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MORE ON BLURRY HOD

GUNTER FUCHS

ABSTRACT. I continue the study of the blurry HOD hierarchy. The technically most involved result is that the theory ZFC + "$\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+1}$ is the least leap" is equiconsistent with the theory ZFC + "there is a measurable cardinal."

CONTENTS

1. INTRODUCTION

In [6], I began in the study of blurry ordinal definability, and the present article is the continuation of that study. The basic idea is, given a cardinal $\kappa \geq 2$, to weaken the notion of ordinal definability of a set a, which can be formulated by saying that a is the unique set x satisfying $\varphi(x, \alpha)$, for some formula φ and ordinal α, to the notion of $<\kappa$-blurry ordinal definability, which says that a is one of fewer than κ many sets x which satisfy $\varphi(x, \alpha)$. The seeming metamathematical complications in formalizing this notion can be overcome in much the same way as with traditional ordinal definability, as is spelled out in detail in Section 2. Precursors to this idea are Hamkins & Leahy [11], where the case $\kappa = \omega$ is considered, and Tzouvaras [23],

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where it is argued that the case $\kappa = \omega_1$ is the one to be focused on, whereas I consider the full spectrum of possible values for κ.

As was observed by Tzouvaras, for every cardinal κ, the class $<\kappa$-OD of all $<\kappa$-blurrily ordinal definable sets is uniformly definable (using κ as parameter), and defining $<\kappa$-HOD to consist of all sets x such that $\mathsf{TC}(\{x\}) \subseteq <\kappa$-$\mathsf{OD}$ gives rise to an inner model. Clearly, increasing κ weakens the notion of $<\kappa$-blurry ordinal definability, so we are dealing with a weakly increasing hierarchy of inner models, leading from $\mathsf{HOD} = <2$-HOD to $\mathrm{V} = <\infty$-$\mathsf{HOD} = \bigcup_{2\le\kappa\in\mathrm{Card}} <\kappa$-$\mathsf{HOD}$, assuming the axiom of choice. The main objective of this project is to understand how this hierarchy grows. Since the concepts involved are highly non-absolute, there are two kinds of results to be expected: ZFC-provable, general results about the hierarchy on the one hand, and relative consistency results on the possible behavior of the hierarchy on the other.

The main results of the first kind are to be found in Section 3, and they concern the question how close $<\kappa$-HOD is to $<\lambda$-HOD, for different cardinals κ and λ. Generalizing results from [6], I show that if $\kappa < \lambda$, then $<\kappa$-HOD satisfies the (external) λ-approximation and -cover properties in $<\lambda$-HOD, Theorem 3.3. This has a very useful consequence, Corollary 3.7, which says that if $<\kappa$-$\mathsf{HOD} \subsetneq <\lambda$-$\mathsf{HOD}$, then there are sets a and b such that $a \in <\lambda$-$\mathsf{HOD} \setminus <\kappa$-$\mathsf{HOD}$, $a \subseteq b \in <\kappa$-$\mathsf{HOD}$ and $\mathsf{card}(b) < \lambda$. This fact proves useful both in understanding the growth of the blurry HOD hierarchy in forcing extensions and in obtaining the lower consistency strength bounds mentioned below.

Most results of the second kind concern the notion of a leap, a central concept in the analysis of the blurry HOD hierarchy introduced in [6]. A leap is a cardinal $\kappa > 2$ such that $<\kappa$-$\mathsf{HOD} \neq <\bar{\kappa}$-$\mathsf{HOD}$ for every cardinal $2 \le \bar{\kappa} < \kappa$. The main result of the article is that the following three statements are equiconsistent (over ZFC):

(1) The least leap is the cardinal successor of a strong limit cardinal of countable cofinality.
(2) $\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+1}$ is the least leap.
(3) There is an inner model with a measurable cardinal.

This follows by taking together Theorems 5.1 and 9.16.

I will conclude this introduction by giving a section by section overview of the contents of this work.

In section 2, I give the formal definition of the blurry HOD hierarchy, along with a definable prewellordering that has not explicitly been introduced in [6], so even readers familiar with that paper are encouraged to skim this part. I also sum up the basic facts on leaps.

Section 3 proves the results on the (external) approximation and cover properties mentioned above, and contains some connections to the question of existence of fresh sequences. There is also an excursion contrasting this with the effects of Bukovsky's approximation property on fresh sets.

In Section 4, I give an overview over Příkrý forcing, Příkrý sequences and their interactions with inner models and covering. These results will be used in the subsequent sections.

Section 5 uses these results to establish the lower bound of an inner model with a measurable cardinal to (a weakening of) the least leap being the successor of a

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singular strong limit cardinal, and it explores what can be said when the cofinality of the cardinal in question is uncountable.

Section 6 summarizes the forcing tools that will be used in the subsequent sections.

Section 7 shows that the statement that the least leap is the successor of a singular cardinal has no consistency strength; methods for producing models in which the singular cardinal in question has countable or uncountable cofinality are developed here. Thus, the assumption that the singular cardinal is a *strong* limit is essential in the main equiconsistency result.

Section 8 produces a model where $\aleph_\omega$ is a strong limit cardinal which is also a limit of leaps, and where $\aleph_{\omega+1}$ is a leap. The basic idea is to add a Příkrý sequence over $L[U]$ and to collapse between the Příkrý points with even indices.

In Section 9, this idea is taken further, forcing over $L[U]$ to produce a model where $\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+1}$ is the least leap. The key idea is to modify Magidor's version of Příkrý forcing with integrated collapses so as to collapse between the Příkrý points with even index, und subsequently coding the collapsing functions in order to make them ordinal definable. This result provides the desired equiconsistency.

Section 10 sketches how to achieve the situation where the least leap is the successor cardinal of $\aleph_\lambda$, for other values of λ.

The article concludes with Section 11, containing some open questions, and in concluding the introduction, I would like to thank the referee for very useful feedback.

2. Basics

Definition 2.1. Let $\langle \varphi_n \mid n < \omega \rangle$ be a recursive enumeration of all internal first order formulas in the language of set theory with two free variables. Let $\mathsf{Sat}(x, y, z)$ be the satisfaction relation, so that for a set u, a natural number n and a pair $\langle a, b \rangle \in u^2$, $\mathsf{Sat}(u, n, \langle a, b \rangle)$ holds iff $\langle u, \in \cap u^2 \rangle \models \varphi_n(a, b)$.

Let a be a set. A *blurry ordinal definition of* a is an OD set A with $a \in A$. Given a cardinal λ, a $<\lambda$-*blurry ordinal definition of* a is a blurry ordinal definition of a of cardinality less than λ. Let $\Sigma(a)$ be the least cardinal κ such that a has $<\kappa^+$-blurry ordinal definition (if such a cardinal exists - assuming AC, it always does).

Let $D(a)$ be the least ordinal of the form $\prec \alpha, n, \beta \succ$ such that the set

$$\{x \in \mathrm{V}_\alpha \mid \mathsf{Sat}(\mathrm{V}_\alpha, \varphi_n, \langle x, \beta \rangle)\}$$

is a $<\Sigma(a)^+$-blurry ordinal definition of a (if such a triple exists).

For a cardinal $\lambda > 1$, define $<\lambda$-OD to be the class of sets that have a $<\lambda$-blurry ordinal definition, and define that $a \in {<\lambda}$-HOD iff $\mathsf{TC}(\{a\}) \subseteq {<\lambda}$-$\mathsf{OD}$.

Observation 2.2. *We have the following facts about* Σ *and* D.

1. *The functions* Σ *and* D *are definable without parameters. Assuming* ZFC, *both are total.*
2. *Let* a *be a set for which* $\Sigma(a)$ *is defined. Let* $D(a) = \rho$. *Then* $D^{-1}(\rho) = \{x \mid D(x) = \rho\}$ *is a blurry ordinal definition of* a, *and* $\mathsf{card}(D^{-1}(\rho)) = \Sigma(a)$.

Proof. (1) is obvious.

(2): Since $\Sigma(a)$ is defined, so is $\rho = D(a)$. The ordinal ρ is of the form $\prec \alpha, n, \beta \succ$, and the set $A = \{x \mid \mathrm{V}_\alpha \models \varphi_n(x, \beta)\}$ is a blurry ordinal definition of a of cardinality

$\Sigma(a)$. Now suppose that $b \in D^{-1}(\rho)$ as well. Then A is also a blurry ordinal definition of b. Thus, $D^{-1}(\rho) \subseteq A$. Hence, $\mathsf{card}(D^{-1}(\rho)) \leq \mathsf{card}(A) = \Sigma(a)$. By minimality of $\Sigma(a)$, we have that $\Sigma(a) \leq \mathsf{card}(D^{-1}(\rho))$. $\square$

So we can let R be the prewellordering associated to the norm D, that is, R is the set-like, well-founded relation on $\mathrm{dom}(D) \times \mathrm{dom}(D)$ defined by setting xRy iff $D(x) < D(y)$. For every cardinal $\lambda \geq 2$, the restriction of R to $<\lambda$-HOD is then a prewellordering of $<\lambda$-HOD such that for every $x \in {<\lambda}$-HOD, the set of y that are incomparable to x is precisely $D^{-1}(D(x)) \cap {<\lambda}$-$\mathsf{HOD}$, a set of cardinality less than λ (more precisely, of cardinality $\Sigma(x)$).

The following "metatheorem" is sometimes useful. It shows that we may allow parameters from $<\kappa$-OD in $<\kappa$-blurry ordinal definitions.

Theorem 2.3. *Let $\kappa \geq \omega$ be a cardinal, and suppose that A is a set of cardinality less than κ definable using ordinals and parameters which are $<\kappa$-OD. Then every $a \in A$ is $<\kappa$-OD.*

Proof. Let $A = \{x \mid \varphi(x, \vec{\alpha}, \vec{b})\}$, where $\mathsf{card}(A) = \bar{\kappa} < \kappa$, $\vec{b} = b_0, \ldots, b_{n-1}$ are $<\kappa$-OD and $\vec{\alpha} = \alpha_0, \ldots, \alpha_{m-1}$ are ordinals. For $i < n$, let B_i be a $<\kappa$-blurry ordinal definition of b_i. Let

$$B = \{\langle c_0, \ldots, c_{n-1}\rangle \in B_0 \times \cdots \times B_{n-1} \mid \mathsf{card}(\{x \mid \varphi(x, \vec{\alpha}, \vec{c})\}) = \bar{\kappa}\}.$$

Then $\langle b_0, \ldots, b_{n-1}\rangle \in B$. Let

$$C = \bigcup_{\vec{c} \in B} \{x \mid \varphi(x, \vec{\alpha}, \vec{c})\}.$$

Then C is OD, $A \subseteq C$, and $\mathsf{card}(C) \leq \omega \cdot \bar{\kappa} \cdot \mathsf{card}(B) < \kappa$, so that C is a $<\kappa$-blurry ordinal definition of any element of A. $\square$

I will need some basic fact on blurry HOD which were established in [6].

Definition 2.4 ([6, Def. 6&7])**.** A cardinal $\lambda > 2$ is a *leap* if

$$<\delta\text{-}\mathsf{HOD} \subsetneq <\lambda\text{-}\mathsf{HOD},$$

for every cardinal $\delta < \lambda$. I write $\langle \Lambda_\alpha \mid \alpha < \Theta\rangle$ for the monotone enumeration of the leaps (it is the empty sequence, i.e., Λ_0 is undefined, if there is no leap). A leap λ is a successor leap if $\lambda = \Lambda_{\xi+1}$, for some ξ, and it is a limit leap if it is of the form $\lambda = \Lambda_\xi$, where ξ is a limit ordinal. A leap γ is a *big leap* if

$$\left(\bigcup_{\delta<\gamma, \delta \in \mathrm{Card}} <\delta\text{-}\mathsf{HOD}\right) \subsetneq <\gamma\text{-}\mathsf{HOD}.$$

The following facts were shown in [6, Lemma 2 and Thm. 9&10].

Fact 2.5. *Leaps have the following properties.*

(1) *The class of leaps is closed in the ordinals.*
(2) Λ_0*, if defined, is an uncountable successor cardinal.*
(3) *Successor leaps are successor cardinals.*
(4) *Every leap is big.*
(5) *If λ is a limit leap, then $<\lambda$-HOD does not satisfy the axiom of choice.*

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3. The closeness between $<\kappa$-HOD and $<\lambda$-HOD

In the analysis of the consistency strength of certain leap constellations, it is crucial to understand how close $<\kappa$-HOD and $<\lambda$-HOD are to each other, for cardinals $\kappa < \lambda$. In [6], the closeness between HOD and $<\lambda$-HOD was already investigated. Let me look at the more general picture now.

3.1. **Approximation and cover properties.** The following are properties expressing the closeness between two models, introduced by Hamkins [10], along with "external" versions thereof, designed to apply in the context where the inner models in question don't necessarily satisfy the axiom of choice, but are situated within an ambient model of ZFC.

Definition 3.1. Let $M \subseteq N$ be transitive classes, and let κ be a cardinal in N.

M satisfies the *κ-cover property* in N if for every set $a \in N$ with $a \subseteq M$ and $\mathsf{card}(a)^N < \kappa$, there is a set $c \in M$ such that $a \subseteq c$ and $\mathsf{card}(c)^M < \kappa$. M satisfies the *external κ-cover property* in N if for every $a \in N$ with $a \subseteq M$ and $\mathsf{card}(a)^{\mathrm{V}} < \kappa$, there is a $c \in M$ with $a \subseteq c$ and $\mathsf{card}(c)^{\mathrm{V}} < \kappa$.

Let $a \in N$ be a set with $a \subseteq M$. A set of the form $a \cap c$, where $c \in M$ and $\mathsf{card}(c)^M < \kappa$, is called a *$\kappa$-approximation to a in M*. If the set is of the form $a \cap c$, where $c \in M$ and $\mathsf{card}(c)^{\mathrm{V}} < \kappa$, then it is called an *external κ-approximation* to a in M. The set a is said to be *(externally) κ-approximated in M* if every (external) κ-approximation to a in M belongs to M. M satisfies the *(external) κ-approximation property in N* if whenever $a \in N$ with $a \subseteq M$ is (externally) κ-approximated in M, then $a \in M$.

The external versions of the approximation and cover properties generalize their original versions.

Observation 3.2. *Assume that M, N are transitive models of* ZFC*, and let κ be a cardinal. Then M satisfies the κ-approximation property in N iff M satisfies the external κ-approximation property in N.*

Similarly for the external κ-cover property.

