\|K_{\alpha,w}\|_{s,q,\alpha}}{\|K_{\alpha,w}\|_{p,\alpha}} \to \infty \text{ as } |w| \to \infty.$$

Thus, we get a contradiction to (3.12). Now, we assume that $\mathcal{F}(s,p,\alpha) \subset \mathcal{F}^q_\alpha$, and hence (3.13) holds. On the other hand, by (3.11), we get

$$\frac{\|z^n\|_{s,q,\alpha}}{\|z^n\|_{p,\alpha}} \sim n^{\frac{1}{2q}-\frac{1}{2p}} \to 0 \text{ as } n \to \infty,$$

which is a contradiction to (3.13). ∎

4 Hadamard lacunary series

This section studies random Hadamard lacunary series in Fock spaces induced by a more general sequence of random variables $(X_n)_{n\ge 0}$. Recall that a Hadamard lacunary sequence is a subsequence $(n_k)_{k\ge1} \subset \mathbb{N}$ such that $\inf_{k\ge1} \frac{n_{k+1}}{n_k} > 1$. The main result in this section is the following.

Theorem 4.1 *Let $p, q, s, \alpha, \beta \in (0,\infty)$, $(n_k)_{k\ge1}$ be a Hadamard lacunary sequence, and $(X_n)_{n\ge0}$ a sequence of independent, identically distributed symmetric random variables with finite variances.*

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(a) *For an entire function* $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$*, the series* $\mathcal{R}f(z) = \sum_{k=1}^{\infty} a_k X_{n_k} z^{n_k}$ *belongs to* $\mathcal{F}(s,p,\alpha)$ *a.s. if and only if* $f \in \mathcal{F}_\alpha^p$.

(b) *The series* $\mathcal{R}f(z)$ *belongs to* $\mathcal{F}(s,q,\beta)$ *a.s. for a lacunary series* $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in \mathcal{F}_\alpha^p$ *if and only if one of the following conditions holds:*
  (i) $\alpha < \beta$*; or*
  (ii) $\alpha = \beta$ *and* $q \geq p$.

To proceed, we need to extend a result of Tung [35, Theorem (Summary)] from $p \in [1,\infty)$ to $p \in (0,\infty)$.

Proposition 4.2 *Let* $p, s, \alpha \in (0,\infty)$, $(n_k)_{k\geq 1}$ *be a Hadamard lacunary sequence. For an entire function* $f(z) = \sum_{n=0}^{\infty} a_k z^{n_k}$*, the following statements are equivalent:*

(i) $f \in \mathcal{F}_\alpha^p$;
(ii) $f \in \mathcal{F}(s,p,\alpha)$*; and*
(iii)

$$\sum_{k=0}^{\infty} |a_k|^p \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{p}{2}} n_k^{-\frac{p}{4}+\frac{1}{2}} < \infty.$$

To prove this proposition, we need the following characterization of functions in $\mathcal{F}(2,p,\alpha)$ through their Taylor coefficients. In particular, the case $p=2$ reduces to the well-known characterization of functions in the Hilbert Fock space $\mathcal{F}_\alpha^2$.

Lemma 4.3 *Let* $p, \alpha \in (0,\infty)$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function.*

(a) *For* $0 < p \leq 2$,

$$\sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty \Rightarrow f \in \mathcal{F}(2,p,\alpha) \Rightarrow \sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty.$$

(b) *For* $2 \leq p < \infty$,

$$\sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty \Rightarrow f \in \mathcal{F}(2,p,\alpha) \Rightarrow \sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty.$$

Proof (a) Let $p \in (0,2]$. Using Stirling's approximation, we get

$$\begin{aligned}\|f\|_{2,p,\alpha}^p &= \int_0^\infty \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n}\right)^{\frac{p}{2}} d\lambda_{p\alpha}(r) \leq \sum_{n=0}^{\infty} |a_n|^p \int_0^\infty r^{np} p\alpha r e^{-\frac{p\alpha r^2}{2}} dr \\ &= \sum_{n=0}^{\infty} |a_n|^p \left(\frac{2}{p\alpha}\right)^{\frac{np}{2}} \int_0^\infty t^{\frac{np}{2}} e^{-t} dt = \sum_{n=0}^{\infty} |a_n|^p \left(\frac{2}{p\alpha}\right)^{\frac{np}{2}} \Gamma\left(\frac{np}{2}+1\right) \\ &\simeq \sum_{n=0}^{\infty} |a_n|^p \left(\frac{n}{e\alpha}\right)^{\frac{np}{2}} n^{\frac{1}{2}} \simeq \sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}}.\end{aligned}$$

From this, the first implication follows, whereas the second one can be reduced from Proposition 3.5 and [34, Theorem 4(i)].

