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NEW MOCK THETA FUNCTIONS AND FORMULAS FOR BASIC HYPERGEOMETRIC SERIES

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Abstract In recent years, mock theta functions in the modern sense have received great attention to seek examples of q-hypergeometric series and find their alternative representations. In this paper, we discover some new mock theta functions and express them in terms of Hecke-type double sums based on some basic hypergeometric series identities given by Z.G. Liu.

Keywords: mock theta functions; Hecke-type double sums; basic hypergeometric series

2020 Mathematics subject classification: Primary 33D15; 05A30; 11B65; 11F11

1. Notation

Throughout this paper, we use the standard q-series notation [13]. Let q be a complex number with $0 < |q| < 1$. Recall that

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$
$$(a_1,a_2,\ldots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty$$

and

$$j(x;q) :=(x,q/x,q;q)_\infty, \qquad J_m := j(q^m;q^{3m}) = (q^m;q^m)_\infty,$$

where m is a positive integer. In addition, the basic hypergeometric series ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right) := \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n\cdots(b_r;q)_n}z^n, \qquad |z|<1.$$

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Recall that a series is called a Hecke-type double sum if it has the following form:

$$\sum_{(m,n)\in D} (-1)^{H(m,n)} q^{Q(m,n)+L(m,n)},$$

where $H(m,n)$ and $L(m,n)$ are linear forms, $Q(m,n)$ is a quadratic form and D is a subset of $\mathbb{Z}\times\mathbb{Z}$ such that $Q(m,n)\geq 0$. In their paper [18], Hickerson and Mortenson gave the following definition for a special type of Hecke-type double sums.

Definition 1. [18] *Let $x,y\in\mathbb{C}^* := \mathbb{C}-\{0\}$ and define $\mathrm{sg}(r) := 1$ for $r\geq 0$ and $\mathrm{sg}(r) := -1$ for $r<0$. Then*

$$f_{a,b,c}(x,y,q) := \sum_{sg(r)=\mathrm{sg}(s)} sg(r)(-1)^{r+s}x^r y^s q^{ar(r-1)/2+brs+cs(s-1)/2}.$$

Note that,

$$f_{a,b,a}(x,y,q) = f_{a,b,a}(y,x,q). \tag{1.1}$$

Moreover, we will use the following definition of Appell-Lerch sums, which were first studied by Appell [2] and Lerch [19].

Definition 2. ([18]). *Let $x,z\in\mathbb{C}^*$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z) := \frac{-z}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} z^n}{1-q^n xz}.$$

2. Introduction

In 1920, Ramanujan introduced 17 functions, which he called ‘mock theta functions’ in his famous deathbed letter to Hardy. He defined each of these functions as a q-series and found that these functions have certain asymptotic properties as q approaches a root of unity, which are similar to theta functions, but that they are not theta functions. Motivated by Ramanujan’s work, mock theta functions have attracted the attention of many mathematicians. With the contribution of many works, Andrews and Hickerson [1], Berndt and Chan [3], Chen and Wang [5], Choi [6–9], Gordon and McIntosh [14, 25] and Waston [28, 29], to name a few, many new mock theta functions were discovered and a number of identities satisfied by those mock theta functions were proved. See [15] for a summary on the classical mock theta functions.

In 2002, Zwegers [33] established the modularity theory for classical mock theta functions by using the Appell-Lerch sums or Hecke-type double sums. With the contribution of the works of Bringmann and Ono [4] and Zwegers [33], we now know that each of Ramanujan’s original 17 mock theta functions is the holomorphic part of a weight 1/2 harmonic weak Maass form with a weight 3/2 unary theta function as its ‘shadow’. In [31], Zagier gave the definition of mock theta functions in the modern sense. A function

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is called a mock modular form of weight k if it is the holomorphic part of a weight k harmonic weak Maass form $f(q)$ (as usual $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$, where $\tau = x + y\mathrm{i} \in \mathbb{H}$). In addition, if the weight of a harmonic weak Maass form $f(q)$ is $1/2$ and its 'shadow' is a unary theta function, then the holomorphic part of $f(q)$ is called a mock theta function. In particular, Zagier [31] and Zwegers [33] showed that specializations of Appell-Lerch sums $m(x, q, z)$ give rise to mock theta functions. In [18], Hickerson and Mortenson proved that if n and p are positive integers with $\gcd(n, p) = 1$, then $f_{n,n+p,n}(x, y, q)$ can be represented as a linear combination of Appell-Lerch sums $m(x, q, z)$ and theta functions.

In his plenary address at the Millennial Conference on Number Theory, Andrews challenged mathematicians in the 21st century to elucidate the overlap between classes of q-series and modular forms. Motivated by Andrews' challenge, seeking examples of q-hypergeometric series which are mock theta functions in the modern sense and finding their alternative representations have become important tasks for studying mock theta functions and have attracted the attention of many mathematicians. In recent years, some mathematicians discovered genuine mock theta functions. Lovejoy and Osburn [22–24] used the Bailey machinery to produce families of q-hypergeometric multisums, which are mock theta functions. Cui *et al.* [11], Gu and Hao [16], Gu and Liu [17] and Zhang and Li [32] also discovered several new mock theta functions on Bailey pairs. Recently, Cui and Gu [10] established three two-parameter mock theta functions and represented them in terms of Appell-Lerch sums by using some formulas of basic hypergeometric series. Very recently, Yao [30] used partial fraction decomposition to give rise to several families of two-parameter mock theta functions and expressed them in the form of Appell-Lerch sums. In particular, she established some identities involving several families of mock theta functions and Appell–Lerch sums, which imply three identities proved by Cui and Gu [10]. Very recently, Mortenson *et al.* [27] obtained new symmetries for string functions by exploiting their natural setting of Hecke-type double sums, where special double sums are expressed in terms of Appell-Lerch sums and theta functions. By expressing Hecke-type double-sums in terms of theta functions and Appell-Lerch functions, Mortenson [26] gave general string function formulas for the affine Lie algebra $A_1^{(1)}$ for levels $N = 1, 2, 3, 4$. Motivated by those works, we present several new mock theta functions and express them in terms of linear combinations of $f_{n,n+p,n}(x, y, q)$ and theta functions by using some basic hypergeometric series identities due to Liu [20, 21] in this paper.

Define

$$M_1(q) := 2\frac{J_1J_4}{J_2}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(-1)^nq^{n^2}}{(-1,q^2;q^2)_n}, \qquad M_2(q) := 2J_1\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}}{(-1,q;q)_n},$$

$$M_3(q) := \frac{J_2^2}{J_4}\sum_{n=0}^{\infty}\frac{(-1;q^2)_n}{(-q,q^2;q^2)_n}(-1)^nq^{n^2+2n}, \qquad M_4(q) := J_2\sum_{n=0}^{\infty}\frac{(-1)^nq^{2n^2+n}}{(-q,q^2;q^2)_n},$$

$$M_5(q) := \frac{J_2^5}{J_1^2J_4^2}\sum_{n=0}^{\infty}\frac{(-1,-q^2;q^2)_n}{(-q,q^2;q^2)_n}(-q)^n, \qquad M_6(q) := \frac{J_2^2}{J_1}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n}{(q^2,-q;q^2)_n}(-1)^nq^{n^2},$$

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$$M_7(q) := 2\frac{J_1J_4}{J_2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{n^2}}{(-1,q^2;q^2)_n}, \qquad M_8(q) := \frac{J_2^2}{J_4}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{n^2+n}}{(-q,q^2;q^2)_n},$$

$$M_9(q) := J_2\sum_{n=0}^{\infty}\frac{(q;q^2)_n}{(-1,-q,q^2;q^2)_n}q^{2n^2}.$$

The main results of the paper can be stated as follows.

Theorem 2.1 *The following identities are true:*

$$M_1(q) = f_{1,5,1}(q,q,q^2) + f_{1,5,1}(q^3,q^3,q^2) - qf_{1,5,1}(q^5,q^5,q^2), \tag{2.1}$$

$$M_2(q) = f_{1,3,1}(q,q,q^2) - f_{1,3,1}(1,1,q^2) - qf_{1,3,1}(q^4,q^4,q^2), \tag{2.2}$$

$$M_3(q) = f_{1,5,1}(q,q,q^2) + 2qf_{1,5,1}(q^9,q^3,q^2) - 2q^2f_{1,5,1}(q^5,q^7,q^2), \tag{2.3}$$

$$M_4(q) = f_{1,3,1}(q^5,q^3,q^4) + q^6f_{1,3,1}(q^{13},q^{11},q^4), \tag{2.4}$$

$$M_5(q) = f_{1,3,1}(q^3,q,q^2) + qf_{1,3,1}(q^3,q^5,q^2), \tag{2.5}$$

$$M_6(q) = \frac{1}{2}f_{1,2,1}(q,q,q^4) + \frac{1}{2}f_{1,2,1}(q^3,q^3,q^4) + qf_{1,2,1}(q^5,q^7,q^4), \tag{2.6}$$

$$M_7(q) = f_{1,2,1}(q^2,q^2,q^4) - f_{1,2,1}(1,1,q^4) - qf_{1,2,1}(q^6,q^6,q^4), \tag{2.7}$$

$$M_8(q) = f_{1,2,1}(q^3,q^3,q^4) + q^4f_{1,2,1}(q^9,q^9,q^4), \tag{2.8}$$

$$M_9(q) = \frac{1}{2}f_{3,5,3}(q^3,q^3,q^2) + \frac{1}{2}f_{3,5,3}(q^5,q^5,q^2) - \frac{1}{2}q^2f_{3,5,3}(q^9,q^9,q^2). \tag{2.9}$$