Proof. The point is that if $b \in M$, then $\mathsf{card}(b)^M < \kappa$ iff $\mathsf{card}(b)^{\mathrm{V}} < \kappa$, and similarly for N, since κ is a cardinal. In detail, since M satisfies the axiom of choice, $\mathsf{card}(b)^M$ exists, and hence, $\mathsf{card}(b)^{\mathrm{V}}$ exists. Clearly, $\mathsf{card}(b)^V \leq \mathsf{card}(b)^M$. Suppose we had that $\bar{\kappa} = \mathsf{card}(b)^V < \kappa$ while $\mathsf{card}(b)^M \geq \kappa$. Let $f \in M$ be a surjection from b onto κ, and let $g : b \longrightarrow \bar{\kappa}$ be a bijection. Then $f \circ g^{-1} : \bar{\kappa} \longrightarrow \kappa$ is a surjection, so κ is not a cardinal, a contradiction. □

In the context in which I want to apply these properties, the ambient universe satisfies ZFC, but the two models, being of the form $<\kappa$-HOD and $<\lambda$-HOD, are merely models of ZF.

Theorem 3.3 (ZFC)**.** *Let $2 \leq \kappa \leq \lambda$ be infinite cardinals. Then $<\kappa$-HOD satisfies the external λ-cover property and the external λ-approximation property in $<\lambda$-HOD.*

Proof. The case $\kappa = 2$ was shown in [6, Theorem 3], and it was pointed out there that this case implies directly that $<\omega$-$\mathsf{HOD} = \mathsf{HOD}$, which was first shown in [11]. Because of this, we may assume that $\omega \leq \kappa < \lambda$.

To verify the external λ-cover property, let $a \in <\lambda$-HOD be such that $a \subseteq <\kappa$-HOD and $\gamma = \mathsf{card}(a)^{\mathrm{V}} < \lambda$. Let A be a $<\lambda$-blurry ordinal definition of a. Since

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$a \subseteq {<}\kappa\text{-}\mathsf{HOD}$ and $\mathsf{card}(a) = \gamma$, we may assume that for all $b \in A$, $b \subseteq {<}\kappa\text{-}\mathsf{HOD}$ and $\mathsf{card}(b) = \gamma$, since these requirements may be added to the definition of A if necessary. Set $c = \bigcup A$. By the axiom of choice in V, $\mathsf{card}(c)^{\mathrm{V}} \le \gamma \cdot \mathsf{card}(A) < \lambda$, c is OD, and $c \subseteq {<}\kappa\text{-}\mathsf{HOD}$. Thus, $c \in {<}\kappa\text{-}\mathsf{HOD}$, and clearly, $a \subseteq c$.

Turning to the external λ-approximation property, let $a \in {<}\lambda\text{-}\mathsf{HOD}$, $a \subseteq {<}\kappa\text{-}\mathsf{HOD}$ be externally λ-approximated in ${<}\kappa\text{-}\mathsf{HOD}$. Let A be a ${<}\lambda$-blurry ordinal definition of a. As before, we may assume that every $b \in A$ is a subset of ${<}\kappa\text{-}\mathsf{HOD}$ that's externally λ-approximated in ${<}\kappa\text{-}\mathsf{HOD}$.

I will now use the functions D and Σ of Definition 2.1, definable without parameters. Given distinct sets $b_0, b_1 \in A$, it follows that $b_0 \triangle b_1 \subseteq {<}\kappa\text{-}\mathsf{HOD}$, so for every $x \in b_0 \triangle b_1$, $\Sigma(x) < \kappa$. So, letting

$$\mu_{b_0,b_1} = \min D[b_0 \triangle b_1]$$

it follows that $D^{-1}(\mu_{b_0,b_1})$ is a subset of ${<}\kappa\text{-}\mathsf{OD}$ of cardinality less than κ; see Observation 2.2. Let

$$\Delta = {<}\kappa\text{-}\mathsf{HOD} \cap \bigcup \{D^{-1}(\mu_{b_0,b_1}) \mid \{b_0, b_1\} \in [A]^2\}.$$

Then Δ has cardinality less than λ. Moreover, Δ is OD, as the function D is definable without parameters, A is OD, and so is the function mapping $b_0, b_1 \in A$ to μ_{b_0,b_1}. Moreover, $\Delta \subseteq {<}\kappa\text{-}\mathsf{HOD}$, so $\Delta \in {<}\kappa\text{-}\mathsf{HOD}$.

Letting $a_0 = a \cap \Delta$, we have that $a_0 \in {<}\kappa\text{-}\mathsf{HOD}$, since a is λ-approximated in ${<}\kappa\text{-}\mathsf{HOD}$ and $\Delta \in {<}\kappa\text{-}\mathsf{HOD}$ has size less than λ. But note that if $b_0, b_1 \in A$ are distinct, then $b_0 \cap \Delta \neq b_1 \cap \Delta$. In other words, a is the unique $c \in A$ such that $c \cap \Delta = a_0$. Thus, a is ordinal definable from the parameters c and a_0, which are ${<}\kappa\text{-}\mathsf{OD}$, so by Theorem 2.3, a is ${<}\kappa\text{-}\mathsf{OD}$. Since a is contained in ${<}\kappa\text{-}\mathsf{HOD}$, it follows that $a \in {<}\kappa\text{-}\mathsf{HOD}$, as wished. □

Definition 3.4 (Hamkins)**.** Let $M \subseteq N$ be inner models, and let θ be an ordinal. A sequence $a : \theta \longrightarrow M$ is a *fresh sequence in N over M* if $a \in N$, $a \notin M$, but for all $\xi < \theta$, $a{\restriction}\xi \in M$.

Lemma 3.5 (ZFC)**.** *Let λ be an infinite cardinal, and suppose $M \subseteq N$ are inner models with $M \subsetneq N$.*

(1) *Suppose that M satisfies the external λ-approximation property in N. If N has a fresh sequence over M of length θ, then $\mathrm{cf}(\theta) < \lambda$.*
(2) *Suppose that M satisfies the external λ-approximation and -cover properties in N. Then there are sets a, b such that $a \in N \setminus M$, $a \subseteq b \in M$ and $\mathsf{card}(b) < \lambda$.*

Proof. For (1), suppose $f : \theta \longrightarrow M$, $f \in N$ is such that for every $\alpha < \theta$, $f{\restriction}\alpha \in M$, and $\mathrm{cf}(\theta) \ge \lambda$. I will show that $f \in M$. As a set, f is a subset of M, and the point is that f is externally λ-approximated in M. To see this, let $a \in M$, $\mathsf{card}(a)^{\mathrm{V}} < \lambda$. Since $\mathrm{cf}(\theta) \ge \lambda$, it follows that $\{\xi < \theta \mid \langle \xi, f(\xi)\rangle \in a\}$ is bounded in θ, say by α. Then $f \cap a = (f{\restriction}\alpha) \cap a \in M$, as $f{\restriction}\alpha \in M$. So f is externally λ-approximated in M, which implies that $f \in M$, since M satisfies the external λ-approximation property in N.

For (2), Let Δ be the class of $\in$-minimal elements of $N \setminus M$. Clearly, $a \in \Delta$ iff a has the following properties:

(a) $a \in N$.
(b) $a \subseteq M$.

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(c) $a \notin M$.

Let $\mu = \min\{\mathsf{card}(b) \mid b \in \Delta\}$, and let $a \in \Delta$ be of cardinality μ. It follows that a is externally μ-approximated in M, because if $b \in M$ has cardinality less than μ, then $\mathsf{card}(a \cap b) < \mu$, $a \cap b \in N$ and $a \cap b \subseteq M$. So it must me that $a \cap b \in M$, because otherwise, $a \cap b$ would be in Δ, but of cardinality less than μ, which is impossible. It follows that $\mu < \lambda$, because if $\mu \geq \lambda$, then it would follow that a is λ-approximated in M, but this in turn would imply that $a \in M$, since M has the external λ-approximation property in N, a contradiction.

Hence, $\mathsf{card}(a) = \mu < \lambda$. Since $a \subseteq M$ and M has the external λ-cover property in N, there is a $b \in M$ with $a \subseteq b$ and $\mathsf{card}(b) < \lambda$, so a and b are as wished. □

Corollary 3.6 (ZFC). *Let $2 \leq \kappa < \lambda$ be a infinite cardinals. If θ is a limit ordinal with $\mathrm{cf}(\theta) \geq \lambda$, then $<\lambda$-HOD has no length θ sequence that's fresh over $<\kappa$-HOD.*

Corollary 3.7 (ZFC). *Let $2 \leq \kappa < \lambda$ be cardinals such that $<\kappa\text{-HOD} \subsetneqq <\lambda\text{-HOD}$. Then there are sets a and b such that $a \in <\lambda\text{-HOD} \setminus <\kappa\text{-HOD}$, $a \subseteq b \in <\kappa\text{-HOD}$ and $\mathsf{card}(b) < \lambda$.*

This last corollary simplifies some arguments of [6], and will be used frequently in what follows.

3.2. **Bukovsky's approximation property.** In this subsection, I would like to explore a second notion of closeness, originally introduced by Bukovsky. This material is not going to be used in the following sections and may be skipped without loss.

Definition 3.8 ([1, (1.6)]). Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathsf{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathsf{card}(g(\xi))^{M_1} < \kappa$.

It was shown in [6] that for any cardinal λ, $\mathsf{Apr}_{\mathsf{HOD},<\lambda\text{-}\mathsf{HOD}}(\lambda)$ holds. In the present context, I would like to extend this to $<\kappa$-HOD instead of HOD, where $\kappa < \lambda$. Since $<\kappa$-HOD may not satisfy the axiom of choice, it is useful to devise a variation of Bukovsky's property, similar to the "external" versions of Hamkins' approximation and cover properties. In fact, a slight strengthening of the obvious external version of the condition holds, formulated below.

Definition 3.9. Let $M_1 \subseteq M_2$ be inner models, and let κ be a cardinal in M_2. Then $\mathsf{Apr}^+_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to M_1, then there is a function $g : \alpha \longrightarrow M_1$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathsf{card}(g(\xi))^{\mathrm{V}} < \kappa$.

In this definition, one could have allowed α to be any set $a \in M_1$, and the following lemma would still hold, but I won't need that version of the condition. Let us check that in the situation where Bukovsky's original condition is usually applied, it follows from its "external" version.

Observation 3.10. *Let $M_1 \subseteq M_2$ be inner models, κ a cardinal. Suppose that $M_1 \models \mathsf{AC}$. Then $\mathsf{Apr}^+_{M_1,M_2}(\kappa)$ implies $\mathsf{Apr}_{M_1,M_2}(\kappa)$.*

Proof. Let $f : \alpha \longrightarrow \beta$, $f \in M_2$. By $\mathsf{Apr}^+_{M_1,M_2}(\kappa)$, let $g : \alpha \longrightarrow \mathcal{P}(\beta)$, $g \in M_1$ be such that for all $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathsf{card}(f(\xi)) < \kappa$. Fixing $\xi < \alpha$, since

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$M_1 \models \mathsf{AC}$, we may let $\kappa' = \mathsf{card}(f(\xi))$. It follows that $\kappa' < \kappa$, or else, there would be in V a bijection between $\mathsf{card}(f(\xi)) < \kappa$ and $\kappa' \geq \kappa$, so that κ couldn't have been a cardinal. $\square$

Observation 3.11. *Let $\kappa < \lambda$ be cardinals. Then $\mathsf{Apr}^+_{<\kappa\text{-}\mathsf{HOD},<\lambda\text{-}\mathsf{HOD}}(\lambda)$ holds.*

Proof. Let $f : \alpha \longrightarrow {<}\kappa\text{-}\mathsf{HOD}$, $f \in {<}\lambda\text{-}\mathsf{HOD}$. Let F be a $<\lambda$-blurry ordinal definition of f. We may assume that for all $f' \in F$, $f' : \alpha \longrightarrow {<}\kappa\text{-}\mathsf{HOD}$. Define $g : \alpha \longrightarrow {<}\kappa\text{-}\mathsf{HOD}$ by $g(\xi) = \{f'(\xi) \mid f' \in F\}$. Then $g \in {<}\kappa\text{-}\mathsf{HOD}$, since g is OD and $g \subseteq {<}\kappa\text{-}\mathsf{HOD}$, and for all $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathsf{card}(g(\xi)) < \lambda$, as wished. $\square$

Let me now show that the external version of Bukovsky's approximation property also limits the existence fresh sequences, but not as strongly as the external version of Hamkins' approximation property. First, I need a version of Kurepa's theorem on the existence of branches through trees. If T is a tree, I will write $T(\alpha)$ for the collection of its nodes on level α, and I will denote the tree ordering by $<_T$. The height of T, $\mathrm{ht}(T)$, is the least α such that $T(\alpha) = \emptyset$, and T is *well-pruned* if for every $\alpha < \beta < \mathrm{ht}(T)$ and every $x \in T(\alpha)$ there is a $y \in T(\beta)$ such that $x <_T y$.

Theorem 3.12 (after Kurepa)**.** *Let $\kappa < \lambda$ be cardinals, λ regular. Let T be a well-pruned tree of height λ all of whose levels have size less than κ. Then there is an $\alpha < \lambda$ such that for all $x \in T(\alpha)$ and all $\beta \in (\alpha, \lambda)$, there is a unique $y = y(x, \beta) \in T(\beta)$ with $x <_T y$.*

Proof. For $x \in T$, let $s(x)$ be the least $\beta < \lambda$ such that there are distinct $y_0, y_1 \in T(\beta)$ such that $x <_T y_0$ and $x <_T y_1$, if there is such a β - otherwise, let $s(x) = 0$. Say that x is *splitting* if $s(x) > 0$. I have to show that there is an α such that no $x \in T(\alpha)$ is splitting.

Define, for $\alpha < \lambda$,

$$f(\alpha) = \sup\{s(x) \mid x \in T(\alpha)\}.$$

Since $\mathsf{card}(T(\alpha)) < \kappa < \lambda = \mathrm{cf}(\lambda)$, $f(\alpha) < \lambda$. Since λ is regular, the set $C = \{\gamma < \lambda \mid f[\gamma] \subseteq \gamma\}$ is club in λ. Let $\langle \xi_i \mid i < \lambda \rangle$ be the monotone enumeration of C, and let $\alpha = \xi_\kappa$ be the κ-th element of C. I claim that α is as wished.