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(b) Let $p \in [2, \infty)$. Similarly as above, we have

$$\|f\|^p_{2,p,\alpha} = \int_0^\infty \left(\sum_{n=0}^\infty |a_n|^2 r^{2n}\right)^{\frac{p}{2}} d\lambda_{p\alpha}(r) \geq \sum_{n=0}^\infty |a_n|^p \int_0^\infty r^{np} p\alpha r e^{-\frac{p\alpha r^2}{2}} dr$$
$$\simeq \sum_{n=0}^\infty |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}}.$$

From this, the second implication follows, whereas the first one can be obtained from Proposition 3.5 and [34, Theorem 4(ii)]. ■

Proof of Proposition 4.2 From Khintchine's inequality for lacunary series ([14, Theorem 6.2.2, p. 114] or [36], [39]), it follows that $M_s(f, r) \simeq M_2(f, r)$ for $s \in (0, \infty)$, which implies that $f \in \mathcal{F}^p_\alpha$ if and only if $f \in \mathcal{F}(2, p, \alpha)$ or, equivalently, $f \in \mathcal{F}(s, p, \alpha)$. In the rest of the proof, we consider $s = 2$. For $p \in [2, \infty)$, (ii) $\Longrightarrow$ (iii) by Lemma 4.3(b) and (iii) $\Longrightarrow$ (i) by [35, Theorem 2.4]. For $p \in (0, 2]$, (ii) follows from (iii) by Lemma 4.3(a). It remains to prove (ii) $\Longrightarrow$ (iii). For $f \in \mathcal{F}(2, p, \alpha)$, we get

$$\begin{aligned}
\|f\|^p_{2,p,\alpha} &= \int_0^\infty \left(\sum_{j=0}^\infty |a_j|^2 r^{2n_j}\right)^{\frac{p}{2}} d\lambda_{p\alpha}(r) \\
&\geq \sum_{k=0}^\infty \int_{\sqrt{\frac{n_k}{\alpha}}}^{\sqrt{\frac{n_{k+1}}{\alpha}}} \left(\sum_{j=0}^\infty |a_j|^2 r^{2n_j}\right)^{\frac{p}{2}} d\lambda_{p\alpha}(r) \\
&\geq \sum_{k=0}^\infty |a_k|^p \int_{\sqrt{\frac{n_k}{\alpha}}}^{\sqrt{\frac{n_{k+1}}{\alpha}}} r^{pn_k} d\lambda_{p\alpha}(r) \\
&\gtrsim \sum_{k=0}^\infty |a_k|^p \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{p}{2}} n_k^{-\frac{p}{4}+\frac{1}{2}},
\end{aligned}$$

where the last inequality is based on [35, Lemma 2.1]. ■

The proof of Theorem 4.1 (a) For a Hadamard lacunary series $f(z) = \sum_{k=0}^\infty a_k z^{n_k}$, by Proposition 4.2, the series $\mathcal{R}f(z)$ belongs to $\mathcal{F}(s, p, \alpha)$ a.s. if and only if

$$\sum_{k=0}^\infty |a_k X_{n_k}|^p \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{p}{2}} n_k^{-\frac{p}{4}+\frac{1}{2}} < \infty \text{ a.s.},$$

which, by [15, Theorem 5, p. 33], is equivalent to

$$\sum_{k=0}^\infty |a_k|^p \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{p}{2}} n_k^{-\frac{p}{4}+\frac{1}{2}} < \infty, \text{ i.e., } f \in \mathcal{F}^p_\alpha,$$

where the last argument is based on Proposition 4.2 again.