Remark. Note that some identities are sign flips away from classical mock theta functions. Sometimes sign flips can be interesting. $M_3(q)$ is related to the sixth order mock theta function $\phi_-(q)$:

$$\frac{J_1^2}{J_2}\phi_-(q) = qf_{1,5,1}(q^3,q^9,q^2) + q^2f_{1,5,1}(q^7,q^5,q^2),$$

where

$$\phi_-(q) := \sum_{n\geq 1}\frac{(-q;q)_{2n-1}q^n}{(q;q^2)_n}$$

and $M_4(q)$ is a sign flip away from

$$\frac{J_1J_4}{J_2}(1+V_0(q)) = 2f_{1,3,1}(q^5,q^3,q^4) - 2q^6f_{1,3,1}(q^{13},q^{11},q^4),$$

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where $V_0(q)$ is the eighth order mock theta function defined by

$$V_0(q) := -1 + 2\sum_{n\geq 1} \frac{(-q;q^2)_n q^{n^2}}{(q;q^2)_n}.$$

Moreover, $M_5(q)$ is related to the second-order mock theta function $A(q)$:

$$\frac{J_1^2}{J_2} A(q) = q f_{1,3,1}(q^5, q^5, q),$$

where

$$A(q) := \sum_{n\geq 0} \frac{(-q^2;q^2)_n q^{n+1}}{(q;q^2)_{n+1}}.$$

Furthermore, $M_8(q)$ is a sign flip away from $f_{1,2,1}(-q,q,-q)$ [18, (6.1)]:

$$\frac{J_1 J_4}{J_2}\phi(-q) = f_{1,2,1}(-q,q,-q) = f_{1,2,1}(q^3,q^3,q^4) - q^4 f_{1,2,1}(q^9,q^9,q^4),$$

where $\phi(q)$ is the sixth-order mock theta function defined by

$$\phi(q) = \sum_{n\geq 0} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q)_{2n}}.$$

Hickerson and Mortenson [18] proved that $f_{n,n+p,n}(x,y,q)$ can be represented as linear combinations of Appell-Lerch sums and theta functions. Therefore, $M_i(q)$ can be represented in terms of Appell-Lerch sums and theta functions. In the following, we list some simple expressions of $M_i(q)$ $(5 \leq i \leq 8)$. The expressions of $M_i(q)$ $(i = 1,2,3,4,9)$ are analogous but lengthy, so are omitted.

Theorem 2.2. *We have*

$$M_5(q) = \left(1 - \frac{2}{q}\right)\frac{J_1^2}{J_2} - 2\frac{J_1^2}{J_2}\left(m(-q^6,q^{16},q^2) - \frac{1}{q}m(-q^{14},q^{16},q^2)\right) + \frac{J_8^4 j(q,q^8)^2}{qJ_{16}^2 j(q^2;q^{16})^2}, \tag{2.10}$$

$$M_6(q) = \frac{J_1 J_4}{J_2}\left(1 - m(q^5,q^{12},q^2) - \frac{1}{q}m(q,q^{12},q^2)\right), \tag{2.11}$$

$$M_7(q) = \frac{J_2^2}{J_4} + J_4^2 + \frac{2J_2^2}{qJ_4}m(q^2,q^{12},-1) - \frac{J_2^2 J_8^4 J_{12}^6}{qJ_4^4 J_6^2 J_{24}^4}, \tag{2.12}$$

$$M_8(q) = \frac{2J_1 J_4}{J_2}\left(m(q^5,q^{12},-1) + \frac{1}{q}m(q,q^{12},-1)\right) - \frac{J_1^2 J_6^2 J_8^2}{qJ_2^2 J_{24}^2}. \tag{2.13}$$

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The paper is organized as follows. In § 3, we recall some basic hypergeometric series identities and prove several lemmas. Sections 4 and 5 are devoted to the proofs of Theorems 2.1 and 2.2, respectively.

3. Preliminaries

In this section, we recall some identities on $j(x;q)$ and $m(x,q,z)$ and present several lemmas, which will be used to prove the main results of this paper.

The following identities will be frequently used without mention [18]:

$$j(-1;q)=2j(-q;q^4)=2\frac{J_2^2}{J_1},\qquad j(-q;q^2)=\frac{J_2^5}{J_1^2J_4^2},\qquad j(q;q^2)=\frac{J_1^2}{J_2},$$

$$j(q;q^4)=\frac{J_1J_4}{J_2},\qquad j(q;q^6)=\frac{J_1J_6^2}{J_2J_3}$$

and

$$j(x;q)=j(q/x;q)=-xj(1/x;q).$$

The Appell-Lerch sum $m(x,q,z)$ satisfies the following identities.

Lemma 3.1. (Hickerson and Mortenson [18]). *For generic* $x,z_0,z_1\in\mathbb{C}-\{0\}$,

$$m(x,q,z)=x^{-1}m(1/x,q,1/z),\tag{3.1}$$

$$m(x,q,z)=x^{-1}-x^{-1}m(xq,q,z),\tag{3.2}$$

$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0J_1^3j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}.\tag{3.3}$$

We also require some identities on $f_{a,b,c}(x,y,q)$.

Lemma 3.2. ([18, (6.2), (6.3), (6.5)]). *For* $x,y\in\mathbb{C}^*$ *and* $k,l\in\mathbb{Z}$,

$$\begin{aligned}f_{a,b,c}(x,y,q)=&(-x)^l(-y)^kq^{al(l-1)/2+bkl+ck(k-1)/2}f_{a,b,c}(q^{al+bk}x,q^{bl+ck}y,q)\\&+\sum_{m=0}^{l-1}(-x)^mq^{m(m-1)/2}j(q^{mb}y;q^c)+\sum_{m=0}^{k-1}(-y)^mq^{cm(m-1)/2}j(q^{mb}x;q^a),\end{aligned}\tag{3.4}$$

$$f_{a,b,c}(x,y,q)=-\frac{q^{a+b+c}}{xy}f_{a,b,c}(q^{2a+b}/x,q^{2c+b}/y,q)\tag{3.5}$$

and

$$f_{a,b,c}(x,y,q)=-yf_{a,b,c}(q^bx,q^cy,q)+j(x;q^a).\tag{3.6}$$

Now, we prove some lemmas.

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Lemma 3.3. *For* $|ab/q| < 1$,

$$\frac{(q,ab/q;q)_\infty}{(a,b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/a,q/b,cd/q^2;q)_n}{(q,c/q,d/q;q)_n}\left(\frac{ab}{q}\right)^n$$
$$=1+\sum_{n=1}^{\infty}(1-q^{2n})q^{n^2-2n}(ab)^n\frac{(q/a,q/b;q)_n}{(a,b;q)_n}F_n(c,d,q), \tag{3.7}$$

where

$$F_n(c,d,q):=\frac{(c+d)q+cd(q-2)-q^3}{(c-q)(d-q)(1-q)}$$
$$+\sum_{j=2}^{n}\frac{(-1)^j(1-q^{2j-1})(q/c,q/d;q)_j}{(1-q^j)(1-q^{j-1})(c/q,d/q;q)_j}(cd)^jq^{-j(j+3)/2}. \tag{3.8}$$

Proof. It follows from [21, Equation (3.14)] that

$${}_3\phi_2\left(\begin{matrix}q^{-n},\alpha q^n,\beta\\ c,d\end{matrix};q,q\right)=(-c)^nq^{n(n-1)/2}\frac{(\alpha q/c;q)_n}{(c;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},\alpha q^n,d/\beta\\ \alpha q/c,d\end{matrix};q,q\beta/c\right). \tag{3.9}$$

Setting $(\alpha,\beta,c,d)\to(1,cd/q^2,c/q,d/q)$ in Equation (3.9) yields

$${}_3\phi_2\left(\begin{matrix}q^{-n},q^n,cd/q^2\\ c/q,d/q\end{matrix};q,q\right)=(-c)^nq^{n(n-3)/2}\frac{(q^2/c;q)_n}{(c/q;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},q^n,q/c\\ d/q,q^2/c\end{matrix};q,d\right). \tag{3.10}$$

We also require the following formula proved by Chen and Wang [5, Lemma 2.3]:

$${}_3\phi_2\left(\begin{matrix}q^{-n},q^n,q/c\\ d/q,q^2/c\end{matrix};q,d\right)=(q/c)^n(1-q^n)\frac{(c/q;q)_n}{(q^2/c;q)_n}F_n(c,d,q), \tag{3.11}$$

where $F_n(c,d,q)$ is defined by Equation (3.8).