Namely, let $x \in T(\alpha)$. Suppose towards a contradiction that x is splitting. Let $b = \{y \mid y <_T x\}$. Then every $y \in b$ is splitting. Let, for $i < \kappa$, y_i be the unique element of $b \cap T(\xi_i)$. Then $s(y_i) < \xi_{i+1} < \alpha$. So there is a $z_i >_T y_i$ in $T(\xi_{i+1})$ such that $z_i \notin b$. And since T is well-pruned, z_i has a successor u_i in $T(\alpha)$. For $i < j$, it follows that $u_i \neq u_j$, because otherwise, we'd have: $y_i <_T z_i <_T u_i$ and $x_j <_T u_i$, so $z_i <_T x_j$, that is, $z_i \in b$, a contradiction. But then, $\{u_i \mid i < \kappa\} \subseteq T(\alpha)$, so $\mathsf{card}(T(\alpha)) \geq \kappa$, a contradiction. $\square$

Observation 3.13. *Let λ be a cardinal, $M \subseteq N$ inner models such that $\mathrm{Apr}^+_{M,N}(\lambda)$ holds, and let θ be a limit ordinal with $\mathrm{cf}(\theta) > \lambda$. Then N has no length θ sequence fresh over M.*

Note:

(1) By Lemma 3.5, if M has the external λ-approximation property in N, then the same conclusion can be drawn just assuming that $\mathrm{cf}(\theta) \geq \lambda$.

(2) This observation cannot be improved in general. For example, if S is a Souslin tree and b is a generic branch for S, then $\mathsf{Apr}^+_{\mathrm{V},\mathrm{V}[b]}(\aleph_1)$ holds, since S is ccc. But of course, b is a fresh ω_1-sequence over V.

Proof. Assume the contrary. Let θ be the least counterexample. It is easy to see that θ must be regular. So $\theta = \mathrm{cf}(\theta) > \lambda$.

Let $f : \theta \longrightarrow M$ be fresh over M, $f \in N$. By applying $\mathsf{Apr}_{M,N}(\lambda)$ to the function $\xi \mapsto f{\restriction}\xi$, it follows that there is a function $g : \theta \longrightarrow M$, $g \in M$, so that for all $\xi < \theta$, $f{\restriction}\xi \in g(\xi)$ and $\mathsf{card}(g(\xi)) < \lambda$. We may assume that for all $\xi < \theta$, $g(\xi)$ consists of functions with domain ξ.

Working in M, define an ordinal ρ and a sequence $\langle g_\alpha \mid \alpha < \rho\rangle$ of functions with domain θ recursively as follows:

(1) $g_0 = g$.
(2) If g_α has already been defined, then define $g'_\alpha : \theta \longrightarrow M$ by setting

$$g'_{\alpha+1}(\xi) = \{h \in g_\alpha(\xi) \quad | \quad \forall \zeta \in (\xi,\theta)\exists h' \in g_\alpha(\zeta) \quad h'{\restriction}\xi = h \\ \text{and } \forall \zeta < \xi \quad h{\restriction}\zeta \in g_\alpha(\zeta)\}.$$

If $g'_{\alpha+1} = g_\alpha$, then let $\rho = \alpha+1$, ending the construction. Otherwise, define $g_{\alpha+1} = g'_{\alpha+1}$.
(3) If α is a limit ordinal and g_ζ has been defined for all $\zeta < \alpha$, then define $g_\alpha : \theta \longrightarrow M$ by

$$g_\alpha(\xi) = \bigcap_{\zeta<\alpha} g_\zeta(\xi).$$

Since $g_{\alpha+1}(\xi) \subseteq g_\alpha(\xi)$ when defined, the sequence has to stabilize, and it has a last element, say $g_{\bar\rho}$, so that $\rho = \bar\rho + 1$. Note that inductively, $f{\restriction}\xi \in g_\alpha(\xi)$, for every $\xi < \theta$, $\alpha < \rho$.

Now let $T = \langle \bigcup \mathrm{ran}(g_{\bar\rho}), \subsetneqq\rangle$. By construction, it is a well-pruned tree of height θ each of whose levels has cardinality less than λ in V. Since $\lambda < \theta$ and θ is regular, Theorem 3.12 applies. So let $\alpha < \theta$ be such that for every $\xi \in (\alpha, \theta)$, every node in $T(\alpha)$ has a unique successor in $T(\xi)$. Then

$$f = \bigcup \left(\bigcup_{\xi \in (\alpha,\theta)} \{h \in T(\xi) \mid f{\restriction}\alpha \subseteq h\} \right) \in M$$

since $T \in M$. □

4. Canonical inner models and Příkrý sequences

Příkrý forcing and its interactions with canonical inner models will play a major role in the arguments to follow, so it seems appropriate to give an overview of the results I am going to use.

The starting point for Příkrý forcing is a measurable cardinal, and I will actually begin a little bit below this assumption. A crucial tool I will use is the Dodd-Jensen covering lemma for their core model K^{DJ}, which was introduced in [2].

An inner model W is said to have the *covering property* if for every uncountable set of ordinals X, there is a set $Y \in W$ of the same cardinality as X, such that $X \subseteq Y$.

Theorem 4.1 (Dodd-Jensen [3, Theorem 5.17])**.** *If there is no inner model with a measurable cardinal, then K^{DJ} has the covering property.*

I will also use some basic properties of the core model, such as the fact that it is forcing invariant (i.e., $K^{\mathrm{V}} = K^{\mathrm{V}[G]}$, when G is set generic over V) and that $K \subseteq \mathsf{HOD}$. For more background, I refer the reader to [21]. Let's now move into the realm of measurable cardinals.

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Definition 4.2 (Příkrý [20])**.** Given a measurable cardinal κ and a normal ultrafilter U on κ, the Příkrý forcing for U, denoted $\mathbb{P}_U$ consists of all pairs of the form $\langle s, T\rangle$ such that s is a finite subset of κ, $T \in U$ and $s \subseteq \min(T)$. The partial ordering of $\mathbb{P}_U$ is defined by $\langle s', T'\rangle \le_{\mathbb{P}_U} \langle s, T\rangle$ iff $s \subseteq s'$, $T' \subseteq T$ and $s' \setminus s \subseteq T$.

Definition 4.3. Let W be an inner model, $\bar{\kappa}$ a measurable cardinal in the sense of W, and $\bar{U}$ a normal ultrafilter on $\bar{\kappa}$ in W. Then a set $C \subseteq \bar{\kappa}$ of order type ω is a Příkrý sequence for $\bar{U}$ iff for every $A \in \bar{U}$, $C \setminus A$ is finite. Such a C is a *maximal* Příkrý sequence for $\bar{U}$ if for every Příkrý sequence D for $\bar{U}$, $D \setminus C$ is finite.

Fact 4.4. *Let κ be measurable, U a normal ultrafilter on κ, and G be $\mathbb{P}_U$-generic over* V*.*

- *The set $C = \bigcup\{a \mid \exists B \in U \quad \langle a, B\rangle \in G\}$ is a Příkrý sequence for U.*
- $\mathrm{V}_\kappa = \mathrm{V}_\kappa^{\mathrm{V}[G]}$.
- $\mathrm{V}[G] = \mathrm{V}[C]$*, that is, $G = G_C$ can be recovered from C, as* $G_C = \{\langle a, B\rangle \in \mathbb{P}_U \mid a \text{ is an initial segment of } C \text{ and } C \setminus a \subseteq B\}$.
- *Mathias* [17] *characterizes when $D \subseteq \kappa$ of order type ω gives rise to a $\mathbb{P}_U$-generic filter G_D: this is the case iff D is a Příkrý sequence for U.*
- *In* $\mathrm{V}[C]$*, C is a maximal Příkrý sequence over* V*. See Fuchs* [5] *for a proof of (a generalization of) this, and Gitik-Kanovei-Koepke* [9] *for much more information in this direction.*

A lot more information on Příkrý forcing and its variations can be found in [8].

I will use the minimal inner model for a measurable cardinal, $L[U]$, which originated in work of Solovay, Silver and Kunen (see Kunen [15]). There is a crucial connection to Příkrý sequences, which is why I include it in the present section.

When I write $L[U]$, I mean to say that $U \in L[U]$ is a normal ultrafilter on an ordinal κ in the sense of $L[U]$, where for no $\bar{\kappa} < \kappa$ is there a $\bar{U}$ such that $\bar{U} \in L[\bar{U}]$ and $\bar{U}$ is a normal ultrafilter on $\bar{\kappa}$ in $L[\bar{U}]$ – Kunen [15] showed that this $L[U]$ is unique, and that κ is the unique measurable cardinal in $L[U]$; in his terminology, it is the minimal ρ-model. For more on the history, Kanamori [13, Chapter 4, §20] is a good source. By the uniqueness of U, it is definable, and hence every element of $L[U]$ is ordinal definable, so $L[U] \subseteq \mathsf{HOD}$.

Later, I will also need the Dodd-Jensen covering lemma for $L[U]$, stated below. The statement "$0^\dagger$ does not exist," which occurs in its formulation, may be taken to to mean that there is no elementary embedding $j : L[U] \longrightarrow L[U]$ with critical point greater than κ, the measurable cardinal of $L[U]$. From a more modern perspective, if $0^\dagger$ does not exist and there is an inner model with a measurable cardinal, then $L[U]$ is the core model K; see Mitchell [18] for a nice introduction to the general theory.

Theorem 4.5 (Dodd-Jensen [4])**.** *Suppose there is an inner model with a measurable cardinal, i.e., $L[U]$ exists. Assume that $0^\dagger$ does not exist. Then either $L[U]$ has the covering property, or there is a maximal Příkrý sequence C for U such that $L[U][C]$ has the covering property.*

The maximality of C is not explicitly stated in the formulation of the theorem in [4], but it is implicit in the proof. It is made more explicit in the excellent overview article Mitchell [19, pp. 1547-1548].

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5. Obtaining consistency strength lower bounds

Having generalized the salient facts about HOD and $<\lambda$-HOD (section 3), and having the necessary inner model theoretic tools (section 4) at our disposal, the next goal is to derive strength from the statement "there is a non-leap, singular strong limit cardinal whose successor cardinal is a leap."

Theorem 5.1 (ZFC). *Suppose λ is a singular strong limit cardinal, λ is not a leap, but λ^+ is a leap. Then there is an inner model with a measurable cardinal.*

Proof. Assume towards a contradiction that there is no inner model with a measurable cardinal.

Since λ is a limit cardinal which is not a leap, the leaps below λ must be bounded in λ, since the class of leaps is closed, by Fact 2.5. Let $\bar{\lambda} < \lambda$ be a cardinal which is greater than all the leaps below λ. It follows that $<\lambda\text{-HOD} = <\bar{\lambda}\text{-HOD}$.

Let κ be the least cardinal such that there is a set $a \in {<\lambda^+\text{-HOD}} \setminus {<\lambda\text{-HOD}}$ with $a \subseteq {<\lambda\text{-HOD}}$ of cardinality κ, and let a_0 be such a set. By Corollary 3.7, $\kappa = \mathsf{card}(a_0) \leq \lambda$.

I will now use the functions D and Σ introduced in Definition 2.1. Let $P = D[a_0]$, $\pi = \mathrm{otp}(P)$, and let $e : \pi \longrightarrow P$ be the monotone enumeration of P. Since $\mathsf{card}(a_0) \leq \lambda$, it follows that $\pi < \lambda^+$. For $\xi \leq \pi$, let

$$a_0{\restriction}\xi = \{x \in a_0 \mid e^{-1}(D(x)) < \xi\} = a_0 \cap D^{-1}[e[\xi]].$$

So $a_0{\restriction}\pi = a_0$. Let $\Omega \leq \pi$ be least such that $a_0{\restriction}\Omega \notin {<\lambda\text{-HOD}}$. Note that $a_0{\restriction}\xi \in {<\lambda^+\text{-HOD}}$, for all $\xi \leq \pi$ - the question is when $a_0{\restriction}\xi$ stops being in $<\lambda$-HOD; this is clearly true for $\xi = 0$ but false for $\xi = \pi$.

(1) Ω is a limit ordinal.

Proof of (1). Otherwise, say $\Omega = \bar{\Omega} + 1$. Let $a'_0 = a_0 \cap D^{-1}(e(\bar{\Omega}))$. Then $a_0{\restriction}\Omega = a{\restriction}\bar{\Omega} \cup a'_0$. Let $\bar{\kappa} = \mathsf{card}(D^{-1}(e(\bar{\Omega})))$. Then $\bar{\kappa} < \lambda$. This is because $e(\bar{\Omega}) = D(x)$ for some $x \in a_0 \subseteq {<\lambda\text{-HOD}}$, so that $\Sigma(x) < \lambda$ – see Observation 2.2. So $\mathcal{P}(D^{-1}(e(\bar{\Omega})))$ is OD (using $e(\bar{\Omega})$ as a parameter), has cardinality less than λ, and contains a'_0, making it a $<\lambda$-blurry ordinal definition of a'_0. Since $a'_0 \subseteq {<\lambda\text{-HOD}}$, it follows that $a'_0 \in {<\lambda\text{-HOD}}$. But also, by minimality of Ω, $a_0{\restriction}\bar{\Omega} \in {<\lambda\text{-HOD}}$. Hence,

$$a_0{\restriction}\Omega = (a_0{\restriction}\bar{\Omega}) \cup a'_0 \in {<\lambda\text{-HOD}},$$

a contradiction. □

(2) $\kappa < \lambda$.

Proof of (2). Since Ω is a limit ordinal, it makes sense to consider its cofinality. As $\Omega \leq \pi < \lambda^+$ and λ is singular, it follows that $\mathrm{cf}(\Omega) < \lambda$. And since $K \subseteq {<\lambda^+\text{-HOD}}$ and K has the covering property, $\bar{\Omega} = \mathrm{cf}^{<\lambda^+\text{-HOD}}(\Omega) < \lambda$.