(b) The sufficiency follows from part (a), (3.1), and Theorem 3.3. For the necessity, we consider $\alpha = \beta$ and $q < p$ first. We construct the following function:

$$f_0(z) := \sum_{k=1}^\infty a_k z^{n_k} \text{ with } a_k := \left(\frac{\alpha^{n_k}}{n_k!}\right)^{\frac{1}{2}} n_k^{\frac{1}{4}-\frac{1}{2q}}, \ k \in \mathbb{N}.$$

Then

$$\sum_{k=0}^{\infty} |a_k|^q \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{q}{2}} n_k^{-\frac{q}{4}+\frac{1}{2}} = \sum_{k=1}^{\infty} 1 = \infty$$

and

$$\sum_{k=0}^{\infty} |a_n|^p \left(\frac{n_k!}{\alpha^{n_k}}\right)^{\frac{p}{2}} n_k^{-\frac{p}{4}+\frac{1}{2}} = \sum_{k=1}^{\infty} n_k^{\frac{q-p}{2q}} < \infty,$$

where the last inequality is based on the fact that $(n_k)_{k\geq 1}$ is a Hadamard lacunary sequence. From this and Proposition 4.2, it follows that $f_0 \in \mathcal{F}^p_\alpha$ and $f_0 \notin \mathcal{F}^q_\alpha$, and hence, by part (a), $\mathcal{R}f_0$ does not belong to $\mathcal{F}(s,q,\alpha)$ a.s. Now, let $\beta < \alpha$. Similarly as above, we consider the function

$$g_0(z) := \sum_{k=1}^{\infty} b_k z^{n_k} \text{ with } b_k := \left(\frac{\beta^{n_k}}{n_k!}\right)^{\frac{1}{2}} n_k^{\frac{1}{4}-\frac{1}{2q}},\ k \in \mathbb{N},$$

and get that the series $\mathcal{R}f_0 \notin \mathcal{F}(s,q,\beta)$ a.s. and $f_0 \in \mathcal{F}^{q'}_\beta$ for $q' > q$, and hence $f_0 \in \mathcal{F}^p_\alpha$ by Theorem 3.3. ∎

5 Random multipliers

In this section, we present a complete description of random multipliers between Fock spaces; namely, we characterize the symbol space $(\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta))_*$, where $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$ denotes the space of multipliers from $\mathcal{F}^p_\alpha$ to $\mathcal{F}^q_\beta$.

Theorem 5.1 *Let $p, q \in (0,\infty]$ and $\alpha, \beta \in (0,\infty)$.*

(a) *If either $\alpha > \beta$ or $\alpha = \beta$ and $q < p$, then $(\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta))_* = \{0\}$.*

(b) *If $\alpha = \beta$ and $p \leq q$, then $(\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta))_*$ consists of only constant functions.*

(c) *If $\alpha < \beta$ and $p \leq q$, then $(\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta))_* = (\mathcal{F}^\infty_{\beta-\alpha})_*$.*

(d) *If $\alpha < \beta$ and $q < p < \infty$, then $(\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta))_* = \mathcal{F}\left(2, \frac{pq}{p-q}, \beta - \alpha\right)$.*

(e) *If $\alpha < \beta$ and $q < p = \infty$, then $(\mathcal{M}(\mathcal{F}^\infty_\alpha, \mathcal{F}^q_\beta))_* = \mathcal{F}(2, q, \beta - \alpha)$.*

The proof of this theorem follows from Theorems 2.6 and 5.2, which characterizes the space $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$. This is clearly of independent interests. Recall that an entire function ϕ is called a *multiplier* from $\mathcal{F}^p_\alpha$ to $\mathcal{F}^q_\beta$, if ϕf belongs to $\mathcal{F}^q_\beta$ for $f \in \mathcal{F}^p_\alpha$. If $\phi \in \mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$, then, by the closed graph theorem, the multiplication operator $M_\phi : f \mapsto \phi f$ is necessarily bounded.