The following identity was proved by Liu [20, Theorem 1.7]:

$$\frac{(uq,uab/q;q)_\infty}{(ua,ub;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/a,q/b,v;q)_n}{(q,c,d;q)_n}\left(\frac{uab}{q}\right)^n$$
$$=\sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b;q)_n}{(1-u)(q,ua,ub;q)_n}(-uab)^nq^{n(n-3)/2}{}_3\phi_2\left(\begin{matrix}q^{-n},uq^n,v\\ c,d\end{matrix};q,q\right), \tag{3.12}$$

where $|uab/q| < 1$. Taking $(u,c,d,v)\to(1,c/q,d/q,cd/q^2)$ in Equation (3.12) and employing Equation (3.10), we arrive at

$$\frac{(q,ab/q;q)_\infty}{(a,b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/a,q/b,cd/q^2;q)_n}{(q,c/q,d/q;q)_n}\left(\frac{ab}{q}\right)^n$$

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$$=1+\sum_{n=1}^{\infty}\frac{(1-q^{2n})(q/a,q/b;q)_n}{(1-q^n)(a,b;q)_n}(-ab)^n q^{n(n-3)/2}{}_3\phi_2\left(\begin{matrix}q^{-n},q^n,cd/q^2\\ c/q,d/q\end{matrix};q,q\right)$$

$$=1+\sum_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n}\frac{(q/a,q/b,q^2/c;q)_n}{(a,b,c/q;q)_n}(abc)^n q^{n^2-3n}{}_3\phi_2\left(\begin{matrix}q^{-n},q^n,q/c\\ d/q,q^2/c\end{matrix};q,d\right), \tag{3.13}$$

where $|ab/q|<1$. Substituting Equation (3.11) into Equation (3.13) yields Equation (3.7). This completes the proof of this lemma. □

Lemma 3.4. *Define*

$$G_n(c,q):=\frac{q+c(q-2)}{(c-q)(1-q)}+\sum_{j=2}^{n}\frac{(1-q^{2j-1})(q/c;q)_j}{(1-q^j)(1-q^{j-1})(c/q;q)_j}c^j q^{-j^2}. \tag{3.14}$$

Then,

$$G_n(-q,q)=\frac{(-1)^n q^{-n^2+n}}{1-q^n}-\frac{1}{2}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}, \tag{3.15}$$

$$G_n(-q^3,q^2)=(-1)^n\frac{q^{-2n^2+3n}}{1-q^{2n}}-\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2-j}, \tag{3.16}$$

$$F_n(q^{3/2},-q^{3/2},q)=\frac{1+q^n}{2(1-q^n)}q^{-n(n-1)/2}-\frac{1}{2}\sum_{j=1-n}^{n-1}q^{-j(j+1)/2}, \tag{3.17}$$

$$F_n(-q^2,-q^3,q^2)=\frac{(-1)^n q^{-n^2+2n}}{1-q^{2n}}-\frac{1}{2}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}, \tag{3.18}$$

where $F_n(c,d,q)$ is defined by Equation (3.8).

Proof. Setting $c=-q$ in Equation (3.14), we have

$$\begin{aligned}G_n(-q,q)=&\frac{q-3}{2(1-q)}+\sum_{j=2}^{n}\frac{1-q^{2j-1}}{(1-q^{j-1})(1-q^j)}(-1)^j q^{-j^2+j}\\
=&\frac{q-3}{2(1-q)}+\sum_{j=2}^{n}\frac{(1-q^{j-1})+q^{j-1}(1-q^j)}{(1-q^{j-1})(1-q^j)}(-1)^j q^{-j^2+j}\\
=&\frac{q-3}{2(1-q)}+\sum_{j=2}^{n}\frac{(-1)^j q^{-j^2+j}}{1-q^j}+\sum_{j=2}^{n}\frac{(-1)^j q^{-j^2+2j-1}}{1-q^{j-1}}\\
=&\frac{q-3}{2(1-q)}+\sum_{j=2}^{n}\frac{(-1)^j q^{-j^2+j}}{1-q^j}-\sum_{j=1}^{n-1}\frac{(-1)^j q^{-j^2}}{1-q^j}\end{aligned}$$

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$$=\frac{q-3}{2(1-q)}+\frac{(-1)^n q^{-n^2+n}}{1-q^n}+\frac{q^{-1}}{1-q}-\sum_{j=2}^{n-1}(-1)^j q^{-j^2}$$
$$=\frac{q-3}{2(1-q)}+\frac{(-1)^n q^{-n^2+n}}{1-q^n}+\frac{q^{-1}}{1-q}-\frac{1}{2}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}+\frac{1}{2}(1-2q^{-1}),$$

which yields Equation (3.15).

Setting $(q,c)\to(q^2,-q^3)$ in Equation (3.14) yields

$$G_n(-q^3,q^2)=\frac{1+q-q^2}{q^2-1}+\sum_{j=2}^{n}\frac{(-1)^j(1+q^{-1})(1-q^{2j-1})}{(1-q^{2j-2})(1-q^{2j})}q^{-2j^2+3j}$$
$$=\frac{1+q-q^2}{q^2-1}+\sum_{j=2}^{n}\frac{(1-q^{2j-2})+q^{-1}(1-q^{2j})}{(1-q^{2j-2})(1-q^{2j})}(-1)^j q^{-2j^2+3j}$$
$$=\frac{1+q-q^2}{q^2-1}+\sum_{j=2}^{n}\frac{(-1)^j q^{-2j^2+3j}}{1-q^{2j}}+\sum_{j=2}^{n}\frac{(-1)^j q^{-2j^2+3j-1}}{1-q^{2j-2}}$$
$$=\frac{1+q-q^2}{q^2-1}+\sum_{j=2}^{n}\frac{(-1)^j q^{-2j^2+3j}}{1-q^{2j}}-\sum_{j=1}^{n-1}\frac{(-1)^j q^{-2j^2-j}}{1-q^{2j}}$$
$$=\frac{1+q-q^2}{q^2-1}+(-1)^n\frac{q^{-2n^2+3n}}{1-q^{2n}}-\sum_{j=2}^{n-1}(-1)^j q^{-2j^2-j}(1+q^{2j})+\frac{q^{-3}}{1-q^2}$$
$$=q^{-1}+q^{-3}-1+(-1)^n\frac{q^{-2n^2+3n}}{1-q^{2n}}-\sum_{j=2}^{n-1}(-1)^j q^{-2j^2-j}-\sum_{j=1-n}^{-2}(-1)^j q^{-2j^2-j}$$
$$=q^{-1}+q^{-3}-1+(-1)^n\frac{q^{-2n^2+3n}}{1-q^{2n}}-\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2-j}+(1-q^{-1}-q^{-3}),$$

which implies Equation (3.16).

Taking $c=q^{3/2}$ and $d=-q^{3/2}$ in Equation (3.8), we arrive at

$$F_n(q^{3/2},-q^{3/2},q)=\frac{-1}{1-q^{-1}}+\sum_{j=2}^{n}\frac{(1-q^{-1})q^{\frac{-j^2+3j}{2}}}{(1-q^{j-1})(1-q^j)}$$
$$=\frac{-1}{1-q^{-1}}+\sum_{j=2}^{n}\frac{q^{\frac{-j^2+3j}{2}}}{1-q^j}-\sum_{j=2}^{n}\frac{q^{\frac{-j^2+3j-2}{2}}}{1-q^{j-1}}$$
$$=\frac{-1}{1-q^{-1}}+\sum_{j=2}^{n}\frac{q^{\frac{-j^2+3j}{2}}}{1-q^j}-\sum_{j=1}^{n-1}\frac{q^{\frac{-j^2+j}{2}}}{1-q^j}$$

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$$=\frac{-1}{1-q^{-1}}+\frac{q^{\frac{-n^2+3n}{2}}}{1-q^n}+\sum_{j=2}^{n-1}\frac{q^{\frac{-j^2+3j}{2}}}{1-q^j}-\frac{1}{1-q}-\sum_{j=2}^{n-1}\frac{q^{\frac{-j^2+j}{2}}}{1-q^j}$$

$$=\frac{q^{\frac{-n^2+3n}{2}}}{1-q^n}-\frac{1}{2}\sum_{j=1}^{n-1}q^{\frac{-j^2+j}{2}}-\frac{1}{2}\sum_{j=0}^{n-2}q^{\frac{-j^2-j}{2}}$$

$$=\frac{q^{\frac{-n^2+3n}{2}}}{1-q^n}-\frac{1}{2}\sum_{j=1-n}^{-1}q^{\frac{-j^2-j}{2}}-\frac{1}{2}\sum_{j=0}^{n-1}q^{\frac{-j^2-j}{2}}+\frac{1}{2}q^{\frac{-n^2+n}{2}},$$

which yields Equation (3.17) after simplification.

At last, setting $(q,c,d)\to(q^2,-q^2,-q^3)$ in Equation (3.8), we obtain

$$F_n(-q^2,-q^3,q^2)=\frac{q^2-2q-1}{2(1-q^2)}+\sum_{j=2}^{n}\frac{(1+q^{-1})(-1)^j(1-q^{2j-1})}{(1-q^{2j-2})(1-q^{2j})}q^{-j^2+2j}$$

$$=\frac{q^2-2q-1}{2(1-q^2)}+\sum_{j=2}^{n}\frac{(1-q^{2j-2})+q^{-1}(1-q^{2j})}{(1-q^{2j-2})(1-q^{2j})}(-1)^jq^{-j^2+2j}$$

$$=\frac{q^2-2q-1}{2(1-q^2)}+\sum_{j=2}^{n}\frac{(-1)^jq^{-j^2+2j}}{1-q^{2j}}+\sum_{j=2}^{n}\frac{(-1)^jq^{-j^2+2j-1}}{1-q^{2j-2}}$$

$$=\frac{q^2-2q-1}{2(1-q^2)}+\sum_{j=2}^{n}\frac{(-1)^jq^{-j^2+2j}}{1-q^{2j}}-\sum_{j=1}^{n-1}\frac{(-1)^jq^{-j^2}}{1-q^{2j}}$$

$$=\frac{q^2-2q-1}{2(1-q^2)}+\frac{(-1)^nq^{-n^2+2n}}{1-q^{2n}}+\frac{q^{-1}}{1-q^2}$$
$$+\sum_{j=2}^{n-1}(-1)^j\left(\frac{q^{-j^2+2j}}{1-q^{2j}}-\frac{q^{-j^2}}{1-q^{2j}}\right)$$