So let $c : \bar{\Omega} \longrightarrow \Omega$ be monotone and cofinal, $c \in {<\lambda^+\text{-HOD}}$. Let $a'_0 = \{a_0{\restriction}c(\xi) \mid \xi < \bar{\Omega}\}$. Then $a'_0 \in {<\lambda^+\text{-HOD}}$, $a'_0 \subseteq {<\lambda\text{-HOD}}$ and $a'_0 \notin {<\lambda\text{-HOD}}$. So by minimality of κ, $\kappa \leq \mathsf{card}(a'_0) \leq \bar{\Omega} < \lambda$, as wished. □

So we have $\kappa = \mathsf{card}(a_0) < \lambda$, and hence, $\mathsf{card}(D[a_0]) < \lambda$. By the Dodd-Jensen Covering Lemma, there is a set $c \in K$ such that $D[a_0] \subseteq c$ and $\mathsf{card}(c) \leq \aleph_1 + \mathsf{card}(D[a_0]) < \lambda$. Let $c' = \{\xi \in c \mid \mathsf{card}(D^{-1}(\xi)) < \bar{\lambda}\}$. Then c' is OD, and we still have that $D[a_0] \subseteq c'$, since for every $x \in a_0$, $x \in {<\bar{\lambda}\text{-HOD}}$, so $\Sigma(x) < \bar{\lambda}$, so $\mathsf{card}(D^{-1}(D(x))) < \bar{\lambda}$, again by Observation 2.2. It then follows

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that $\mathsf{card}(D^{-1}[c']) \leq \bar{\lambda} \cdot \mathsf{card}(c') < \lambda$, and that $a_0 \subseteq D^{-1}[c']$. But $\mathcal{P}(D^{-1}[c'])$ is an OD set of cardinality $2^{\mathsf{card}(D^{-1}[c'])} < \lambda$, and it contains a_0, so a_0 has a $<\lambda$-blurry ordinal definition and $a_0 \subseteq <\lambda$-HOD. This means that $a_0 \in <\lambda$-HOD. This is a contradiction. □

The assumptions of the previous lemma are satisfied in particular in the focal case that the least leap, Λ_0, is the successor cardinal of the singular strong limit cardinal λ. In the remainder of this section, I will show that assuming the cofinality of λ to be uncountable increases the consistency strength of this constellation. The results presented here are merely a first step; a much higher consistency strength can be obtained, as I will show in a follow-up paper.

Observation 5.2. *If λ is a strong limit cardinal, then $H_\lambda \subseteq <\lambda$-$\mathsf{HOD}$.*

Proof. Since every member of H_λ is coded by a bounded subset of λ, it suffices to show that every bounded subset of λ is in $<\lambda$-HOD. But if $\alpha < \lambda$, then $\mathcal{P}(\alpha)$ is an OD set of cardinality less than λ, so $\mathcal{P}(\alpha) \subseteq <\lambda$-$\mathsf{HOD}$. □

Lemma 5.3. *Suppose λ is a singular strong limit cardinal, and that the following conditions all hold true:*

(1) $\lambda^+ = \Lambda_0$.
(2) $(\lambda^+)^{\mathsf{HOD}} = \lambda^+$.
(3) λ *is singular in* HOD.

Then there is a set in $<\lambda^+$-$\mathsf{HOD} \setminus \mathsf{HOD}$ which is a subset of λ of cardinality less than λ.

Proof. By Corollary 3.7, there are an $a \in <\lambda^+$-$\mathsf{HOD} \setminus \mathsf{HOD}$ and a set b with $\mathsf{card}(b) \leq \lambda$ such that $a \subseteq b \in \mathsf{HOD}$. Since HOD satisfies the axiom of choice and $\lambda^+ = (\lambda^+)^{\mathsf{HOD}}$, it follows that $\theta = \mathsf{card}(b)^{\mathsf{HOD}} \leq \lambda$. So let $f \in \mathsf{HOD}$, $f : \theta \longrightarrow b$ a bijection. Let $\bar{a} = f^{-1}[a]$. Then $\bar{a} \subseteq \theta \leq \lambda$, and $\bar{a} \notin \mathsf{HOD}$ (or else $a = f[\bar{a}] \in \mathsf{HOD}$), but $\bar{a} \in <\lambda^+$-HOD. By Observation 5.2, $\bar{a}$ must be unbounded in λ, since $\mathsf{HOD} = <\lambda$-HOD contains all bounded subsets of λ. That is, $\theta = \lambda$.

Let $\bar{\lambda} = \mathrm{cf}^{\mathsf{HOD}}(\lambda)$. By assumption, $\bar{\lambda} < \lambda$. Let $g : \bar{\lambda} \longrightarrow \lambda$ be cofinal, $g \in \mathsf{HOD}$. Then the set $\bar{a}' = \{\bar{a} \cap g(\xi) \mid \xi < \bar{\lambda}\}$ is in $<\lambda^+$-$\mathsf{HOD} \setminus \mathsf{HOD}$. Clearly, $\mathsf{card}(\bar{a}') \leq \bar{\lambda} < \lambda$ and $\bar{a}' \subseteq \mathsf{HOD}$ (the latter again since $\mathsf{HOD} = <\lambda$-HOD contains all bounded subsets of λ, by Observation 5.2.)

By that same observation again, $H_\lambda = H_\lambda^{\mathsf{HOD}}$. Since λ is a strong limit cardinal, we have that H_λ has cardinality λ. But since $(\lambda^+)^{\mathsf{HOD}} = \lambda^+$, H_λ also has cardinality λ in HOD. So let $h : \lambda \longrightarrow H_\lambda$, $h \in \mathsf{HOD}$. Note that $\bar{a}' \subseteq H_\lambda$. Let $\tilde{a} = h^{-1}[\bar{a}']$. Then $\tilde{a} \subseteq \lambda$ and $\mathsf{card}(\tilde{a}) < \lambda$. Moreover, $\tilde{a} \in <\lambda^+$-$\mathsf{HOD} \setminus \mathsf{HOD}$. So $\tilde{a}$ is as wished. □

Theorem 5.4. *If Λ_0 is the successor of a singular strong limit cardinal λ of uncountable cofinality, then $0^\dagger$ exists.*

Proof. Assume $0^\dagger$ does not exist. By Theorem 5.1, there is an inner model with a measurable cardinal. By the Dodd-Jensen covering lemma for $L[U]$, either $L[U]$ has the covering property, or there is a maximal Příkrý sequence C for U and $L[U][C]$ has the covering property. It cannot be that $L[U]$ has the covering property, because if so, we could run the argument of Theorem 5.1 with $L[U]$ in place of K^{DJ}, getting a contradiction to the covering property of $L[U]$. So let C be a maximal Příkrý sequence for U, so that $L[U][C]$ has the covering property. Let κ be the measurable

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cardinal of $L[U]$. The existence of C implies that $\mathrm{cf}(\kappa) = \omega$. Since $\mathrm{cf}(\lambda) \geq \omega_1$, $\kappa \neq \lambda$.

Case 1: $\kappa < \lambda$

There are only κ many maximal Příkrý sequences over U, and C is one of them. This is because if D is a maximal Příkrý sequence over U, then both $C \setminus D$ and $D \setminus C$ are finite, so D is obtained from C by removing some finite number of elements of C and adding some finite number of ordinals less than κ, and there are only κ ways of choosing these finite sets. Since U is OD, this shows that $C \in {<}\kappa^+\text{-}\mathsf{HOD} \subseteq {<}\lambda\text{-}\mathsf{HOD}$. So $L[U][C] \subseteq {<}\lambda\text{-}\mathsf{HOD} = \mathsf{HOD}$. This allows us again to run the argument of the proof of Theorem 5.1 with $L[U][C]$ in place of K^{DJ}, getting a contradiction to the covering property of $L[U][C]$.

Case 2: $\lambda < \kappa$.

The conditions of Lemma 5.3 are all satisfied: $\lambda^+ = \Lambda_0$ by assumption, $(\lambda^+)^{\mathsf{HOD}} = \lambda^+$ because $0^\dagger$ does not exist, so $(\lambda^+)^K = \lambda^+$, and $K \subseteq \mathsf{HOD}$, and λ is singular in HOD, because there is a cofinal subset of λ of order type less than λ in V, which is covered by such a subset in $L[U][C]$, but as $\lambda < \kappa$, $\mathcal{P}(\lambda)^{L[U][C]} = \mathcal{P}(\lambda)^{L[U]} \subseteq \mathsf{HOD}$, so such a subset exists in HOD as well. So by Lemma 5.3, there is an $a \subseteq \lambda$ of cardinality less than λ such that $a \in {<}\lambda^+\text{-}\mathsf{HOD} \setminus \mathsf{HOD}$.

But by the covering property of $L[U][C]$, let $b \in L[U][C]$ be such that $a \subseteq b$, $\mathsf{card}(b) \leq \mathsf{card}(a) + \aleph_1$, $b \subseteq \lambda$. Then $\mathsf{card}(b) < \lambda$, and since $b \in \mathcal{P}(\lambda)^{L[U][C]} = \mathcal{P}(\lambda)^{L[U]}$, $b \in L[U] \subseteq \mathsf{HOD}$. But then $a \in \mathcal{P}(b)$, and $\mathsf{card}(\mathcal{P}(b)) < \lambda$, since λ is a strong limit cardinal. So $\mathcal{P}(b)$ is a ${<}\lambda$-blurry ordinal definition of a, and $a \in {<}\lambda\text{-}\mathsf{HOD} = \mathsf{HOD}$. This is a contradiction. □

6. Forcing basics

In this section, I will collect some facts on the interaction between forcing and blurry ordinal definability. Most of these were established in Fuchs [6].

Observation 6.1 (ZFC, [6, Proposition 10])**.** *Suppose that $\mathbb{P}$ is a notion of forcing, G is generic for $\mathbb{P}$ over V, κ is a cardinal in $\mathrm{V}[G]$, and V is definable in $\mathrm{V}[G]$ from a parameter in ${<}\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$. Then*

$$<\kappa\text{-}\mathsf{OD}^{\mathrm{V}} \subseteq <\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$$

and so, ${<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}} \subseteq {<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]}$ as well.

The following observation will complement this fact. In order to formulate it, I'll introduce some terminology.

Definition 6.2. Let $\mathbb{P}$ be a forcing notion. For $p \in \mathbb{P}$, let *the cone below p in $\mathbb{P}$* be the set

$$\mathbb{P}_{\leq p} = \{q \in \mathbb{P} \mid q \leq p\}$$

equipped with the restriction of the ordering of $\mathbb{P}$.

$\mathbb{P}$ is called *cone homogeneous* if for any two conditions $p, q \in \mathbb{P}$, there are $p' \leq p$ and $q' \leq q$ such that $\mathbb{P}_{\leq p'}$ and $\mathbb{P}_{\leq q'}$ are isomorphic.

Observation 6.3 (ZFC)**.** *Suppose that $\mathbb{P}$ is an ordinal definable, cone homogeneous notion of forcing, G is generic for $\mathbb{P}$ over V, and κ is a cardinal in $\mathrm{V}[G]$.*

If $a \in \mathrm{V}$ is not ${<}\kappa$-OD in V, then a is not ${<}\kappa$-OD in $\mathrm{V}[G]$. (So if $a \in \mathrm{V}$ is not ${<}\kappa$-HOD in V, then it is not ${<}\kappa$-HOD in $\mathrm{V}[G]$ either.)

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Proof. Suppose $A = \{x \mid \varphi(x, \alpha)\}^{\mathrm{V}[G]}$ were a $<\kappa$-blurry ordinal definition of a. Let $\delta = \mathsf{card}(A)^{\mathrm{V}[G]} < \kappa$.

Since $\mathbb{P}$ is cone homogeneous, every condition in $\mathbb{P}$ forces with respect to $\mathbb{P}$ that $\varphi(\check{a}, \check{\alpha})$ holds and that the cardinality of $\{x \mid \varphi(x, \check{\alpha})\}$ is $\check{\delta}$.

Working in V, let

$$B = \{x \mid \text{every } p \in \mathbb{P} \text{ forces } \varphi(\check{x}, \check{\alpha})\}.$$

Clearly, $a \in B \subseteq A$. It follows that $\mathsf{card}(B)^{\mathrm{V}} < \kappa$, because otherwise, κ would be collapsed as a cardinal in $\mathrm{V}[G]$ (there would be an injection from κ into B in V, and an injection from B into $\delta < \kappa$ in $\mathrm{V}[G]$ - the composition of these would inject κ into δ, making κ not a cardinal in $\mathrm{V}[G]$). □

It is easy to see that the argument goes through even if it is only assumed that $\mathbb{P}$ is $<\kappa$-OD.

Lemma 6.4 ([6, Lemma 3])**.** *Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over* V*. Then*

$$<\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}.$$

7. Making the least leap the successor of a singular without large cardinals

It was shown in [6, Theorem 16] that if one adds a Příkrý sequence to the measurable cardinal κ of $L[U]$, then in the forcing extension, κ^+ is the least leap, so it is the successor of a singular cardinal in that model. We have seen that the consistency strength of a measurable cardinal cannot be dropped in this result. Let me show that no large cardinals are needed to produce a model in which the least leap is the successor of a singular (not strong limit) cardinal. First, let's achieve this just using Cohen forcing. I'll begin by establishing some terminology and observations on products of forcing notions.

Definition 7.1. Let $\mathbb{P}$ be a forcing poset, and let κ, λ be cardinals. Then $\mathbb{P}^\lambda$ *with $<\kappa$ support* is the poset consisting of functions $p : \lambda \longrightarrow \mathbb{P}$ such that for all but less than κ many $i < \lambda$, $p(i) = 1_{\mathbb{P}}$. The support of $p \in \mathbb{P}^\lambda$, denoted support(p), is the set of $i < \lambda$ such that $p(a) \neq 1_{\mathbb{P}}$. Finite support means $<\omega$ support. The ordering on $\mathbb{P}^\lambda$ is the ordering of $\mathbb{P}$ in each coordinate, that is, for $p, q \in \mathbb{P}^\lambda$, $p \leq_{\mathbb{P}^\lambda} q$ iff for each $i < \lambda$, $p(i) \leq_{\mathbb{P}} q(i)$.

In this situation, let $g : \lambda \longrightarrow \lambda$ be a bijection. Then g induces an automorphism π_g of $\mathbb{P}^\lambda$, defined by

$$\pi_g(p) = p \circ g^{-1}$$

Note that the support of $\pi_g(p)$ is the pointwise image of the support of p under g.

For $i < \lambda$, let Γ_i be the canonical $\mathbb{P}^\lambda$-name for the i-th slice of the $\mathbb{P}^\lambda$-generic filter. Any automorphism of $\mathbb{P}^\lambda$ induces a transformation of $\mathbb{P}^\lambda$-names, and I will make no notational difference between the automorphism and the induced transformation. Clearly, $\pi_g(\Gamma_i) = \Gamma_{g(i)}$.

Now suppose $a, b \subseteq \lambda$ are disjoint sets of the same cardinality, and let $f : a \longrightarrow b$ be a bijection. Then f determines a bijection $f^+ : \lambda \longrightarrow \lambda$ defined by letting $f^+{\restriction}a = f$, $f^+{\restriction}b = f^{-1}$, and $f^+{\restriction}(\lambda \setminus (a \cup b))$ is the identity. Note that $f^+ = (f^+)^{-1}$. The *coordinate-swapping automorphism* of $\mathbb{P}^\lambda$ induced by f is the automorphism π_{f^+}. I will also denote it by π_f. This automorphism swaps the coordinates in a with the coordinates in b, as prescribed by f.