Theorem 5.2 *Let $p, q \in (0,\infty]$ and $\alpha, \beta \in (0,\infty)$.*

(a) *If either $\alpha > \beta$ or $\alpha = \beta$ and $q < p$, then $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) = \{0\}$.*

(b) *If $\alpha = \beta$ and $p \leq q$, then $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$ consists of only constant functions.*

(c) *If $\alpha < \beta$ and $p \leq q$, then $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) = \mathcal{F}^\infty_{\beta-\alpha}$ and*

$$\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \simeq \|\phi\|_{\infty,\beta-\alpha}.$$

(d) *If* $\alpha < \beta$ *and* $q < p < \infty$, *then* $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) = \mathcal{F}^{\frac{pq}{p-q}}_{\beta-\alpha}$ *and*

$$\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \simeq \|\phi\|_{\frac{pq}{p-q},\beta-\alpha}.$$

(e) *If* $\alpha < \beta$ *and* $q < p = \infty$, *then* $\mathcal{M}(\mathcal{F}^\infty_\alpha, \mathcal{F}^q_\beta) = \mathcal{F}^q_{\beta-\alpha}$ *and*

$$\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \simeq \|\phi\|_{q,\beta-\alpha}.$$

Proof Fix a nonzero function $\phi \in \mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$. Then

$$\|M_\phi f\|_{q,\beta} \le \|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \|f\|_{p,\alpha} \text{ for } f \in \mathcal{F}^p_\alpha.$$

For each $w \in \mathbb{C}$, we consider the normalized reproducing kernel

$$k_{\alpha,w} := e^{\alpha \overline{w} z - \frac{\alpha |w|^2}{2}}$$

of the Hilbert Fock space $\mathcal{F}^2_\alpha$. Then, $\|k_{\alpha,w}\|_{p,\alpha} = 1$ for $p \in (0,\infty]$. By (3.3), for $z, w \in \mathbb{C}$,

$$\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \ge \|M_\phi k_{\alpha,w}\|_{q,\beta} \ge |\phi(z) k_{\alpha,w}(z)| e^{\frac{-\beta|z|^2}{2}}.$$

In particular, with $w = z$, we get

$$|\phi(z)| \le \|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} e^{\frac{(\beta-\alpha)|z|^2}{2}} \text{ for } z \in \mathbb{C}. \tag{5.1}$$

If $\alpha > \beta$, then, by (5.1), $\phi(z) \to 0$ as $|z| \to \infty$, that is, $\phi(z) \equiv 0$ on $\mathbb{C}$. If $\alpha = \beta$, then, by (5.1), $\phi(z)$ is a constant function. If $q < p$, then, by (3.1), $\mathcal{F}^q_\beta$ is a proper subspace of $\mathcal{F}^p_\alpha$, and hence $\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) = \{0\}$. Thus, (a) and (b) are proved.

(c) Let $\alpha < \beta$ and $p \le q$. Then, by (5.1),

$$\mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) \subset \mathcal{F}^\infty_{\beta-\alpha} \text{ and } \|\phi\|_{\infty,\beta-\alpha} \le \|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta}.$$

On the other hand, for $\phi \in \mathcal{F}^\infty_{\beta-\alpha}$, using (3.1) and (3.3), we get

$$\begin{aligned}\|M_\phi f\|_{q,\beta} &= \left(\frac{q\beta}{2\pi} \int_{\mathbb{C}} |\phi(z)|^q |f(z)|^q e^{-\frac{q\beta|z|^2}{2}} dA(z)\right)^{\frac{1}{q}} \\ &\le \|\phi\|_{\infty,\beta-\alpha} \left(\frac{q\beta}{2\pi} \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q\alpha|z|^2}{2}} dA(z)\right)^{\frac{1}{q}} \\ &\simeq \|\phi\|_{\infty,\beta-\alpha} \|f\|_{q,\alpha} \lesssim \|\phi\|_{\infty,\beta-\alpha} \|f\|_{p,\alpha}.\end{aligned}$$

Thus,

$$\phi \in \mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta) \text{ and } \|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \lesssim \|\phi\|_{\infty,\beta-\alpha}.$$