$$=\frac{q^2-2q-1+2q^{-1}}{2(1-q^2)}+\frac{(-1)^nq^{-n^2+2n}}{1-q^{2n}}-\sum_{j=2}^{n-1}(-1)^jq^{-j^2}$$

$$=\frac{q^2-2q-1+2q^{-1}}{2(1-q^2)}+\frac{(-1)^nq^{-n^2+2n}}{1-q^{2n}}$$
$$-\frac{1}{2}\sum_{j=1-n}^{n-1}(-1)^jq^{-j^2}+\frac{1-2q^{-1}}{2},$$

which implies Equation (3.18). This completes the proof. □

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Lemma 3.5. *Suppose that $2\alpha,\ 2\beta,\ 2\lambda,\ 2\mu$ be integers with $\alpha > \lambda > 0$. Then,*

$$\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}-\sum_{n=1}^{\infty}\sum_{j=1-n}^{n-1}(-1)^j q^{\alpha n^2-\beta n-\lambda j^2-\mu j}$$
$$=f_{t_1,t_2,t_1}(q^{\alpha+\beta-\lambda-\mu},q^{\alpha+\beta-\lambda+\mu},q^k)+q^{\alpha+\beta}f_{t_1,t_2,t_1}(q^{3\alpha+\beta-\lambda-\mu},q^{3\alpha+\beta-\lambda+\mu},q^k) \quad (3.19)$$

and

$$\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{\alpha n^2+\beta n-\lambda j^2-\mu j}-\sum_{n=1}^{\infty}\sum_{j=1-n}^{n-1}(-1)^n q^{\alpha n^2-\beta n-\lambda j^2-\mu j}$$
$$=f_{t_1,t_2,t_1}(q^{\alpha+\beta-\lambda-\mu},q^{\alpha+\beta-\lambda+\mu},q^k)-q^{\alpha+\beta}f_{t_1,t_2,t_1}(q^{3\alpha+\beta-\lambda-\mu},q^{3\alpha+\beta-\lambda+\mu},q^k), \quad (3.20)$$

where

$$k=\gcd(2\alpha-2\lambda,2\alpha+2\lambda),\qquad t_1=\frac{2(\alpha-\lambda)}{k},\qquad t_2=\frac{2(\alpha+\lambda)}{k}.$$

Proof. Here we only prove Equation (3.19). One can use the same method to prove Equation (3.20), so we omit the details. It is easy to check that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}-\sum_{n=1}^{\infty}\sum_{j=1-n}^{n-1}(-1)^j q^{\alpha n^2-\beta n-\lambda j^2-\mu j}\\ &=\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-1-n}(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}\\ &=\left(\sum_{\substack{n+j\geq 0,\\ n-j\geq 0}}-\sum_{\substack{n+j<0,\\ n-j<0}}\right)(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}. \qquad (3.21)\end{aligned}$$

Setting $n=\frac{r+s}{2}$ and $j=\frac{r-s}{2}$, we have

$$\begin{aligned}&\left(\sum_{\substack{n+j\geq 0,\\ n-j\geq 0}}-\sum_{\substack{n+j<0,\\ n-j<0}}\right)(-1)^j q^{\alpha n^2+\beta n-\lambda j^2-\mu j}\\ &=\left(\sum_{\substack{r,s\geq 0,\\ r\equiv s\ (\mathrm{mod}\ 2)}}-\sum_{\substack{r,s<0,\\ r\equiv s\ (\mathrm{mod}\ 2)}}\right)(-1)^{\frac{r-s}{2}}q^{\frac{(\alpha-\lambda)r^2+2(\alpha+\lambda)rs+(\alpha-\lambda)s^2+2(\beta-\mu)r+2(\beta+\mu)s}{4}}\\ &=\sum_{\mathrm{sg}(u)=\mathrm{sg}(v)}\mathrm{sg}(u)(-1)^{u+v}q^{(\alpha-\lambda)u^2+2(\alpha+\lambda)uv+(\alpha-\lambda)v^2+(\beta-\mu)u+(\beta+\mu)v}\end{aligned}$$

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$$
\begin{aligned}
&+ q^{\alpha+\beta} \sum_{\mathrm{sg}(u)=\mathrm{sg}(v)} \mathrm{sg}(u)(-1)^{u+v} q^{(\alpha-\lambda)u^2+2(\alpha+\lambda)uv+(\alpha-\lambda)v^2+(2\alpha+\beta-\mu)u+(2\alpha+\beta+\mu)v} \\
=& f_{t_1,t_2,t_1}(q^{\alpha+\beta-\lambda-\mu}, q^{\alpha+\beta-\lambda+\mu}, q^k) + q^{\alpha+\beta} f_{t_1,t_2,t_1}(q^{3\alpha+\beta-\lambda-\mu}, q^{3\alpha+\beta-\lambda+\mu}, q^k), \quad (3.22)
\end{aligned}
$$

which yields Equation (3.19) after combining Equation (3.21). This completes the proof. □

4. Proofs of the main results

The aim of this section is to present proofs of the main results of this paper.

4.1. Proof of Equation (2.1)

Taking $d \to \infty$ in Equation (3.7) yields

$$
\begin{aligned}
&\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n}{(q, c/q; q)_n} \left(\frac{abc}{q^2}\right)^n \\
=& 1 + \sum_{n=1}^{\infty} (1-q^{2n}) q^{n^2-2n} (ab)^n \frac{(q/a, q/b; q)_n}{(a, b; q)_n} G_n(c, q), \qquad (4.1)
\end{aligned}
$$

where $G_n(c, q)$ is defined by Equation (3.14). Setting $(b, c) \to (0, -q)$ in Equation (4.1), we get

$$
\begin{aligned}
&\frac{(q; q)_\infty}{(a; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a; q)_n}{(q, -1; q)_n} a^n q^{n(n-1)/2} \\
=& 1 + \sum_{n=1}^{\infty} (1-q^{2n}) q^{3(n^2-n)/2} (-a)^n \frac{(q/a; q)_n}{(a; q)_n} G_n(-q, q). \qquad (4.2)
\end{aligned}
$$

Taking $(q, a) \to (q^2, -q)$ in Equation (4.2) and using Equation (3.15), we arrive at

$$
\begin{aligned}
&\frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(-1, q^2; q^2)_n} (-1)^n q^{n^2} \\
=& 1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{3n^2-2n} \left(\frac{(-1)^n q^{-2n^2+2n}}{1-q^{2n}} - \frac{1}{2} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2} \right) \\
=& 1 + \sum_{n=1}^{\infty} (1+q^{2n})(-1)^n q^{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} (1-q^{4n}) q^{3n^2-2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2} \\
=& 1 + \sum_{n=1}^{\infty} (1+q^{2n})(-1)^n q^{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} q^{3n^2-2n} \left(\sum_{j=-n}^{n} (-1)^j q^{-2j^2} - 2(-1)^n q^{-2n^2} \right)
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{2}\sum_{n=1}^{\infty} q^{3n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2}\\
=&\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n}-\frac{1}{2}\sum_{n=0}^{\infty} q^{3n^2-2n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2}\\
&+\frac{1}{2}\sum_{n=1}^{\infty} q^{3n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2}. \qquad (4.3)
\end{aligned}
$$

We recall Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} z^n=(z,q/z,q;q)_\infty. \qquad (4.4)$$

By Equation (4.4),

$$\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n}=\left(\frac{1}{2}-q^{-1}\right)\frac{J_1^2}{J_2}. \qquad (4.5)$$

In view of Equation (3.19),

$$
\begin{aligned}
&\sum_{n=0}^{\infty} q^{3n^2-2n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2}-\sum_{n=1}^{\infty} q^{3n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2}\\
&\qquad=f_{1,5,1}(1/q,1/q,q^2)+qf_{1,5,1}(q^5,q^5,q^2). \qquad (4.6)
\end{aligned}
$$

It follows from Equations (4.3), (4.5) and (4.6) that

$$M_1(q)=\left(1-\frac{2}{q}\right)\frac{J_1^2}{J_2}-f_{1,5,1}(1/q,1/q,q^2)-qf_{1,5,1}(q^5,q^5,q^2). \qquad (4.7)$$

Setting $(a,b,c,q,x,y,k,l)\to(1,5,1,q^2,q,q,1,1)$ in Equation (3.4) yields

$$
\begin{aligned}
2\frac{J_1^2}{J_2}=&f_{1,5,1}(q,q,q^2)-q^{12}f_{1,5,1}(q^{13},q^{13},q^2)\\
=&2f_{1,5,1}(q,q,q^2). \qquad \text{(by Equation (3.5))} \qquad (4.8)
\end{aligned}
$$

Taking $(a,b,c,q,x,y,k,l)\to(1,5,1,q^2,1/q,1/q,1,1)$ in Equation (3.4) yields

$$
\begin{aligned}
\frac{2}{q}\frac{J_1^2}{J_2}=&-f_{1,5,1}(1/q,1/q,q^2)+q^8f_{1,5,1}(q^{11},q^{11},q^2)\\
=&-f_{1,5,1}(1/q,1/q,q^2)-f_{1,5,1}(q^3,q^3,q^2). \qquad \text{(by Equation (3.5))}
\end{aligned}
$$

from which with Equations (4.7) and (4.8), Equation (2.1) follows.