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Lemma 7.2. *Let $\mathbb{P}$ be a forcing poset, $\kappa \leq \lambda$ infinite cardinals, and let $\mathbb{Q} = \mathbb{P}^\lambda$ with $<\kappa$ support. Let G be generic for $\mathbb{Q}$. Let $p \in \mathbb{Q}$ be some fixed condition with support a, and let $I \subseteq \lambda$ be a subset of cardinality at least κ in V. Then there is a coordinate swapping automorphism $\pi = \pi_f$ of $\mathbb{Q}$, where $f : a \longrightarrow b$ is a bijection, $a \cap b = \emptyset$, $\pi \in \mathrm{V}$, $\pi(p) \in G$, $b \subseteq I$.*

Proof. Let D be the set of conditions q in $\mathbb{P}^\lambda$ such that there are a $b \subseteq I$, b disjoint from a, and a bijection $f : a \longrightarrow b$ such that $q \leq \pi_f(p)$. Then D is dense in $\mathbb{P}^\lambda$, because given a condition r, say with support c, we can find $b \subseteq I$ with $b \cap (a \cup c) = \emptyset$ and $\mathsf{card}(b) = \mathsf{card}(a)$, as $\mathsf{card}(a), \mathsf{card}(c) < \kappa \leq \mathsf{card}(I)$. Letting f be any bijection between a and b, it follows that the support of $\pi_f(p)$ is b, hence disjoint from c, the support of r. Hence, $\pi_f(p)$ and r are compatible. Any common extension q is in D and below r, as wished. By genericity, there is a $q \in G \cap D$. Letting $f : a \longrightarrow b$ witness this, $\pi = \pi_f$ is as wished. □

Theorem 7.3. *Assume $\mathrm{V} = L$, say, and let λ be a singular limit cardinal of uncountable cofinality. Let G be generic for $\mathbb{P} = \mathrm{Add}(\omega, \lambda)$, the forcing to add λ Cohen reals. Then $\mathrm{V}[G]$ has the same cardinals as V, and in $\mathrm{V}[G]$, λ is a singular limit cardinal and the least leap is λ^+.*

Proof. Since $\mathbb{P}$ is ccc, it preserves cardinals and cofinalities, so λ is a singular limit cardinal in $L[G]$, and since $\mathbb{P}$ is weakly homogeneous and ordinal definable in L, $\mathsf{HOD}^{L[G]} = L$.

Let's show that $<\lambda\text{-}\mathsf{HOD}^{L[G]} = \mathsf{HOD}^{L[G]}$, that is, that there are no leaps less than or equal to λ in $L[G]$. Assume towards a contradiction that $\mathsf{HOD}^{L[G]} \subsetneq <\lambda\text{-}\mathsf{HOD}^{L[G]}$. By Lemma 3.7, let a, b be such that $a \in <\lambda\text{-}\mathsf{HOD}^{L[G]} \setminus \mathsf{HOD}^{L[G]}$, $a \subseteq b \in \mathsf{HOD}^{L[G]}$ and $\beta = \mathsf{card}(b) < \lambda$. We may assume that $\beta \geq \omega$. By replacing a and b with their pointwise images under a bijection in $\mathsf{HOD}^{L[G]}$ between b and $\beta = \mathsf{card}(b)$, we may also assume that $a \subseteq \beta = b$ (using that $L = \mathsf{HOD}^{L[G]}$ and $L[G]$ have the same cardinals).

Now let $\dot{a} \in \mathrm{V}$ be such that $\dot{a}^G = a$ and $\dot{a}$ is a nice name for a subset of β. Say $\langle A_\alpha \mid \alpha < \beta \rangle$ is a sequence of antichains in $\mathbb{P}$ such that $\dot{a} = \bigcup_{\alpha<\beta} \{\check{\alpha}\} \times A_\alpha$. Since $\mathbb{P}$ has the c.c.c., each A_α is countable. Let us view $\mathbb{P}$ as the finite support product of λ copies of the Cohen forcing $\mathrm{Add}(\omega, 1)$. For $\alpha < \beta$, let $I_\alpha = \bigcup_{p \in A_\alpha} \mathrm{support}(p)$, and let $I = \bigcup_{\alpha<\beta} I_\alpha$. Each I_α is a countable union of finite sets, hence countable, so that I has cardinality at most β.

Since $a \in <\lambda\text{-}\mathsf{HOD}^{L[G]}$, we may fix a $<\lambda$-blurry ordinal definition A of a, say $A = \{x \mid \psi(x, \rho)\}^{L[G]}$. We may also fix a condition $p \in G$ which forces that $\dot{a} \subseteq \check{\beta}$ is not in L, and that $\psi(\dot{a}, \check{\rho})$ holds.

Working in L, let $\lambda = \bigcup_{\alpha<\lambda} T_\alpha$ be a partition, with $I \cup \mathrm{support}(p) \subseteq T_0$, and each T_α of the same cardinality as T_0, but at least ω.

Now, for every $\alpha < \lambda$, $\alpha > 0$, there is a coordinate-swapping isomorphism π_α of $\mathbb{P}^\lambda$ that's based on a bijection $f_\alpha : T_0 \longrightarrow T_\alpha$ and such that $\pi_\alpha(p) \in G$: since T_α is infinite, Lemma 7.2 applies, showing that there is such an isomorphism based on a bijection between $\mathrm{support}(p)$ and some subset of T_α of the same size as $\mathrm{support}(p)$, but if one expands this bijection to a bijection between T_0 and T_α, then it will still have the desired property.

Writing $\mathbb{P}^{T_\alpha}$ for the restriction of $\mathbb{P}^\lambda$ to the coordinates in T_α, clearly these posets are isomorphic, and G_{T_α}, the restriction of G to those coordinates, is $\mathbb{P}^{T_\alpha}$-generic.

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Since $\pi_\alpha(p) \in G$, it follows that $a_\alpha = \pi_\alpha(\dot{a})^G = \pi_\alpha(\dot{a})^{G_{T_\alpha}}$ is a subset of β not in L and that $\psi(a_\alpha, \rho)$ holds in $L[G]$, because this was forced by p about $\dot{a}$.

If $\alpha < \beta < \lambda$, then $a_\alpha \neq a_\beta$, because if we had that $c = a_\alpha = a_\beta$, then we'd get that $c \in L[G_{T_\alpha}] \cap L[G_{T_\beta}] \subseteq L$, as G_{T_α} and G_{T_β} are mutually generic (see [22, Lemma 2.5 + the following remark]), a contradiction.

But since $\{a_\alpha \mid \alpha < \lambda\} \subseteq A$, this means that A has cardinality at least λ in $L[G]$. This is a contradiction.

To complete the proof, it remains to see that λ^+ is a leap in $L[G]$. But in $L[G]$, the cardinality of $\mathcal{P}(\omega)$ is λ (it is crucial here that $\mathrm{cf}(\lambda)$ is uncountable). So in $L[G]$, the OD set $\mathcal{P}(\omega)$ can serve as a $<\lambda^+$-blurry ordinal definition of every real. It follows that $\mathcal{P}(\omega)^{L[G]} \subseteq {<}\lambda^+\text{-}\mathsf{HOD}^{L[G]}$. Since there clearly are reals in $L[G]$ that don't belong to $L = \mathsf{HOD}^{L[G]}$, this shows that in $L[G]$, $\Lambda_0 = \lambda^+$. □

This theorem raises the question whether it is possible to arrange that the least leap is the successor of a singular cardinal with countable cofinality. It turns out that this can be done by a very similar technique, but using as the base forcing a poset constructed in L by Jensen [12], and subsequently slightly modified by Kanovei and Lyubetsky [14]. Following [14, p. 346], let's call this forcing $\mathbb{P}$. The forcing $\mathbb{P}$ is ccc, but the crucial realization of Kanovei and Lyubetsky was that $\mathbb{P}^{<\omega}$, the product of ω copies of $\mathbb{P}$ with finite support, is also ccc. Moreover, another property of $\mathbb{P}$ carries over to $\mathbb{P}^{<\omega}$ as well: Jensen had shown that his forcing adds a real number which is the unique generic real for the forcing in its extension. The same is true of $\mathbb{P}^{<\omega}$ in a sense made precise below. If $G \subseteq \mathbb{P}^{<\omega}$ is a generic filter, then G_i denotes the set of i-th components of conditions in G. The forcing $\mathbb{P}$ consists of perfect subtrees of ${}^{<\omega}2$, ordered by inclusion, and the filter G_i, generic for $\mathbb{P}$, corresponds to a real number x^{G_i}, the binary sequence of length ω which is a branch through every perfect tree in G_i, or $\bigcup(\bigcap G_i)$. Such a real number is called a $\mathbb{P}$-generic real.

Theorem 7.4 (Kanovei & Lyubetsky [14, Lemma 29])**.** *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over L, then in $L[G]$, the set of $\mathbb{P}$-generic reals over L is exactly $\{x^{G_i} \mid i < \omega\}$.*

This can be used to solve our problem.

Theorem 7.5. *Assume $\mathrm{V} = L$, and let λ be any infinite cardinal. Let $\mathbb{P}^{<\lambda}$ be the finite support product of λ copies of $\mathbb{P}$. If $G \subseteq \mathbb{P}^{<\lambda}$ is generic over L, then $L[G]$ has the same cardinals and cofinalities as L, and in $L[G]$, $\Lambda_0 = \lambda^+$.*

Proof. First, it is easy to see that $\mathbb{P}^{<\lambda}$ is ccc: suppose $A \subseteq \mathbb{P}^{<\lambda}$ is an antichain of size $\aleph_1$. Find a subset $\bar{A} \subseteq A$ of size $\aleph_1$ such that the set of supports of conditions in $\bar{A}$ forms a Δ-system with some root $r \in [\lambda]^n$, for some $n < \omega$. Then the set of restrictions of conditions in $\bar{A}$ to r gives rise to an uncountable antichain in the n-fold product of $\mathbb{P}$, contradicting the fact shown by Kanovei and Lyubetsky that $\mathbb{P}^{<\omega}$ is ccc.

So $\mathbb{P}^{<\lambda}$ preserves cardinals and cofinalities.

Next, the argument of the proof of Theorem 7.3 shows that ${<}\lambda\text{-}\mathsf{HOD}^{L[G]} = \mathsf{HOD}^{L[G]} = L$.

Now let's show that λ^+ is a leap (and hence the least leap) in $L[G]$. This will be achieved by showing that x^{G_i} is in ${<}\lambda^+\text{-}\mathsf{HOD}^{L[G]}$, for each $i < \lambda$.

The main point is that $\{x^{G_i} \mid i < \lambda\}$ is exactly the set of $\mathbb{P}$-generic reals in $L[G]$. To see this, let y be a $\mathbb{P}$-generic real in $L[G]$. Let $y = \dot{y}^G$, where $\dot{y}$ is a nice

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name for a subset of ω. Since $\mathbb{P}^{<\lambda}$ is ccc, the set $I \subseteq \lambda$ of coordinates $\dot{y}$ refers to is countable. If I is finite, let's add some coordinates to make it countably infinite. Letting G_I be the restriction of G to coordinates in I, and letting $\mathbb{P}^I$ be the finite support product of $\mathbb{P}$ over I, $y = \dot{y}^G = \dot{y}^{G_I} \in L[G_I]$. But since G_I is generic for $\mathbb{P}^I$, which is isomorphic to $\mathbb{P}^{<\omega}$, Theorem 7.4 applies, showing that $y = x^{G_i}$, for some $i \in I$.

Since $\mathbb{P} \in L$, $\mathbb{P}$ is ordinal definable in $L[G]$, and thus

$$\{x \in {}^{\omega}2 \mid x \text{ is } \mathbb{P}\text{-generic over } L\}$$

is a $<\lambda^+$-blurry ordinal definition of x_i^G, for every $i < \lambda$. Thus, $\{x^{G_i} \mid i < \lambda\} \subseteq$ $<\lambda^+$-$\mathsf{HOD}^{L[G]}$.

So λ^+ is a leap in $L[G]$, and it is the least one. □

8. Making $\aleph_\omega$ a strong limit leap whose successor is a leap

I turn now to the problem of making $\aleph_\omega$ a strong limit cardinal whose successor cardinal is a leap. As a warm-up, I'll arrange that $\aleph_\omega$ is also a limit of leaps. The method will be refined in the next section to arrange that there are no leaps below $\aleph_\omega$.

Definition 8.1. Let κ be a regular cardinal, and let α be an ordinal. By $\mathrm{Col}(\kappa, \alpha)$, I denote the poset consisting of partial functions from κ to α of cardinality less than κ, ordered by inclusion. For a cardinal λ, $\mathrm{Col}(\kappa, <\lambda)$ is the product $\prod_{\alpha<\lambda} \mathrm{Col}(\kappa, \alpha)$, taken with $<\kappa$ support.

For concreteness, $\mathrm{Col}(\kappa, <\lambda)$ consists of all functions f with $\mathsf{card}(f) < \kappa$, $\mathrm{dom}(f) \subseteq \{\langle \alpha, \beta \rangle \mid \beta < \kappa \le \alpha < \lambda\}$, such that for all $\langle \alpha, \beta \rangle \in \mathrm{dom}(f)$, $f(\alpha, \beta) < \alpha$. The ordering is reverse inclusion.

Thus, $\mathrm{Col}(\kappa, <\lambda)$ adds surjections from κ onto α, for each $\alpha < \lambda$. If λ is inaccessible, then after forcing with this poset, λ will be the cardinal successor of κ.

Theorem 8.2. *Assume* $\mathrm{V} = L[U]$*,* U *the normal ultrafilter on* κ*. There is a forcing poset that forces that* $\aleph_\omega$ *is a strong limit cardinal, a limit of leaps, and* $\aleph_{\omega+1}$ *is a leap.*

Proof. Let $C = \{c_n \mid n < \omega\}$ be $\mathbb{P}_U$-generic, such that $c_0 = \omega_1$ and for every $n > 0$, c_n is inaccessible. In $\mathrm{V}[C]$, consider the full support product

$$\mathbb{Q} = \prod_{n<\omega} \mathrm{Col}(c_{2n}, <c_{2n+2})$$

Let $\dot{\mathbb{Q}}$ be a canonical $\mathbb{P}_U$-name for $\mathbb{Q}$, so that the trivial condition of $\mathbb{P}_U$ forces that $\dot{\mathbb{Q}}$ satisfies the definition given, and $\mathbb{Q} = \dot{\mathbb{Q}}^C$. Let G be $\mathbb{Q}$-generic over $\mathrm{V}[C]$.