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(d) Let $\alpha < \beta$ and $q < p < \infty$. Fixing a nonzero function $\phi \in \mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$, for $f \in \mathcal{F}^p_\alpha$, we get

$$\begin{aligned}\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta}\|f\|_{p,\alpha} \geq \|M_\phi f\|_{q,\beta} &= \left(\frac{q\beta}{2\pi}\int_{\mathbb{C}}|f(z)\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}}dA(z)\right)^{\frac{1}{q}} \\ &\simeq \left(\int_{\mathbb{C}}\left|f(z)e^{-\frac{\alpha|z|^2}{2}}\right|^q|\phi(z)|^q e^{-\frac{q(\beta-\alpha)|z|^2}{2}}dA(z)\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{C}}\left|f(z)e^{-\frac{\alpha|z|^2}{2}}\right|^q d\mu_{\phi,q,\beta-\alpha}(z)\right)^{\frac{1}{q}},\end{aligned}$$

where $d\mu_{\phi,q,\beta-\alpha}(z) := |\phi(z)|^q e^{-\frac{q(\beta-\alpha)|z|^2}{2}}dA(z)$. The last inequality means that $\mu_{\phi,q,\beta-\alpha}$ is a (p,q)-Fock Carleson measure [13, Section 3]. Then, by [13, Theorem 3.3],

$$\widetilde{\mu_{\phi,q,\beta-\alpha}}(w) := \int_{\mathbb{C}}|k_{\alpha,w}(z)|^q e^{-\frac{q\alpha|z|^2}{2}}d\mu_{\phi,q,\beta-\alpha}(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

For $w, z \in \mathbb{C}$, by (3.3),

$$\begin{aligned}\widetilde{\mu_{\phi,q,\beta-\alpha}}(w) &= \int_{\mathbb{C}}|k_{\alpha,w}(z)|^q|\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}}dA(z) \\ &\geq |k_{\alpha,w}(z)|^q|\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}}.\end{aligned}$$

Thus, $\widetilde{\mu_{\phi,q,\beta-\alpha}}(z) \geq |\phi(z)|^q e^{-\frac{q(\beta-\alpha)|z|^2}{2}}$ for $z \in \mathbb{C}$. Hence,

$$\begin{aligned}\|\phi\|_{\frac{pq}{p-q},\beta-\alpha} &\simeq \left(\int_{\mathbb{C}}|\phi(z)|^{\frac{pq}{p-q}}e^{-\frac{pq(\beta-\alpha)|z|^2}{2(p-q)}}dA(z)\right)^{\frac{p-q}{pq}} \\ &\leq \left(\int_{\mathbb{C}}\left(\widetilde{\mu_{\phi,q,\beta-\alpha}}(z)\right)^{\frac{p}{p-q}}dA(z)\right)^{\frac{p-q}{pq}} = \|\widetilde{\mu_{\phi,q,\beta-\alpha}}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}(\mathbb{C},dA)} < \infty.\end{aligned}$$

Therefore, $\phi \in \mathcal{F}^{\frac{pq}{p-q}}_{\beta-\alpha}$ and, by [13, Theorem 3.3] again,

$$\|\phi\|_{\frac{pq}{p-q},\beta-\alpha} \lesssim \|\widetilde{\mu_{\phi,q,\beta-\alpha}}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}(\mathbb{C},dA)} \lesssim \|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta}.$$

Conversely, fixing a nonzero function $\phi \in \mathcal{F}^{\frac{pq}{p-q}}_{\beta-\alpha}$, for $f \in \mathcal{F}^p_\alpha$,

$$\begin{aligned}\|M_\phi f\|_{q,\beta} &\simeq \left(\int_{\mathbb{C}}\left|f(z)e^{-\frac{\alpha|z|^2}{2}}\right|^q\left|\phi(z)e^{-\frac{(\beta-\alpha)|z|^2}{2}}\right|^q dA(z)\right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{C}}\left|f(z)e^{-\frac{\alpha|z|^2}{2}}\right|^p dA(z)\right)^{\frac{1}{p}}\left(\int_{\mathbb{C}}\left|\phi(z)e^{-\frac{(\beta-\alpha)|z|^2}{2}}\right|^{\frac{pq}{p-q}}dA(z)\right)^{\frac{p-q}{pq}} \\ &\lesssim \|\phi\|_{\frac{pq}{p-q},\beta-\alpha}\|f\|_{p,\alpha}.\end{aligned}$$

Thus, $\phi \in \mathcal{M}(\mathcal{F}^p_\alpha, \mathcal{F}^q_\beta)$ and $\|M_\phi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta} \lesssim \|\phi\|_{\frac{pq}{p-q},\beta-\alpha}$.