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4.2. Proof of Equation (2.2)

Taking $a \to 0$ in Equation (4.2) and utilizing Equation (3.15), we get

$$\begin{aligned}
J_1 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-1,q;q)_n} =& 1 + \sum_{n=1}^{\infty} (1-q^{2n}) q^{2n^2-n} G_n(-q,q) \\
=& 1 + \sum_{n=1}^{\infty} (1-q^{2n}) q^{2n^2-n} \left(\frac{(-1)^n q^{-n^2+n}}{1-q^n} - \frac{1}{2} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \right) \\
=& 1 + \sum_{n=1}^{\infty} (1+q^n)(-1)^n q^{n^2} \\
& - \frac{1}{2} \sum_{n=1}^{\infty} q^{2n^2-n} \left(\sum_{j=-n}^{n} (-1)^j q^{-j^2} - 2(-1)^n q^{-n^2} \right) \\
& + \frac{1}{2} \sum_{n=1}^{\infty} q^{2n^2+n} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \\
=& \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n q^{n^2} + \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} + \sum_{n=1}^{\infty} (-1)^n q^{n^2-n} \\
& - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{2n^2-n-j^2} + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{j=1-n}^{n-1} (-1)^j q^{2n^2+n-j^2}. \quad (4.9)
\end{aligned}$$

Thanks to Equation (4.4),

$$\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{J_1^2}{2J_2}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} + \sum_{n=1}^{\infty} (-1)^n q^{n^2-n} = 0. \tag{4.11}$$

Moreover, by Equation (3.19),

$$\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{2n^2-n-j^2} - \sum_{n=1}^{\infty} \sum_{j=1-n}^{n-1} (-1)^j q^{2n^2+n-j^2} = f_{1,3,1}(1,1,q^2) + q f_{1,3,1}(q^4,q^4,q^2). \tag{4.12}$$

Combining Equations (4.9)–(4.12), we arrive at

$$M_2(q) = \frac{J_1^2}{J_2} - f_{1,3,1}(1,1,q^2) - q f_{1,3,1}(q^4,q^4,q^2). \tag{4.13}$$

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Putting $(a, b, c, q, x, y, k, l) \to (1, 3, 1, q^2, q, q, 1, 1)$ in Equation (3.4), we get

$$2\frac{J_1^2}{J_2} = f_{1,3,1}(q, q, q^2) - q^8 f_{1,3,1}(q^9, q^9, q^2)$$
$$= 2f_{1,3,1}(q, q, q^2), \qquad \text{(by Equation (3.5))}$$

from which with Equation (4.13), Equation (2.2) follows.

4.3. Proof of Equation (2.3)

Putting $(q, c) \to (q^2, -q^3)$ in Equation (4.1), we have

$$\frac{(q^2, ab/q^2; q^2)_\infty}{(a, b; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^2/a, q^2/b; q^2)_n}{(q^2, -q; q^2)_n} \left(-\frac{ab}{q}\right)^n$$
$$= 1 + \sum_{n=1}^{\infty} (1 - q^{4n}) q^{2n^2-4n} (ab)^n \frac{(q^2/a, q^2/b; q^2)_n}{(a, b; q^2)_n} G_n(-q^3, q^2). \qquad (4.14)$$

Setting $b \to 0$ in Equation (4.14) yields

$$\frac{(q^2; q^2)_\infty}{(a; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^2/a; q^2)_n}{(q^2, -q; q^2)_n} a^n q^{n^2} = 1 + \sum_{n=1}^{\infty} (1 - q^{4n}) q^{3n^2-3n} (-a)^n \frac{(q^2/a; q^2)_n}{(a; q^2)_n} G_n(-q^3, q^2). \qquad (4.15)$$

Setting $a = -q^2$ in Equation (4.15) and employing Equation (3.16), we deduce that

$$\frac{(q^2; q^2)_\infty}{(-q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1; q^2)_n}{(-q, q^2; q^2)_n} (-1)^n q^{n^2+2n}$$
$$= 1 + \sum_{n=1}^{\infty} (1 - q^{4n}) q^{3n^2-n} \frac{(-1; q^2)_n}{(-q^2; q^2)_n} \left(\frac{(-1)^n q^{-2n^2+3n}}{1 - q^{2n}} - \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \right)$$
$$= 1 + 2\sum_{n=1}^{\infty} (1 - q^{2n}) q^{3n^2-n} \left(\frac{(-1)^n q^{-2n^2+3n}}{1 - q^{2n}} - \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \right)$$
$$= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2+2n} - 2\sum_{n=1}^{\infty} (1 - q^{2n}) q^{3n^2-n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j}$$
$$= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2+2n} + 2\sum_{n=1}^{\infty} q^{3n^2+n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j}$$
$$- 2\sum_{n=1}^{\infty} q^{3n^2-n} \left(\sum_{j=-n}^{n} (-1)^j q^{-2j^2-j} - (-1)^n q^{-2n^2-n} - (-1)^n q^{-2n^2+n} \right)$$

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$$\begin{aligned}=&2\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\\ &-2\sum_{n=0}^{\infty}q^{3n^2-n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2-j}+2\sum_{n=1}^{\infty}q^{3n^2+n}\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2-j}.\end{aligned} \tag{4.16}$$

Thanks to Equation (4.4),

$$2\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=(1-2/q)\frac{J_1^2}{J_2}. \tag{4.17}$$

In light of Equation (3.19),

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{3n^2-n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2-j}-\sum_{n=1}^{\infty}q^{3n^2+n}\sum_{j=1-n}^{n-1}(-1)^j q^{-2j^2-j}\\ =&f_{1,5,1}(1/q,q,q^2)+q^2 f_{1,5,1}(q^5,q^7,q^2),\end{aligned}$$

from which with Equations (4.16) and (4.17), we obtain

$$M_3(q)=\left(1-\frac{2}{q}\right)\frac{J_1^2}{J_2}-2f_{1,5,1}(1/q,q,q^2)-2q^2 f_{1,5,1}(q^5,q^7,q^2). \tag{4.18}$$

Taking $(a,b,c,x,y,q)\to(1,5,1,1/q,q,q^2)$ in Equation (3.6), we obtain

$$f_{1,5,1}(1/q,q,q^2)=-qf_{1,5,1}(q^9,q^3,q^2)-\frac{J_1^2}{qJ_2}. \tag{4.19}$$

Combining Equations (4.18) and (4.19) yields

$$M_3(q)=\frac{J_1^2}{J_2}+2qf_{1,5,1}(q^9,q^3,q^2)-2q^2 f_{1,5,1}(q^5,q^7,q^2). \tag{4.20}$$

Setting $(a,b,c,q,x,y,k,l)\to(1,5,1,q^2,q,q,1,1)$ in Equation (3.4), we have

$$\begin{aligned}2\frac{J_1^2}{J_2}=&f_{1,5,1}(q,q,q^2)-q^{12}f_{1,5,1}(q^{13},q^{13},q^2)\\ =&2f_{1,5,1}(q,q,q^2),\qquad \text{(by Equation (3.5))}\end{aligned}$$

from which with Equation (4.20), Equation (2.3) follows.

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4.4. Proof of Equation (2.4)

Taking $a \to 0$ in Equation (4.15) and using Equation (3.16) yields

$$\begin{aligned}
J_2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+n}}{(q^2,-q;q^2)_n} =& 1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{4n^2-2n} G_n(-q^3,q^2) \\
=& 1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{4n^2-2n} \left(\frac{(-1)^n q^{-2n^2+3n}}{1-q^{2n}} - \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \right) \\
=& 1 + \sum_{n=1}^{\infty} (1+q^{2n})(-1)^n q^{2n^2+n} \\
& - \sum_{n=1}^{\infty} (1-q^{4n}) q^{4n^2-2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \\
=& 1 + \sum_{n=1}^{\infty} (1+q^{2n})(-1)^n q^{2n^2+n} + \sum_{n=1}^{\infty} q^{4n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \\
& - \sum_{n=1}^{\infty} q^{4n^2-2n} \left(\sum_{j=-n}^{n} (-1)^j q^{-2j^2-j} - (-1)^n q^{-2n^2-n} \right. \\
& \left. - (-1)^n q^{-2n^2+n} \right) \\
=& \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+3n} \\
& - \sum_{n=0}^{\infty} q^{4n^2-2n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2-j} \\
& + \sum_{n=1}^{\infty} q^{4n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j}.
\end{aligned} \tag{4.21}$$

In view of Equation (4.4),

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+3n} = (1-1/q) \frac{J_1 J_4}{J_2}. \tag{4.22}$$

By Equation (3.19)

$$\begin{aligned}
\sum_{n=0}^{\infty} q^{4n^2-2n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2-j} - \sum_{n=1}^{\infty} q^{4n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-2j^2-j} \\
= f_{1,3,1}(1/q, q, q^4) + q^2 f_{1,3,1}(q^7, q^9, q^4).
\end{aligned} \tag{4.23}$$

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It follows from Equations (4.21) to (4.23) that

$$M_4(q) = \left(1 - \frac{1}{q}\right)\frac{J_1J_4}{J_2} - f_{1,3,1}(1/q, q, q^4) - q^2 f_{1,3,1}(q^7, q^9, q^4). \tag{4.24}$$

Taking $(a, b, c, x, y, q, k, l) \to (1, 3, 1, 1/q, q, q^4, 1, 1)$ in Equation (3.4), we obtain

$$f_{1,3,1}(1/q, q, q^4) = q^{12} f_{1,3,1}(q^{15}, q^{17}, q^4) + \left(1 - \frac{1}{q}\right)\frac{J_1J_4}{J_2}. \tag{4.25}$$

Substituting Equation (4.25) into Equation (4.24) yields

$$M_4(q) = -q^{12} f_{1,3,1}(q^{15}, q^{17}, q^4) - q^2 f_{1,3,1}(q^7, q^9, q^4). \tag{4.26}$$

Equation (2.4) follows from Equations (3.5) and (4.26).