Then $\mathrm{V}[C][G]$ is our model. In $\mathrm{V}[C][G]$, $\aleph_\omega = \kappa$, and $(\kappa^+)^{\mathrm{V}} = (\kappa^+)^{\mathrm{V}[C][G]}$.

(1) In $\mathrm{V}[C][G]$, $\kappa^+ = \aleph_{\omega+1}$ is a leap.

Proof of (1). We know that C is, in $\mathrm{V}[C]$, one of κ many Příkrý sequences over $L[U]$. Since G does not add new ω-sequences, this $<\kappa^+$-blurry ordinal definition of C still works in $\mathrm{V}[C][G]$. So $C \in (<\kappa^+\text{-}\mathsf{HOD})^{\mathrm{V}[C][G]}$. We have to show that $C \notin (<\kappa\text{-}\mathsf{HOD})^{\mathrm{V}[C][G]}$. Assume the contrary. In $\mathrm{V}[C][G]$, let A and $\gamma < \kappa$ be such that $C \in A$, where $A = \{x \mid \varphi(x, \xi)\}$ and $\mathsf{card}(A) = \gamma$. Arguing in $\mathrm{V}[C]$, $\mathbb{Q}$ is cone-homogeneous, so that the statement "$\varphi(\check{C}, \check{\xi})$" is forced by $1_{\mathbb{Q}}$. Let $\dot{C}$ be the

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canonical name for the Příkrý sequence. So there is a condition $p_0 = \langle s_0, A_0\rangle \in G_C$ (the $\mathbb{P}_U$-generic filter corresponding to C) such that p_0 forces over V that in every further forcing extension by $\dot{Q}$, $\varphi(\dot{C}, \check{\xi})$ holds. We may assume that A_0 consists of inaccessible cardinals.

Let $A_0^* = A_0 \cap \{\alpha < \kappa \mid \alpha \text{ is a limit point of } A_0\}$. Then $A_0^* \in U$. Let D be a Příkrý sequence over V with $p_0^* = \langle s_0, A_0^*\rangle \in G_D$. Everything said about C above is true of D now.

For $\nu < \kappa$ with $\nu \in A^* \setminus \{d_{2n} \mid n < \omega\}$, $\nu > d_0$, let $n(\nu)$ be such that $d_{2n(\nu)} < \nu < d_{2n(\nu)+2}$, and define $d^\nu(m) = d(m)$ for $m < \omega$ unless $m = 2n(\nu) + 1$, in which case we let $d^\nu(m) = \nu$. Observe that $D^\nu = \{d^\nu_m \mid m < \omega\}$ is a Příkrý sequence over V and $\mathrm{V}[D] = \mathrm{V}[D^\nu]$. Moreover,

$$\dot{\mathbb{Q}}^{G_D} = \dot{\mathbb{Q}}^{G_{D^\nu}}.$$

Fix G' generic over $\mathrm{V}[D]$ for this forcing. Since $p_0 \in G_{D^\nu}$ we have that

$$\mathrm{V}[D][G'] = \mathrm{V}[D^\nu][G'] \models \varphi(D^\nu, \xi).$$

This is true for every ν as above, so for κ many $\nu < \kappa$. This is a contradiction, since the forcing $\mathbb{P}_U * \dot{\mathbb{Q}}$ preserves κ as a cardinal. □

(2) In $\mathrm{V}[C][G]$, $\aleph_\omega$ is a limit of leaps.

Proof of (2). For every $n \in \omega$, the n-th coordinate of G, let's call it G_n, codes a sequence $\langle F^n_\alpha \mid \alpha < c_{2n+2}\rangle$ such that $F^n_\alpha : c_{2n} \longrightarrow \alpha$ is surjective. This sequence can clearly be coded by a subset of c_{2n+2}, (which is $\aleph_{n+2}^{\mathrm{V}[C][G]}$). Thus, G_n is in $({<}c^{++}_{2n+2}\text{-}\mathsf{HOD})^{\mathrm{V}[C][G]}$ (that is, $({<}\aleph_{n+4}\text{-}\mathsf{HOD})^{\mathrm{V}[C][G]}$. Now if in $\mathrm{V}[C][G]$, the leaps were bounded below $\aleph_\omega = \kappa$, say by γ, then for every $n < \omega$, G_n would be in ${<}\gamma\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$. Now pick n so that $c_{2n} > \gamma$. Since $\mathbb{Q}{\restriction}[n, \omega]$ is c_{2n}-closed in $\mathrm{V}[C]$ and cone-homogeneous, it follows that ${<}\gamma\text{-}\mathsf{HOD}^{\mathrm{V}[C][G\restriction[n,\omega)]} \subseteq \mathrm{V}[C]$. So $G_n \notin {<}\gamma\text{-}\mathsf{HOD}^{\mathrm{V}[C][G\restriction[n,\omega)]}$. But then, G_n cannot be in ${<}\gamma\text{-}\mathsf{HOD}^{\mathrm{V}[C][G\restriction[n,\omega)][G\restriction[0,n)]}$ either, by applying Observation 6.3 in $\mathrm{V}[C][G{\restriction}[n, \omega)]$ to the forcing $\mathbb{Q}{\restriction}n$. Note here that $\mathbb{Q}{\restriction}n$ is definable in $\mathrm{V}[C][G{\restriction}[n, \omega)]$ from a finite set of ordinals. It is important here that for $\gamma < \delta < c_{2n}$, $\mathrm{Col}(\gamma, {<}\delta)^{\mathrm{V}[C]} = \mathrm{Col}(\gamma, {<}\delta)^{\mathrm{V}[C][G\restriction[n,\omega)]}$, and that γ is preserved as a cardinal by $\mathbb{Q}{\restriction}n$. □

□

9. Making $\aleph_\omega$ a strong limit cardinal and $\aleph_{\omega+1}$ the least leap

Fix a measurable cardinal κ and a normal ultrafilter U on κ. The strategy is going to be similar to that of the previous subsection: add a Příkrý sequence, then collapse between the even indexed Příkrý points. Unfortunately, this introduces leaps below κ. So as a last step, I will encode the collapsing functions into the continuum function above κ.

To have better control over the collapsing functions, I will use Magidor's idea from [16] of adding a Příkrý sequence while simultaneously collapsing between the Příkrý points. But the forcing needs to be adjusted so that it collapses between the even indexed Příkrý points only.

Another difference is that I merely work with a measurable cardinal, while Magidor's construction was designed to work with cardinals with potentially higher degrees of supercompactness. Namely, the use of Magidor's original forcing was to produce a model where $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \aleph_{\omega+k}$, for a fixed

$k < \omega$ with $k \geq 2$, thus violating the singular cardinal hypothesis. The starting assumption in the original setting was a cardinal κ that's $\kappa^{+(k-1)}$-supercompact.

9.1. **Příkrý forcing with interleaving alternating collapses á la Magidor.** The development follows Magidor's exposition [16] very closely. See Definition 8.1 for the meaning of $\mathrm{Col}(\kappa, {<}\lambda)$.

Definition 9.1. For a set $A \subseteq \mathrm{On}$, denote by $[A]^2$ the collection of two-element subsets of A. When I write $\{\alpha, \beta\} \in [A]^2$, it is understood that $\alpha < \beta$.

$\mathbb{M}$ is the forcing notion consisting of conditions of the form

$$\pi = \langle \langle \kappa_i \mid i < l \rangle, \langle f_i \mid i < l, \ i \text{ is even} \rangle, A, F \rangle$$

with the following properties:

(1) $2 \leq l \in \omega$ is an even number, the *length* of π, denoted $\mathrm{lh}(\pi)$.
(2) The sequence $\langle \kappa_i \mid i < l \rangle$ is strictly increasing, $\kappa_0 = \omega$, and for $0 < i < l$, κ_i is inaccessible. Note that $l - 1$, the largest index for which κ_i is defined, is *odd*.
(3) For even i with $i + 2 < l$,

$$f_i \in \mathrm{Col}(\kappa_i^+, {<}\kappa_{i+2}).$$

(4) $f_{l-2} \in \mathrm{Col}(\kappa_{l-2}, {<} \min(A))$.
(5) $A \in U$, and for all $\alpha \in A$, α is inaccessible.
(6) For $i < l$, $\kappa_i < \min(A)$.
(7) F is a function with domain $[A]^2$, and if $\alpha < \beta < \gamma$ and $\alpha, \beta, \gamma \in A$, then $F(\{\alpha, \beta\}) \in \mathrm{Col}(\alpha^+, {<}\gamma)$. I will often write $F(\alpha, \beta)$ instead of $F(\{\alpha, \beta\})$.

I write $\vec{\kappa}^\pi = \langle \kappa_i \mid i < \mathrm{lh}(\pi) \rangle$, $\vec{f}^\pi = \langle f_i \mid i < \mathrm{lh}(\pi) \text{ is even} \rangle$, $A^\pi = A$, $F^\pi = F$.

For future reference, if $j \leq l$ is even, then I write

$$\pi{\restriction}j = \langle \langle \kappa_i \mid i < j \rangle, \langle f_i \mid i < j, i \text{ even} \rangle \rangle (= \langle \vec{\kappa}^\pi{\restriction}j, \vec{f}^\pi{\restriction}j \rangle).$$

Note that $\pi{\restriction}j$ is not in $\mathbb{M}$.

The ordering of $\mathbb{M}$ is defined as follows. Let π be as above, and let

$$\pi' = \langle \langle \kappa'_i \mid i < l' \rangle, \langle f'_i \mid i < l' \text{ is even} \rangle, A', F' \rangle \in \mathbb{M}.$$

Then $\pi' \leq \pi$ if:

(1) $l \leq l'$.
(2) For $i < l$, $\kappa_i = \kappa'_i$.
(3) For $l \leq i < l'$, $\kappa'_i \in A$.
(4) For all even $i < l$, $f_i \subseteq f'_i$.
(5) For all even $i \in [l, l')$, $F(\kappa'_i, \kappa'_{i+1}) \subseteq f'_i$.
(6) $A' \subseteq A$.
(7) For all $\{\alpha, \beta\} \in [A']^2$, $F(\alpha, \beta) \subseteq F'(\alpha, \beta)$.

In this situation, if $j \leq l$, then π' is a *j-direct extension of* π ($\pi' \leq^j \pi$) if $\pi' \leq \pi$ and

(1) For all even $i \in [j, l)$, $f'_i = f_i$.
(2) For all even $i \in [l, l')$, $f'_i = F(\kappa'_i, \kappa'_{i+1})$.
(3) $A' = A \setminus (\kappa'_{l'-1} + 1)$ and $F' = F{\restriction}[A']^2$.

And π' is a *j-length preserving extension of* π ($\pi' \leq_j \pi$) if $\pi' \leq \pi$ and

(1) $j \leq l = l'$.
(2) For all even $i < j$, $f'_i = f_i$.

The poset $\mathbb{M}$, as defined, has no weakest condition. One could of course add such a condition, but I chose not to, because it seemed to introduce notational complications.

Note that if $\pi' \le \pi$, where the components of π and π' are named as in the previous definition, and if $j \le l$, then there is a unique "interpolant" $\tilde{\pi}$ such that

$$\pi' \le_j \tilde{\pi} \le^j \pi.$$

Namely,

$$\tilde{\pi} = \langle\langle \tilde{\kappa}_i \mid i < \tilde{l}\rangle, \langle \tilde{f}_i \mid i < \tilde{l} \text{ is even}\rangle, \tilde{A}, \tilde{F}\rangle$$

where

(1) $\tilde{l} = l'$.
(2) $\tilde{\kappa}_i = \kappa'_i$ for $i < \tilde{l}$.
(3) $\tilde{f}_i = f'_i$ for even $i < j$.
 [this all to ensure that $\pi' \le_j \tilde{\pi}$]
(4) $\tilde{A} = A \setminus (\tilde{\kappa}_{\tilde{l}-1} + 1)$ and $\tilde{F} = f{\restriction}[\tilde{A}]^2$.
(5) For all even $i \in [j, l)$, $\tilde{f}_i = f_i$.
(6) For all even $i \in [l, \tilde{l})$, $\tilde{f}_i = F(\tilde{\kappa}_i, \tilde{\kappa}_{i+1})$.
 [these least three points to ensure that $\tilde{\pi} \le^j \pi$]

Let's introduce some convenient notation.

Definition 9.2. Let $\pi, \tilde{\pi} \in \mathbb{M}$ with $\tilde{\pi} \le \pi$. Let $\vec{\lambda}$ be such that $\vec{\kappa}^{\tilde{\pi}} = \vec{\kappa}^{\pi} {}^\frown \vec{\lambda}$ (that is, the sequence $\vec{\kappa}^{\tilde{\pi}}$ is the sequence $\vec{\kappa}^{\pi}$ extended by $\vec{\lambda}$). Let $m = \mathrm{lh}(\vec{\lambda})$, noting that m is even. Then there is a weakest extension π' of π such that $\vec{\kappa}^{\pi'} = \vec{\kappa}^{\pi} {}^\frown \vec{\lambda}$. I write $\pi {}^\frown \vec{\lambda}$ for this condition. It is defined by

- $\mathrm{lh}(\pi') = \mathrm{lh}(\pi) + m$.
- $\vec{\kappa}^{\vec{\pi}'} = \vec{\kappa}^{\pi} {}^\frown \vec{\lambda}$.
- $\vec{f}^{\pi'}{\restriction}\mathrm{lh}(\pi) = \vec{f}^{\pi'}$.
- For $i < m$ even, $f^{\pi'}_{\mathrm{lh}(\pi)+i} = F^{\pi}(\lambda_i, \lambda_{i+1})$.
- $A^{\pi'} = A^{\pi} \setminus (\lambda_{m-1} + 1)$ if $m > 0$, $A^{\pi'} = A^{\pi}$ otherwise.
- $F^{\pi'} = F^{\pi}{\restriction}[A^{\pi'}]^2$.

Further, if $\vec{g} = \vec{f}^{\tilde{\pi}}{\restriction}j$, where $j \le \mathrm{lh}(\pi)$ is even, then I write $\pi[\vec{g}]$ for the condition σ such that

- $\mathrm{lh}(\sigma) = \mathrm{lh}(\pi)$.
- $\vec{\kappa}^{\sigma} = \vec{\kappa}^{\pi}$.
- $\vec{f}^{\sigma}{\restriction}j = \vec{g}$.
- $\vec{f}^{\sigma}{\restriction}[j, \mathrm{lh}(\pi)) = \vec{f}^{\pi}{\restriction}[j, \mathrm{lh}(\pi))$.
- $A^{\sigma} = A^{\pi}$.
- $F^{\sigma} = F^{\pi}$.