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(e) Let $\alpha < \beta$ and $q < p = \infty$. Fixing a nonzero function $\phi \in \mathcal{M}(\mathcal{F}^\infty_\alpha, \mathcal{F}^q_\beta)$, similar to part (d), for $f \in \mathcal{F}^\infty_\alpha$, we get

$$\begin{aligned}\|M_\phi\|_{\mathcal{F}^\infty_\alpha \to \mathcal{F}^q_\beta} \|f\|_{\infty,\alpha} \ge \|M_\phi f\|_{q,\beta} &= \left(\frac{q\beta}{2\pi}\int_{\mathbb{C}} |f(z)\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}}\, dA(z)\right)^{\frac{1}{q}}\\ &\simeq \left(\int_{\mathbb{C}} |f(z)|^q |\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}}\, dA(z)\right)^{\frac{1}{q}}\\ &= \left(\int_{\mathbb{C}} |f(z)|^q d\lambda_{\phi,q,\beta}(z)\right)^{\frac{1}{q}},\end{aligned}$$

where $d\lambda_{\phi,q,\beta} := |\phi(z)|^q e^{-\frac{q\beta|z|^2}{2}} dA(z)$. The last inequality means that $\lambda_{\phi,q,\beta}$ is a q-Carleson measure for $\mathcal{F}^\infty_\alpha$ [1, Section 3.2]. Then

$$\|\phi\|^q_{q,\beta-\alpha} = \int_{\mathbb{C}} |\phi(z)|^q e^{-\frac{q(\beta-\alpha)|z|^2}{2}}\, dA(z) = \int_{\mathbb{C}} e^{\frac{q\alpha|z|^2}{2}}\, d\lambda_{\phi,q,\beta}(z) < \infty,$$

where the last step is based on [1, Section 3.2.2]. Therefore, $\phi \in \mathcal{F}^q_{\beta-\alpha}$. Moreover, following the arguments in the proof of [1, Theorem 1.2], we conclude that

$$\|\phi\|_{q,\beta-\alpha} \lesssim \|M_\phi\|_{\mathcal{F}^\infty_\alpha \to \mathcal{F}^q_\beta}.$$

Conversely, fixing a nonzero function $\phi \in \mathcal{F}^q_{\beta-\alpha}$, for $f \in \mathcal{F}^\infty_\alpha$, using (3.3), we get

$$\begin{aligned}\|M_\phi f\|_{q,\beta} &\simeq \left(\int_{\mathbb{C}} \left|f(z)e^{-\frac{\alpha|z|^2}{2}}\right|^q \left|\phi(z)e^{-\frac{(\beta-\alpha)|z|^2}{2}}\right|^q dA(z)\right)^{\frac{1}{q}}\\ &\le \|f\|_{\infty,\alpha}\left(\int_{\mathbb{C}} \left|\phi(z)e^{-\frac{(\beta-\alpha)|z|^2}{2}}\right|^q dA(z)\right)^{\frac{1}{q}}\\ &\lesssim \|\phi\|_{q,\beta-\alpha}\|f\|_{\infty,\alpha}.\end{aligned}$$

Thus, $\phi \in \mathcal{M}(\mathcal{F}^\infty_\alpha, \mathcal{F}^q_\beta)$ and $\|M_\phi\|_{\mathcal{F}^\infty_\alpha \to \mathcal{F}^q_\beta} \lesssim \|\phi\|_{q,\beta-\alpha}$. ∎

Remark 5.1 We end this paper with a remark on the definition of standard random sequences. It is perhaps natural to wonder whether one can replace them by more general sequences of i.i.d. random variables, which are centered ($\mathbb{E}(X) = 0$), symmetric ($X \overset{d}{=} -X$), and with finite second moment ($\mathbb{E}(X^2) < \infty$). This is perhaps true, but not obvious to us. The obstacle lies in the Fernique theorem, which holds for Gaussian vectors. For the Bernoulli and Rademacher cases, one has Kahane's inequality to roughly the same effect. These two inequalities are needed, in particular, for Proposition 2.3. Extending them to more general random sequences (in a meaningful way) is a nontrivial task in probability. On the other hand, since there exist no other obviously contending methods of randomization to the standard random sequences, we are content with our choice so far. Another small extension, to complex Gaussian variables, however, holds true easily for every result in this work by considering the real and imaginary parts separately.

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