4.5. Proof of Equation (2.5)

Setting $c = q^{3/2}$ and $d = -q^{3/2}$ in Equation (3.7) yields

$$\begin{aligned}&\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty}\sum_{n=0}^{\infty}\frac{(q/a, q/b, -q; q)_n}{(q; q)_n(q; q^2)_n}(ab/q)^n\\ =&1 + \sum_{n=1}^{\infty}(1 - q^{2n})q^{n^2-2n}(ab)^n\frac{(q/a, q/b; q)_n}{(a, b; q)_n}F(q^{3/2}, -q^{3/2}, q).\end{aligned} \tag{4.27}$$

Putting $(q, a, b) \to (q^2, q, -q^2)$ in Equation (4.27) and utilizing Equation (3.17), we obtain

$$\begin{aligned}&\frac{(-q, q^2; q^2)_\infty}{(q, -q^2; q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1, -q^2; q^2)_n}{(-q, q^2; q^2)}(-q)^n\\ =&1 + \sum_{n=1}^{\infty}(1 - q^{2n})(-1)^n q^{2n^2-n}\left(\frac{1 + q^{2n}}{1 - q^{2n}}q^{-n^2+n} - \sum_{j=1-n}^{n-1} q^{-j^2-j}\right)\\ =&1 + \sum_{n=1}^{\infty}(-1)^n q^{n^2} + \sum_{n=1}^{\infty}(-1)^n q^{n^2+2n} + \sum_{n=1}^{\infty}(-1)^n q^{2n^2+n}\sum_{j=1-n}^{n-1} q^{-j^2-j}\\ &- \sum_{n=1}^{\infty}(-1)^n q^{2n^2-n}\left(\sum_{j=-n}^{n} q^{-j^2-j} - q^{-n^2-n} - q^{-n^2+n}\right)\\ =&\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} + \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n} - \sum_{n=0}^{\infty}(-1)^n q^{2n^2-n}\sum_{j=-n}^{n} q^{-j^2-j}\\ &+ \sum_{n=1}^{\infty}(-1)^n q^{2n^2+n}\sum_{j=1-n}^{n-1} q^{-j^2-j}.\end{aligned} \tag{4.28}$$

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Based on Equation (4.4),

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+2n}=(1-1/q)\frac{J_1^2}{J_2}. \tag{4.29}$$

In light of Equation (3.20),

$$\sum_{n=0}^{\infty}(-1)^n q^{2n^2-n}\sum_{j=-n}^{n}q^{-j^2-j}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2+n}\sum_{j=1-n}^{n-1}q^{-j^2-j}$$
$$=f_{1,3,1}(1/q,q,q^2)-qf_{1,3,1}(q^3,q^5,q^2). \tag{4.30}$$

It follows from Equations (4.28) to (4.30) that

$$M_5(q)=\left(1-\frac{1}{q}\right)\frac{J_1^2}{J_2}-f_{1,3,1}(1/q,q,q^2)+qf_{1,3,1}(q^3,q^5,q^2). \tag{4.31}$$

Putting $(a,b,c,x,y,q,k,l)\to(1,3,1,1/q,q,q^2,1,1)$ in Equation (3.4), we arrive at

$$f_{1,3,1}(1/q,q,q^2)=q^6f_{1,3,1}(q^7,q^9,q^2)+\left(1-\frac{1}{q}\right)\frac{J_1^2}{J_2}. \tag{4.32}$$

Substituting Equation (4.32) into Equation (4.31) yields

$$M_5(q)=-q^6f_{1,3,1}(q^7,q^9,q^2)+qf_{1,3,1}(q^3,q^5,q^2),$$

from which with Equation (2.2), Equation (2.5) follows.

4.6. Proof of Equation (2.6)

Setting $b\to 0$ in Equation (4.27) and employing Equation (3.17), we deduce that

$$\frac{(q;q)_\infty}{(a;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/a,-q;q)_n}{(q;q)_n(q;q^2)_n}(-a)^n q^{n(n-1)/2}$$
$$=1+\frac{1}{2}\sum_{n=1}^{\infty}(1-q^{2n})(-a)^n q^{3n(n-1)/2}\frac{(q/a;q)_n}{(a;q)_n}\left(\frac{1+q^n}{1-q^n}q^{-n(n-1)/2}-\sum_{j=1-n}^{n-1}q^{-j(j+1)/2}\right). \tag{4.33}$$

Taking $(q,a)\to(q^2,q)$ in Equation (4.33) yields

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n}{(q^2,-q;q^2)_n}(-1)^n q^{n^2}$$

$$
\begin{aligned}
&=1+\frac{1}{2}\sum_{n=1}^{\infty}(1-q^{4n})(-1)^n q^{3n^2-2n}\left(\frac{1+q^{2n}}{1-q^{2n}}q^{-n^2+n}-\sum_{j=1-n}^{n-1}q^{-j^2-j}\right)\\
&=1+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n q^{2n^2-n}(1+2q^{2n}+q^{4n})+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n q^{3n^2+2n}\sum_{j=1-n}^{n-1}q^{-j^2-j}\\
&\quad-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n q^{3n^2-2n}\left(\sum_{j=-n}^{n}q^{-j^2-j}-q^{-n^2-n}-q^{-n^2+n}\right)\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n}+\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n}\\
&\quad-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2-2n}\sum_{j=-n}^{n}q^{-j^2-j}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n q^{3n^2+2n}\sum_{j=1-n}^{n-1}q^{-j^2-j}. \qquad (4.34)
\end{aligned}
$$

By Equation (4.4),

$$
\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n}+\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n}=\left(1-\frac{1}{2q}\right)\frac{J_1J_4}{J_2}. \qquad (4.35)
$$

Moreover, in view of Equation (3.20),

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n q^{3n^2-2n}\sum_{j=-n}^{n}q^{-j^2-j}-\sum_{n=1}^{\infty}(-1)^n q^{3n^2+2n}\sum_{j=1-n}^{n-1}q^{-j^2-j}&\\
=f_{1,2,1}(1/q,q,q^4)-qf_{1,2,1}(q^5,q^7,q^4).& \qquad (4.36)
\end{aligned}
$$

Combining Equations (4.34)–(4.36) yields

$$
M_6(q)=\left(1-\frac{1}{2q}\right)\frac{J_1J_4}{J_2}-\frac{1}{2}f_{1,2,1}(1/q,q,q^4)+\frac{q}{2}f_{1,2,1}(q^5,q^7,q^4). \qquad (4.37)
$$

Putting $(a,b,c,q,x,y,k,l)\to(1,2,1,q^4,q,q,1,1)$ in Equation (3.4), we get

$$
\begin{aligned}
f_{1,2,1}(q,q,q^4)=&q^{10}f_{1,2,1}(q^{13},q^{13},q^4)+2\frac{J_1J_4}{J_2}\\
=&-f_{1,2,1}(q^3,q^3,q^4)+2\frac{J_1J_4}{J_2}, \qquad \text{(by Equation (3.5))}, \qquad (4.38)
\end{aligned}
$$

which yields

$$
\frac{J_1J_4}{J_2}=\frac{f_{1,2,1}(q,q,q^4)}{2}+\frac{f_{1,2,1}(q^3,q^3,q^4)}{2}. \qquad (4.39)
$$

Taking $(a,b,c,q,x,y) \to (1,2,1,q^4,1/q,q)$ in Equation (3.6), we arrive at

$$\begin{aligned} -\frac{J_1 J_4}{qJ_2} &= f_{1,2,1}(1/q,q,q^4) + qf_{1,2,1}(q^7,q^5,q^4) \\ &= f_{1,2,1}(1/q,q,q^4) + qf_{1,2,1}(q^5,q^7,q^4), \qquad \text{(by Equation (1.1))} \end{aligned} \tag{4.40}$$

from which with Equations (4.37) and (4.39), Equation (2.6) follows.

4.7. Proof of Equation (2.7)

Taking $(q,b,c,d) \to (q^2,0,-q^2,-q^3)$ in Equation (3.7) yields

$$\begin{aligned} &\frac{(q^2;q^2)_\infty}{(a;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^2/a,q;q^2)_n}{(-1,-q,q^2;q^2)_n} (-a)^n q^{n^2-n} \\ =&1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{3n^2-3n} (-a)^n \frac{(q^2/a;q^2)_n}{(a;q^2)_n} F_n(-q^2,-q^3,q^2). \end{aligned} \tag{4.41}$$

Taking $a=-q$ in Equation (4.41) and using Equation (3.18), we get

$$\begin{aligned} &\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n^2}}{(-1,q^2;q^2)_n} \\ =&1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{3n^2-2n} \left(\frac{(-1)^n q^{-n^2+2n}}{1-q^{2n}} - \frac{1}{2} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \right) \\ =&1 + \sum_{n=1}^{\infty} (-1)^n q^{2n^2} + \sum_{n=1}^{\infty} (-1)^n q^{2n^2+2n} + \frac{1}{2} \sum_{n=1}^{\infty} q^{3n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \\ &- \frac{1}{2} \sum_{n=1}^{\infty} q^{3n^2-2n} \left(\sum_{j=-n}^{n} (-1)^j q^{-j^2} - 2(-1)^n q^{-n^2} \right) \\ =&1 + \sum_{n=1}^{\infty} (-1)^n q^{2n^2+2n} + \sum_{n=1}^{\infty} (-1)^n q^{2n^2-2n} + \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} \\ &- \frac{1}{2} \sum_{n=0}^{\infty} q^{3n^2-2n} \sum_{j=-n}^{n} (-1)^j q^{-j^2} + \frac{1}{2} \sum_{n=1}^{\infty} q^{3n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2}. \end{aligned} \tag{4.42}$$

By Equation (4.4)

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{2n^2+2n} + \sum_{n=1}^{\infty} (-1)^n q^{2n^2-2n} = 0, \tag{4.43}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = \frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}. \tag{4.44}$$

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In addition, thanks to Equation (3.19),

$$\sum_{n=0}^{\infty} q^{3n^2-2n} \sum_{j=-n}^{n} (-1)^j q^{-j^2} - \sum_{n=1}^{\infty} q^{3n^2+2n} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2}$$
$$= f_{1,2,1}(1,1,q^4) + qf_{1,2,1}(q^6,q^6,q^4),$$

which yields

$$M_7(q) = \frac{J_2^2}{J_4} - f_{1,2,1}(1,1,q^4) - qf_{1,2,1}(q^6,q^6,q^4) \tag{4.45}$$

after combining Equations (4.42)–(4.44).