Observation 9.3. *In the situation of the previous definition, if $\pi'' \le \pi$ is such that $\vec{\kappa}^{\pi''} \supseteq \vec{\kappa}^{\pi} {}^\frown \vec{\lambda}$ and for each even $i < j$, $f^{\pi''}_i \supseteq g_i$, then*

$$\pi'' \le (\pi {}^\frown \vec{\lambda})[\vec{g}]$$

Observation 9.4. *In the situation of the previous definition, the j-interpolant of π and $\tilde{\pi}$ is*

$$(\pi {}^\frown \vec{\lambda})[\vec{f}^{\tilde{\pi}}{\restriction}j]$$

which is the same as $(\pi[\vec{f}^{\tilde{\pi}}{\restriction}j]) {}^\frown \vec{\lambda}$.

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Since the length of a condition is finite and can be increased by extending the condition, $\mathbb{M}$ is not countably closed. If a longer decreasing sequence of conditions $\langle \pi_\xi \mid \xi < \lambda \rangle$ (with limit λ) is to have a lower bound, then the lengths of the conditions π_ξ have to stabilize, i.e., be ultimately constant. So let us focus on this case.

Observation 9.5. *Let $\langle \pi_\xi \mid \xi < \lambda \rangle$ be a $\leq_j$-decreasing sequence in $\mathbb{M}$, $j \leq l < \omega$, where l is the common length of the conditions in the sequence and j is even. Let $\kappa_j = \kappa_j^{\pi_0}$.*

If $j < l$, then assume that $\lambda < \kappa_j^+$, and if $j = l$, then assume that $\lambda < \kappa$.

Then there is a $\tilde{\pi} \in \mathbb{M}$ such that $\tilde{\pi} \leq_j \pi_\xi$, for all $\xi < \lambda$.

Proof. For $\xi < \lambda$, let

$$\pi_\xi = \langle \langle \kappa_i \mid i < l \rangle, \langle f_i^\xi \mid i < l \text{ even} \rangle, A_\xi, F_\xi \rangle$$

Define $\tilde{\pi} = \langle \langle \kappa_i \mid i \leq l \rangle, \langle \tilde{f}_i \mid i \leq l \text{ even} \rangle, \tilde{A}, \tilde{F} \rangle$, by cases.
Case 1: $j < l$.

In this case, set:

- $\tilde{A} = \bigcap_{\xi<\lambda} A_\xi$.
  Since $\lambda < \kappa$, $\tilde{A} \in U$.
- For $i < j$, i even, $\tilde{f}_i = f_i^0$.
- For $j \leq i < l$, i even,

$$\tilde{f}_i = \bigcup \{ f_i^\xi \mid \xi < \lambda \}$$

  Since each $f_i^\xi \in \mathrm{Col}(\kappa_i^+, {<}\theta)$, where θ is either κ_{i+2} or the minimum of $\tilde{A}$, so is $\tilde{f}_i$, as $\lambda < \kappa_j^+ \leq \kappa_i^+$ and $\mathrm{Col}(\kappa_i^+, {<}\theta)$ is ${<}\kappa_i^+$-closed.
- $\tilde{F} : [\tilde{A}]^2 \longrightarrow \mathrm{V}$ is defined by

$$\tilde{F}(\alpha, \beta) = \bigcup_{\xi<\lambda} F_\xi(\alpha, \beta).$$

  Note that $\lambda < \kappa_j^+ \leq \kappa_{l-1}^+ < \alpha$, since $\kappa_{l-1} < \min(\tilde{A})$. Since each $F_\xi(\alpha, \beta)$ is in $\mathrm{Col}(\alpha^+, {<}\gamma)$, where $\gamma = \min(\tilde{A} \setminus (\beta + 1))$, so is the union.

Case 2: $j = l$.

In this case, set:

- $\tilde{A} = (\bigcap_{\xi<\lambda} A_\xi) \setminus (\lambda + 1)$.
  Since $\lambda < \kappa$, $\tilde{A} \in U$.
- For $i < l$, i even, $\tilde{f}_i = f_i^0$.
- $\tilde{F} : [\tilde{A}]^2 \longrightarrow \mathrm{V}$ is defined by

$$\tilde{F}(\alpha, \beta) = \bigcup_{\xi<\lambda} F_\xi(\alpha, \beta).$$

  Note that if $\{\alpha, \beta\} \in [A]^2$, then $\lambda < \alpha$. Since each $F_\xi(\alpha, \beta)$ is in $\mathrm{Col}(\alpha^+, {<}\gamma)$, where $\gamma = \min(\tilde{A} \setminus (\beta + 1))$, so is the union.

□

Lemma 9.6. *Let $k \in \omega$ be even. Let τ be an $\mathbb{M}$-name for an ordinal. Let $\pi \in \mathbb{M}$, with $\mathrm{lh}(\pi) = l$. Let $j \leq l$ be even. Let η be the restriction to j of some extension of*

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π. Then there is a $\pi' \le_j \pi$ such that for every $\pi'' \le \pi'$ of length $l+k$, if π'' decides τ and $\pi''{\restriction}j = \eta$, then already the j-interpolant of π'' and π' decides τ.

In other words, in this situation, letting $\vec{\lambda}$ be such that $\vec{\kappa}^{\pi''} = \vec{\kappa}^{\pi'} {}^\frown \vec{\lambda}$, then $(\pi' {}^\frown \vec{\lambda})[\vec{f}^{\pi''}{\restriction}j]$ decides τ.

Proof. The proof is by induction on even k. The inductive statement is that the lemma holds for all π, j, η (but we may fix τ).

$\boxed{\text{k=0}}$

Let $\pi = \langle\langle\kappa_i \mid i < l\rangle, \langle f_i \mid i < l \text{ even}\rangle, A, F\rangle$, $j \le l$ even.

Case 1: there is a $\pi'' \le \pi$ of length $l = l+k$ such that $\pi''{\restriction}j = \eta$ and π'' decides τ.

In this case, letting $\pi'' = \langle\langle\kappa_i \mid i < l\rangle, \langle f''_i \mid i < l \text{ even}\rangle, A'', F''\rangle$, we can define

$$\pi' = \langle\langle\kappa_i \mid i < l\rangle, \langle f'_i \mid i < l \text{ even}\rangle, A'', F''\rangle,$$

where for $i < j$ even, $f'_i = f_i$ (which is dictated by the requirement that π' be a j-length preserving extension of π), for $j \le i < l$, i even, $f'_i = f''_i$, $A' = A''$ and $F' = F''$. Then π' is as wished: Clearly, π' is a j-length preserving extension of π. Now suppose $\tilde{\pi} \le \pi'$ has length l, decides τ, and has $\tilde{\pi}{\restriction}j = \eta$. So $\tilde{\pi}{\restriction}j = \pi''{\restriction}j$, and for $j \le i < l$, i even, $f_i^{\tilde{\pi}} \supseteq f'_i = f''_i$.

Suppose $j > 0$. Then the j-interpolant of π' and $\tilde{\pi}$ is

$$\langle\langle\kappa_i \mid i < l\rangle, \langle f''_i \mid i < j,\ i \text{ even}\rangle, A'', F''\rangle$$

which is the same as π'', and hence, it decides τ.

Case 2: Case 1 fails.

In this case, we can set $\pi' = \pi$. Then π' vacuously behaves as wished.

$\boxed{k \longrightarrow k+2}$

Again, let $\pi = \langle\langle\kappa_i \mid i < l\rangle, \langle f_i \mid i < l \text{ even}\rangle, A, F\rangle$, $j \le l$ even.

I will use the well-ordering $\prec$ of $[A]^2$ defined by $\{\alpha,\beta\} \prec \{\gamma,\delta\}$ iff $\beta < \delta$ or ($\beta = \delta$ and $\alpha < \gamma$). Since A consists of infinite cardinals, it is clear that the set of predecessors of $\{\gamma,\delta\}$ has cardinality at most δ; in fact, it is contained in the set of $a \in [A]^2$ with $\max(a) \le \delta$, which also has cardinality at most δ.

For $a = \{\alpha,\beta\} \in [A]^2$, I am going to define a condition π_a, which I will also denote by $\pi_{\alpha,\beta}$, of length $l+2$, such that the following conditions hold.

(1) $\pi_{\alpha,\beta}$ is of the form

$$\pi_{\alpha,\beta} = \langle\langle\kappa_0,\ldots,\kappa_{l-1},\alpha,\beta\rangle, \langle f_0,\ldots,f_{j-2}, f_j^{\alpha,\beta}, f_{j+2}^{\alpha,\beta},\ldots,f_{l-2}^{\alpha,\beta}, f^{\alpha,\beta}\rangle, B^{\alpha,\beta}, H^{\alpha,\beta}\rangle$$

(i.e., for $i < j$ even, $f_i^{\pi_{\alpha,\beta}} = f_i$, for $j \le i \le l-2$, i even, $f_i^{\pi_{\alpha,\beta}} = f_i^{\alpha,\beta}$ (to be defined), and $f_l^{\pi_{\alpha,\beta}} = f^{\alpha,\beta}$ (also to be defined). And $\kappa_l^{\pi_{\alpha,\beta}} = \alpha$, $\kappa_{l+1}^{\pi_{\alpha,\beta}} = \beta$).

(2) $\pi_{\alpha,\beta} \le \pi$.

(3) If $\{\alpha,\beta\} \prec \{\gamma,\delta\}$, then
   (a) $B^{\alpha,\beta} \supseteq B^{\gamma,\delta}$.
   (b) If $x \in [B^{\gamma,\delta}]^2$, then $H^{\alpha,\beta}(x) \subseteq H^{\gamma,\delta}(x)$.

(So $\pi_{\alpha,\beta}{\restriction}j = \pi{\restriction}j$.)

Note that since $\pi_{\alpha,\beta} \le \pi$, it must be that $F(\alpha,\beta) \subseteq f^{\alpha,\beta}$, and if $x \in B^{\alpha,\beta}$, then $F(x) \subseteq H^{\alpha,\beta}(x)$.

I will define $\pi_{\alpha,\beta}$ by $\prec$-recursion on $\{\alpha,\beta\}$. So assume π_b is already defined, for all $b \prec \{\alpha,\beta\}$.

In order to define $\pi_{\alpha,\beta}$ I will apply the inductive assumption to a preliminary condition $\chi_{\alpha,\beta}$, also of length $l+2$, which is defined as follows.

$$\chi_{\alpha,\beta} = \langle\langle\kappa_0,\ldots,\kappa_{l-1},\alpha,\beta\rangle,\langle f_0,f_2,\ldots,f_{l-2},g^{\alpha,\beta}\rangle,A^{\alpha,\beta},G^{\alpha,\beta}\rangle$$

(so that $f_l^{\chi_{\alpha,\beta}} = g^{\alpha,\beta}$), where

- $g^{\alpha,\beta} = \bigcup\{H^b(\alpha,\beta) \mid b \prec \{\alpha,\beta\} \text{ and } \{\alpha,\beta\} \in B^b\} \cup F(\alpha,\beta)$
  [Note: Let $P = \{b \prec \{\alpha,\beta\} \mid \{\alpha,\beta\} \in B^b\}$. P is downward closed wrt. $\prec$. Moreover, if $b \in P$, then $\max(b) < \alpha$, because it must be that $\max(b) < \min(B^b)$, and $\alpha \in B^b$. So the cardinality of P is at most α.
  If $P \neq \emptyset$, then let $b \in P$. By the note above, $F(\alpha,\beta) \subseteq H^b(\alpha,\beta)$. So in this case, we wouldn't have to explicitly add in $F(\alpha,\beta)$ as we did in the definition of $g^{\alpha,\beta}$. Moreover, in this case, we are looking at an increasing union of elements of $\mathrm{Col}(\alpha^+,{<}\kappa)$ of size at most α, so we get a condition in this poset.
  On the other hand, if $P=\emptyset$, then the first part of the union defining $g^{\alpha,\beta}$ is empty, and we get $g^{\alpha,\beta} = F(\alpha,\beta)$, also a valid condition in $\mathrm{Col}(\alpha^+,{<}\kappa)$.]
- $A^{\alpha,\beta} = \left(A \cap \bigcap_{b\prec\{\alpha,\beta\}} B^b\right) \setminus (\beta+1)$.
- For $x \in [A^{\alpha,\beta}]^2$, $G^{\alpha,\beta}(x) = \bigcup_{b\prec\{\alpha,\beta\}} H^b(x)$.

Note that $\chi_{\alpha,\beta} \leq \pi$.

Now apply the inductive assumption to the number k, the condition $\chi_{\alpha,\beta}$, which has length $l+2$, and the same j, η and τ. The inductive assumption guarantees the existence of a condition $\pi' \leq_j \chi_\alpha$ such that whenever $\pi'' \leq \pi'$ has $\pi''\restriction j = \eta$ and π'' decides τ, then already the j-interpolant of π' and π'' decides τ.

Define $\pi_{\alpha,\beta}$ to be this π'. It is of the form described above, since $\pi'' \leq_j \chi_\alpha$.

Having defined $\langle\pi_{\alpha,\beta} \mid \{\alpha,\beta\} \in [A]^2\rangle$, we are going to amalgamate all of these into one condition, which is going to be the condition π' we want.

By thinning out A, we can find $A_0 \subseteq A$ with $A_0 \in U$, such that $a \mapsto f_i^a$ is constant for $a \in [A_0]^2$, $j \leq i < l-2$, i even. This is because for each $a \in [A]^2$, f_i^a is a condition in $\mathrm{Col}(\kappa_i^+,{<}\kappa_{i+2})$, a poset of size less than κ. Note that this does not immediately work for $i = l-2$, because $f_{l-2}^{\alpha,\beta} \in \mathrm{Col}(\kappa_{l-2}^+,{<}\alpha)$. However, for every $a \in [A_0]^2$, there is an ordinal $\rho_a < \min(a)$ such that $f_{l-2}^a \in \mathrm{Col}(\kappa_{l-2}^+,{<}\rho_a)$. Since the function $a \mapsto \rho_a$ is regressive on $[A_0]^2$, there is an $A_1 \in U$, $A_1 \subseteq A_0$, such that it is constant on A_1, say with value ρ. Now, for all $a \in [A_1]^2$, $f_{l-2}^a \in \mathrm{Col}(\kappa_{l-2}^+,{<}\rho)$, and this poset has cardinality less than κ, so we can find a $B \subseteq A_1$ with $B \in U$, on which the function $a \mapsto f_{l-2}^a$ is also constant. Let $g_j,\ldots,g_{l-2}$ be the constant values of $f_j^a,\ldots,f_{l-2}^a$, for $a \in [B]^2$.