Taking $(a,b,c,q,x,y) \to (1,2,1,q^2,q,q)$ in Equation (3.6) yields

$$\frac{J_1^2}{J_2} = f_{1,2,1}(q,q,q^2) + qf_{1,2,1}(q^5,q^3,q^2). \tag{4.46}$$

In view of Equation (3.5),

$$\begin{aligned} f_{1,2,1}(q^5,q^3,q^2) &= - f_{1,2,1}(q^3,q^5,q^2) \\ &= - f_{1,2,1}(q^5,q^3,q^2), \qquad \text{(by Equation (1.1))}, \end{aligned}$$

which yields

$$f_{1,2,1}(q^5,q^3,q^2) = 0. \tag{4.47}$$

Equation (2.7) follows from Equations (4.45) to (4.47).

4.8. Proof of Equation (2.8)

Putting $a = -q^2$ in Equation (4.41) and employing Equation (3.18), we see that

$$\begin{aligned} &\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n^2+n}}{(-q,q^2;q^2)_n} \\ =&1 + \sum_{n=1}^{\infty} (1-q^{4n}) q^{3n^2-n} \frac{(-1;q^2)_n}{(-q^2;q^2)_n} F_n(-q^2,-q^3,q^2) \\ =&1 + 2\sum_{n=1}^{\infty} (1-q^{2n}) q^{3n^2-n} \left(\frac{(-1)^n q^{-n^2+2n}}{1-q^{2n}} - \frac{1}{2} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \right) \\ =&1 + 2\sum_{n=1}^{\infty} (-1)^n q^{2n^2+n} + \sum_{n=1}^{\infty} q^{3n^2+n} \sum_{j=1-n}^{n-1} (-1)^j q^{-j^2} \end{aligned}$$

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$$-\sum_{n=1}^{\infty} q^{3n^2-n}\left(\sum_{j=-n}^{n}(-1)^j q^{-j^2}-2(-1)^n q^{-n^2}\right)$$
$$=2\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n}-\sum_{n=0}^{\infty} q^{3n^2-n}\sum_{j=-n}^{n}(-1)^j q^{-j^2}+\sum_{n=1}^{\infty} q^{3n^2+n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}. \tag{4.48}$$

The following identity follows from Jacobi's triple product identity Equation (4.4),

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}=\frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{4.49}$$

In view of Equation (3.19),

$$\sum_{n=0}^{\infty} q^{3n^2-n}\sum_{j=-n}^{n}(-1)^j q^{-j^2}-\sum_{n=1}^{\infty} q^{3n^2+n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}$$
$$=f_{1,2,1}(q,q,q^4)+q^2 f_{1,2,1}(q^7,q^7,q^4). \tag{4.50}$$

It follows from Equations (4.48) to (4.50) that

$$M_8(q)=2\frac{J_1J_4}{J_2}-f_{1,2,1}(q,q,q^4)-q^2 f_{1,2,1}(q^7,q^7,q^4). \tag{4.51}$$

Taking $(a,b,c,q,x,y,k,l)=(1,2,1,q^4,q,q,1,1)$ in Equation (3.4), we arrive at

$$2\frac{J_1J_4}{J_2}=f_{1,2,1}(q,q,q^4)-q^{10} f_{1,2,1}(q^{13},q^{13},q^4). \tag{4.52}$$

Combining Equations (4.51) and (4.52) yields

$$M_8(q)=-q^{10} f_{1,2,1}(q^{13},q^{13},q^4)-q^2 f_{1,2,1}(q^7,q^7,q^4). \tag{4.53}$$

Equation (2.8) follows from Equations (3.5) and (4.53).

4.9. Proof of Equation (2.9)

Setting $a\to 0$ in Equation (4.41) and utilizing Equation (3.18), we arrive at

$$(q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{(q;q^2)_n}{(-1,-q,q^2;q^2)_n}q^{2n^2}$$
$$=1+\sum_{n=1}^{\infty}(1+q^{2n})(-1)^n q^{3n^2}-\frac{1}{2}\sum_{n=1}^{\infty}(1-q^{4n})q^{4n^2-2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}$$

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$$
\begin{aligned}
&=1+\sum_{n=1}^{\infty}(-1)^n q^{3n^2}+\sum_{n=1}^{\infty}(-1)^n q^{3n^2+2n}+\frac{1}{2}\sum_{n=1}^{\infty} q^{4n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}\\
&\quad -\frac{1}{2}\sum_{n=1}^{\infty} q^{4n^2-2n}\left(\sum_{j=-n}^{n}(-1)^j q^{-j^2}-2(-1)^n q^{-n^2}\right)\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}+\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}\\
&\quad -\frac{1}{2}\sum_{n=0}^{\infty} q^{4n^2-2n}\sum_{j=-n}^{n}(-1)^j q^{-j^2}+\frac{1}{2}\sum_{n=1}^{\infty} q^{4n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}. \qquad (4.54)
\end{aligned}
$$

In light of Equation (4.4)

$$
\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}+\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}=\frac{J_1J_6^2}{J_2J_3}+\frac{J_3^2}{2J_6}. \qquad (4.55)
$$

Moreover, by Equation (3.19),

$$
\begin{aligned}
\sum_{n=0}^{\infty} q^{4n^2-2n}\sum_{j=-n}^{n}(-1)^j q^{-j^2}-\sum_{n=1}^{\infty} q^{4n^2+2n}\sum_{j=1-n}^{n-1}(-1)^j q^{-j^2}&\\
=f_{3,5,3}(q,q,q^2)+q^2 f_{3,5,3}(q^9,q^9,q^2).& \qquad (4.56)
\end{aligned}
$$

Substituting Equations (4.55) and (4.56) into Equation (4.54), we get

$$
M_9(q)=\frac{J_1J_6^2}{J_2J_3}+\frac{J_3^2}{2J_6}-\frac{1}{2}f_{3,5,3}(q,q,q^2)-\frac{1}{2}q^2 f_{3,5,3}(q^9,q^9,q^2). \qquad (4.57)
$$

Setting $(a,b,c,q,x,y,k,l)=(3,5,3,q^2,q^3,q^3,1,1)$ in Equation (3.4), we obtain

$$
\begin{aligned}
2\frac{J_3^2}{J_6}&=f_{3,5,3}(q^3,q^3,q^2)-q^{16}f_{3,5,3}(q^{19},q^{19},q^2)\\
&=2f_{3,5,3}(q^3,q^3,q^2). \qquad \text{(by Equation (3.5))} \qquad (4.58)
\end{aligned}
$$

Setting $(a,b,c,q,x,y,k,l)=(3,5,3,q^2,q,q,1,1)$ in Equation (3.4) yields

$$
\begin{aligned}
2\frac{J_1J_6^2}{J_2J_3}&=f_{3,5,3}(q,q,q^2)-q^{12}f_{3,5,3}(q^{17},q^{17},q^2)\\
&=f_{3,5,3}(q,q,q^2)+f_{3,5,3}(q^5,q^5,q^2). \qquad \text{(by Equation (3.5))} \qquad (4.59)
\end{aligned}
$$

Equation (2.9) follows from Equations (4.57) to (4.59). This completes the proof of Theorem 2.1. □

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5. Proof of Theorem 2.2

In [18], Hickerson and Mortenson proved that for generic $x, y \in \mathbb{C} - \{0\}$,

$$f_{n,n+2,n}(x,y,q) = g_{n,n+2,n}(x,y,q,y^n/x^n,x^n/y^n) - \Theta_{n,2}(x,y,q), \tag{5.1}$$

where n is a positive integer with

$$\begin{aligned} g_{a,b,c}(x,y,q,z_1,z_0) = &\sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a)m \\ &\left(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\right) \\ &+\sum_{t=0}^{a-1}(-x)^t q^{c\binom{t}{2}} j(q^{bt}y;q^a)m \\ &\left(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\right) \end{aligned} \tag{5.2}$$

and

$$\Theta_{n,2}(x,y,q) = \frac{y^{(n+1)/2}J_{2n}^2 J_{4(n+1)}^2 j(y/x;q^{4(n+1)})j(xyq^{n+2};q^{4(n+1)})j(q^{2n}/(x^2y^2);q^{8(n+1)})}{q^{(n^2-3)/2}x^{(n-3)/2}J_{4n}J_{8(n+1)}j(y^n/x^n;q^{4n(n+1)})j(-x^2q^{n+1};q^{4(n+1)})j(-y^2q^{n+1};q^{4(n+1)})}.$$

Setting $(n,p,x,y,q) \to (1,2,1/q,q,q^2)$ in Equation (5.1) yields

$$\begin{aligned} f_{1,3,1}(1/q,q,q^2) = &\frac{J_1^2}{J_2}\left(m(-q^6,q^{16},q^2) - \frac{1}{q}m(-q^{14},q^{16},q^{-2})\right) \\ &- q^2\frac{J_4^3 J_{32} j(q^6;q^{16})j(q^4;q^{32})}{J_8 J_{16}^3}. \end{aligned} \tag{5.3}$$

Taking $(n,p,x,y,q) \to (1,2,q^3,q^5,q^2)$ in Equation (5.1) yields

$$\begin{aligned} f_{1,3,1}(q^3,q^5,q^2) =& j(q^3;q^2)m(-q^6,q^{16},q^{-2}) + j(q^5;q^2)m(-1/q^2,q^{16},q^2) - \Theta_{1,2}(q^3,q^5,q^2) \\ =& j(q^3;q^2)m(-q^6,q^{16},q^{-2}) + j(q^5;q^2)(-q^2 + q^2 m(-q^{14},q^{16},q^2)) \\ &- \Theta_{1,2}(q^3,q^5,q^2) \\ =& \frac{J_1^2}{J_2}\left(-\frac{1}{q}m(-q^6,q^{16},q^{-2}) - \frac{1}{q^2} + \frac{1}{q^2}m(-q^{14},q^{16},q^2)\right) \\ &+ q^{-2}\frac{J_4^3 J_{16}^3 j(q^2;q^{16})j(q^{12};q^{32})}{2J_8^3 J_{32}^3}. \end{aligned} \tag{5.4}$$

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Substituting Equations (5.3) and (5.4) into Equation (4.31), we arrive at

$$M_5(q) = \left(1 - \frac{2}{q}\right)\frac{J_1^2}{J_2} - 2\frac{J_1^2}{J_2}\left(m(-q^6, q^{16}, q^2) - \frac{1}{q}m(-q^{14}, q^{16}, q^2)\right)$$
$$- \frac{J_1^2J_4^2J_{32}^2 j(-q^6; q^{16})}{J_2J_8J_{16}j(q^2; q^{16})^2} + \frac{J_1^2J_4^2J_{16}^5 j(-q^2; q^{16})}{2qJ_2J_8^3J_{32}^2 j(q^2; q^{16})^2}$$
$$+ \frac{J_4^3J_{16}^3}{2qJ_8^3J_{32}^3} j(q^2; q^{16}) j(q^{12}; q^{32}) + q^2\frac{J_4^3J_{32} j(q^6; q^{16}) j(q^4; q^{32})}{J_8J_{16}^3}. \tag{5.5}$$

The following identities can be proved by using the MAPLE package *thetaids* [12]:

$$- \frac{J_1^2J_4^2J_{32}^2 j(-q^6; q^{16})}{J_2J_8J_{16}j(q^2; q^{16})^2} + \frac{J_1^2J_4^2J_{16}^5 j(-q^2; q^{16})}{2qJ_2J_8^3J_{32}^2 j(q^2; q^{16})^2}$$
$$+ \frac{J_4^3J_{16}^3}{2qJ_8^3J_{32}^3} j(q^2; q^{16}) j(q^{12}; q^{32}) + q^2\frac{J_4^3J_{32} j(q^6; q^{16}) j(q^4; q^{32})}{J_8J_{16}^3} = \frac{J_8^4 j(q; q^8)^2}{qJ_{16}^2 j(q^2; q^{16})^2}. \tag{5.6}$$

Equation (2.10) follows from Equations (5.5) and (5.6).

Hickerson and Mortenson [18] also proved that

$$f_{1,2,1}(x, y, q) = g_{n,n+1,n}(x, y, q, y^n/x^n, x^n/y^n), \tag{5.7}$$

where $g_{a,b,c}(x, y, q, z_1, z_0)$ is defined by Equation (5.2). Setting $(x, y, q) \to (1/q, q, q^4)$ in Equation (5.7) yields

$$\begin{aligned} f_{1,2,1}(1/q, q, q^4) =& j(q; q^4)m(q^5, q^{12}, q^2) + j(1/q; q^4)m(q^{11}, q^{12}, q^{-2}) \\ =& \frac{J_1J_4}{J_2}m(q^5, q^{12}, q^2) - \frac{J_1J_4}{q^{12}J_2}m(q^{-11}, q^{12}, q^2) \\ =& \frac{J_1J_4}{J_2}m(q^5, q^{12}, q^2) - \frac{J_1J_4}{qJ_2}(1 - m(q, q^{12}, q^2)) \\ =& \frac{J_1J_4}{J_2}\left(m(q^5, q^{12}, q^2) - \frac{1}{q} + \frac{1}{q}m(q, q^{12}, q^2)\right). \end{aligned} \tag{5.8}$$

Taking $(x, y, q) \to (q^5, q^7, q^4)$ in Equation (5.7), we deduce that

$$f_{1,2,1}(q^5, q^7, q^4) = -\frac{J_1J_4}{q^2J_2}m(q, q^{12}, q^{-2}) - \frac{J_1J_4}{qJ_2}m(q^5, q^{12}, q^{-2}). \tag{5.9}$$

In view of Equation (3.3),

$$m(q, q^{12}, q^2) - m(q, q^{12}, q^{-2}) = q\frac{J_4^3J_6^3}{J_2^2J_3J_{12}^2} \tag{5.10}$$

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and

$$m(q^5, q^{12}, q^2) - m(q^5, q^{12}, q^{-2}) = -\frac{J_4^3 J_6^3}{J_2^2 J_3 J_{12}^2}. \tag{5.11}$$

Substituting Equations (5.8) and (5.9) into Equation (4.37) and using Equations (5.10) and (5.11), we get Equation (2.11).

Hickerson and Mortenson [18] proved that

$$\begin{aligned} f_{1,2,1}(x, y, q) =& j(y; q)m(xq^2/y^2, q^3, -1) + j(x; q)m(yq^2/x^2, q^3, -1) \\ &- \frac{yJ_3J_6 j(-x/y; q)j(xyq^2; q^3)}{2j(-qy^2/x; q^3)j(-qx^2/y; q^3)}. \end{aligned} \tag{5.12}$$

Taking $(x, y, q) \to (1, 1, q^4)$ in Equation (5.12), we arrive at

$$f_{1,2,1}(1, 1, q^4) = -J_4^2. \tag{5.13}$$

Setting $(x, y, q) \to (q^6, q^6, q^4)$ in Equation (5.12), we have

$$f_{1,2,1}(q^6, q^6, q^4) = -\frac{2J_2^2}{q^2 J_4} m(q^2, q^{12}, -1) + \frac{J_2^2 J_8^4 J_{12}^6}{q^2 J_4^4 J_6^2 J_{24}^4}. \tag{5.14}$$

Substituting Equations (5.13) and (5.14) into Equation (4.45), we arrive at Equation (2.12).

Putting $(x, y, q) \to (q, q, q^4)$ in Equation (5.12), we obtain

$$f_{1,2,1}(q, q, q^4) = 2\frac{J_1 J_4}{J_2} m(q^7, q^{12}, -1) - q\frac{J_2 J_8^2 J_{12}^6}{J_4^2 J_6 J_{24}^2 j(-q^5, q^{12})^2}. \tag{5.15}$$

By Equations (3.7) and (3.8),

$$\begin{aligned} m(q^7, q^{12}, -1) &= q^{-7} m(q^{-7}, q^{12}, -1) \\ &= q^{-7}(q^7 - q^7 m(q^5, q^{12}, -1)) \\ &= 1 - m(q^5, q^{12}, -1). \end{aligned} \tag{5.16}$$

Setting $(x, y, q) \to (q^7, q^7, q^4)$ in Equation (5.12) yields

$$f_{1,2,1}(q^7, q^7, q^4) = -\frac{2}{q^3}\frac{J_1 J_4}{J_2} m(q, q^{12}, -1) + \frac{J_2 J_8^2 J_{12}^6}{q^3 J_4^2 J_6 J_{24}^2 j(-q; q^{12})^2}. \tag{5.17}$$

Substituting Equations (5.15) and (5.17) into Equation (4.51) and using Equation (5.16), we deduce that

$$\begin{aligned} M_8(q) = & \frac{2J_1 J_4}{J_2}\left(m(q^5, q^{12}, -1) + \frac{1}{q} m(q, q^{12}, -1)\right) \\ &+ \frac{J_2 J_8^2 J_{12}^6}{q J_4^2 J_6 J_{24}^2}\left(\frac{q^2}{j(-q^5; q^{12})^2} - \frac{1}{j(-q; q^{12})^2}\right). \end{aligned} \tag{5.18}$$

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With the MAPLE package *thetaids* [12], one can prove that

$$\frac{J_2J_8^2J_{12}^6}{qJ_4^2J_6J_{24}^2}\left(\frac{q^2}{j(-q^5;q^{12})^2}-\frac{1}{j(-q;q^{12})^2}\right)=-\frac{J_1^2J_6^2J_8^2}{qJ_2^2J_{24}^2},$$

which yields Equation (2.13) after combining Equation (5.18). This completes the proof.

6. Concluding remarks

As seen in § 2 'Introduction', seeking examples of q-hypergeometric series which are mock theta functions in the modern sense and finding their alternative representations have received a lot of attention in recent years. In this paper, we pose several new mock theta factions and express them in terms of Appell-Lerch sums and Hecke-type double sum $f_{a,b,c}(x, y, q)$. It would be interesting to find new methods to establish new mock theta functions and find their finding their alternative representations.

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