Define the condition π' of length l by

$$\pi' = \langle\langle\kappa_0,\ldots,\kappa_{l-1}\rangle,\langle f_0,f_2,\ldots,f_{j-2},g_j,g_{j+2},\ldots,g_{l-2}\rangle,C,H\rangle$$

where $C = B \cap \triangle_{a\in[A]^2} B^a$ and $H : [C]^2 \longrightarrow \mathrm{V}$ is defined by $H(a) = f^a$.

We claim that π' is as wished.

Note that for $a \in [C]^2$,

$$\bigcup\{H^b(a) \mid (b \in [A]^2) \wedge (b \prec a) \wedge (a \in [B^b]^2)\} \cup F(a) = g^a \subseteq f^a\}.$$

Let's check that π' has all the desired properties.

It's clear that $\pi' \leq_j \pi$.

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Now suppose $\pi'' \le \pi'$, $\mathrm{lh}(\pi'') = l + k + 2$, $\pi''{\restriction}j = \eta$ and π'' decides τ.

We have to show that already χ, the j-interpolant of π' and π'', decides τ. That is, letting $\vec{\kappa}^{\pi''} = \langle\kappa_0, \ldots, \kappa_{l-1}\rangle^\frown\vec{\lambda}$, and $\vec{h} = \vec{f}^{\pi''}{\restriction}j$,

$$\chi = (\pi'^\frown\vec{\lambda})[\vec{h}].$$

Note that $\vec{\lambda} = \langle\lambda_0, \ldots, \lambda_{k+1}\rangle$ is from C.

Note that since $\pi''{\restriction}j = \eta$, we have that $\eta = \langle\langle\kappa_0, \ldots, \kappa_{j-1}\rangle, \vec{h}\rangle$.

Let $a = \{\lambda_0, \lambda_1\}$.

We have that $\chi \le \pi_a$: since $\lambda_2, \ldots, \beta_{k+1} \in C$ and $\lambda_1 < \lambda_2$, it follows that $\beta_2, \ldots, \beta_k, \beta_{k+1} \in B^a$. To see that $f^a \subseteq f_l^\chi$, note that $f_l^\chi = H(\alpha, \beta_1) = f^a$. It remains to check that $H^a(x) \subseteq H^\chi(x)$ for all $x \in [C \setminus (\beta_1 + 1)]^2$. But for such x, $H^\chi(x) = H(x) = f^x \supseteq g^x \supseteq H^a(x)$ since $\max(a) = \beta_1 < \min(x)$.

So since $\pi'' \le \chi \le \pi_a$, we have that $\pi'' \le \pi_a$. The length of π'' is k more than the length of π_a, which is $l+2$, and $\pi''{\restriction}j = \eta$. Since π'' decides τ, by the properties of π_a, already the j-interpolant of π_a and π'', call it χ', decides τ.

We have:

- χ is the j-interpolant of π' and π''.
- χ' is the j-interpolant of π_a and π''.
- $\chi \le \pi_a$.

It follows that $\chi \le \chi'$: note that χ' is the weakest condition extending π_a whose $\vec{\kappa}$-sequence is $\vec{\kappa}^{\pi_a\frown}\langle\lambda_2, \ldots, \lambda_{k+1}\rangle$ and whose $\vec{f}$-sequence begins with $\vec{h}$. But since $\chi \le \pi_a$, χ is such a condition. As a result, $\chi \le \chi'$. So since χ' decides τ, so does χ (in the same way). □

Lemma 9.7. *Let τ be an $\mathbb{M}$-name for an ordinal. Let $\pi \in \mathbb{M}$, $j \le \mathrm{lh}(\pi)$ even. Let η be the restriction to j of some extension of π. Then there is a $\pi' \le_j \pi$ such that for every $\pi'' \le \pi'$, if π'' decides τ and $\pi''{\restriction}j = \eta$, then already the j-interpolant of π'' and π' decides τ.*

In other words, in this situation, letting $\vec{\lambda}$ be such that $\vec{\kappa}^{\pi''} = \vec{\kappa}^{\pi'\frown}\vec{\lambda}$, then $(\pi'^\frown\vec{\lambda})[\vec{f}^{\pi''}{\restriction}j]$ decides τ.

Proof. Define a sequence $\langle\pi_k \mid k < \omega \text{ even}\rangle$ by recursion on k such that

- $\pi_0 = \pi$.
- $\pi_{k+2} \le_j \pi_k$.
- Whenever $\pi'' \le \pi_{k+2}$ is such that $\pi''{\restriction}j = \eta$, $\mathrm{lh}(\pi'') = \mathrm{lh}(\pi_k) + k$, and π'' decides τ, then the j-interpolant of π_{k+2} and π'' decides τ.

π_{k+2} is defined by applying Lemma 9.6 to π_k, j and k. Note that all conditions π_k have the same length.

By Observation 9.5, let π' be a lower $\le_j$-bound of $\vec{\pi}$ in $\mathbb{M}$. We claim that π' is as wished. So suppose $\pi'' \le \pi'$ is such that $\pi''{\restriction}j = \eta$ and π'' decides τ. Let $k = \mathrm{lh}(\pi'') - \mathrm{lh}(\pi)$. Since $\pi'' \le \pi_{k+2}$, it follows that χ, the j-interpolant of π_{k+2} and π'', decides τ. But $\pi'' \le \pi' \le \pi_{k+2}$, so χ', the j-interpolant of π'' and π', is at least as strong as χ, the j-interpolant of π_{k+2} and π''. So χ' decides τ, as wished. □

Lemma 9.8. *Let τ be an $\mathbb{M}$-name for an ordinal. Let $\pi \in \mathbb{M}$, and let j be an even number such that $j = 0$ or $j < \mathrm{lh}(\pi)$. Then there is a $\pi' \le_j \pi$ such that for every $\pi'' \le \pi'$, if π'' decides τ, then already the j-interpolant of π'' and π' decides τ.*

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In other words, in this situation, letting $\vec{\lambda}$ *be such that* $\vec{\kappa}^{\pi''} = \vec{\kappa}^{\pi'}{}^\frown\vec{\lambda}$, *then* $(\pi'{}^\frown\vec{\lambda})[\vec{f}^{\pi''}\restriction j]$ *decides* τ.

Proof. Let $l = \mathrm{lh}(\pi)$.

Note that the lemma follows immediately from Lemma 9.7 if $j = 0$, using $\eta = \emptyset$. So let's assume that $0 < j < l$. Consider the set $X = \{\pi''\restriction j \mid \pi'' \leq \pi\}$. The cardinality of X is the same as the cardinality of $Y = \{\vec{f}^{\pi''}\restriction j \mid \pi'' \leq \pi\}$, and

$$Y \subseteq \mathrm{Col}(\kappa_0^+, {<}\kappa_2) \times \cdots \times \mathrm{Col}(\kappa_{j-2}^+, {<}\kappa_j).$$

Since κ_j is inaccessible, Y has cardinality at most κ_j. Recall that by Observation 9.5, any $\leq_j$-decreasing sequence in $\mathbb{M}$ starting with π, of length at most κ_j, has a lower bound in $\mathbb{M}$.

We construct such a sequence as follows. Let $\langle \eta_\xi \mid \xi < \kappa_j\rangle$ enumerate X. By recursion on ξ, define π_ξ such that

- $\pi_0 = \pi$.
- If $\xi < \zeta < \kappa_j$, then $\pi_\zeta \leq_j \pi_\xi$.
- Whenever $\pi'' \leq \pi_{\xi+1}$ is such that π'' decides τ and $\pi''\restriction j = \eta_\xi$, then already the j-interpolant of $\pi_{\xi+1}$ and π'' decides τ.

In the successor step of the construction, the existence of $\pi_{\xi+1}$ is guaranteed by applying Lemma 9.7 to π_ξ and η_ξ, and in the limit step, one uses Observation 9.5.

Having constructed the sequence $\vec{\pi}$, another application of Observation 9.5 yields the existence of a π' such that $\pi' \leq_j \pi_\xi$ for all $\xi < \kappa_j$. This π' is as wished: clearly, $\pi' \leq_j \pi$. And if $\pi'' \leq \pi'$ decides τ, then letting ξ be such that $\eta_\xi = \pi''\restriction j$, it follows that χ, the j-interpolant of $\pi_{\xi+1}$ and π'' decides τ, since $\pi'' \leq \pi_{\xi+1}$. But since $\pi' \leq \pi_{\xi+1}$, the j-interpolant of π' and π'' is at least as strong as χ and hence also decides τ. □

Corollary 9.9. *Let* $\pi \in \mathbb{M}$ *with* $\mathrm{lh}(\pi) = n$, *and let* j *be an even number such that* $j = 0$ *or* $j < n$. *Let* $\mu \leq (\kappa_j^\pi)^+$ *and let* $\dot{b}$ *be an* $\mathbb{M}$*-name such that* π *forces wrt.* $\mathbb{M}$ *that* $\dot{b} : \check{\mu} \longrightarrow \mathrm{On}$. *Then there is a* $\pi' \leq_j \pi$ *such that whenever* $\pi'' \leq \pi'$ *and* $\lambda < \mu$ *are such that for some* β, $\pi'' \Vdash_{\mathbb{M}} \dot{b}(\check{\lambda}) = \check{\beta}$, *then this is already forced by the* j*-interpolant of* π' *and* π''.

Proof. Construct a $\leq_j$-decreasing sequence $\langle \pi_\lambda \mid \lambda < \mu\rangle$ by setting $\pi_0 = \pi$, defining π_λ to be a lower bound of $\vec{\pi}\restriction\lambda$ in the case that λ is a limit (using Observation 9.5), and by applying Lemma 9.8 to π_λ and the name "$\dot{b}(\check{\lambda})$" to get $\pi_{\lambda+1}$. Let π' be a $\leq_j$-lower bound of $\vec{\pi}$. This π' is as wished. □

Definition 9.10. Let $G \subseteq \mathbb{M}$ be V-generic. Let $\vec{f}^G = \langle f_i^G \mid i < \omega \text{ even}\rangle$ where for even $i < \omega$, $f_i^G = \bigcup_{\pi\in G,\ i<\mathrm{lh}(\pi)} f_i^\pi$, and let $\vec{\kappa}^G = \bigcup_{\pi\in G}\vec{\kappa}^\pi$. Let $\dot{\vec{f}}$ and $\dot{\vec{\kappa}}$ be canonical names for $\vec{f}$ and $\vec{\kappa}$, respectively.

Clearly, if $G \subseteq \mathbb{M}$ is V-generic, then $\mathrm{V}[G] = \mathrm{V}[\vec{f}^G, \vec{\kappa}^G]$. The following is the version of [16, Thm. 3.2] for the current forcing.

Theorem 9.11. *Let* $p \in \mathbb{M}$ *be a condition of length* n, *let* $0 < j < n$ *be even,* $\mu < (\kappa_j^\pi)^+$ *and* $\dot{b}$ *an* $\mathbb{M}$*-name such that* π *forces that* $\dot{b}$ *is subset of* $\check{\mu}$. *Then* π *forces that* $\dot{b} \in \check{\mathrm{V}}[\dot{\vec{f}}\restriction j]$.

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Proof. This proof is basically identical to that of [16, Thm. 3.2]. I include it for the reader's convenience.

For $i < n$, let $\kappa_i = \kappa_i^\pi$, and write

$$\mathbb{P}_j = \mathrm{Col}(\kappa_0^+, {<}\kappa_2) \times \cdots \times \mathrm{Col}(\kappa_{j-2}^+, {<}\kappa_j)$$

where for convenience I'll index elements of $\mathbb{P}_j$ with even numbers between 0 and $j-2$, so members of $\mathbb{P}_j$ are of the form $\langle g_0, g_2, \ldots, g_{j-2}\rangle$ such that for $i < j$ even, $g_i \in \mathrm{Col}(\kappa_i^+, {<}\kappa_{i+2})$.

Note that if G is $\mathbb{M}$-generic over V and $\pi \in G$, then $G_j = \{\vec{f}^\pi{\restriction}j \mid \pi \in G,\ \mathrm{lh}(\pi) > j\}$ is $\mathbb{P}_j$-generic over V. The filter G_j is definable from $\vec{f}{\restriction}j$ and vice versa, so $\mathrm{V}[\vec{f}{\restriction}j]$ can be viewed as a $\mathbb{P}_j$-generic extension of V.

Let's show that the set of conditions that force that $\dot{b}$ is in $\mathrm{V}[\dot{\vec{f}}{\restriction}j]$ is dense below π in $\mathbb{M}$. So let $\pi' \le \pi$. We'll find a $\chi \le \pi'$ that does the job. Let $\chi' \le \pi'$ be as in Corollary 9.9 (identifying $\dot{b}$ with a name for the characteristic function of $\dot{b}$). That is, $\chi' \le_j \pi$ has the property that whenever $\xi < \mu$ and $\chi'' \le \chi'$ decides the statement "$\check{\xi} \in \dot{b}$," then already the j-interpolant of χ' and χ'' decides it (the same way χ'' does).

Define a partition $F : [A^{\chi'}]^{<\omega} \longrightarrow X$, where $X = \mathcal{P}(\mathbb{P}_j \times \mu \times 3)$. Note that $X \in \mathrm{V}_\kappa$, hence X has cardinality less than κ. F is defined by letting $F(\{\lambda_0, \ldots, \lambda_{i-1}\})$ (where it is understood that $\vec{\lambda} = \langle\lambda_0, \ldots, \lambda_{i-1}\rangle$ is increasing) consists of all triples $\langle\vec{g}, \xi, i\rangle$ such that $\vec{g} \le_{\mathbb{P}_j} \vec{f}^{\chi'}{\restriction}j$, $\xi < \mu$ and $i < 3$ is such that

- if $(\chi'^\frown\vec{\lambda})[\vec{g}] \Vdash \check{\xi} \notin \dot{b}$, then $i = 0$.
- if $(\chi'^\frown\vec{\lambda})[\vec{g}] \Vdash \check{\xi} \in \dot{b}$, then $i = 0$.
- if $(\chi'^\frown\vec{\lambda})[\vec{g}]$ does not decide "$\check{\xi} \in \dot{b}$" then $i = 2$.

By Rowbottom's theorem, there is a $B \in U$, $B \subseteq A^{\chi'}$, such that for every $l < \omega$, F is constant on $[B]^l$, say with value E_l.

(1) If $\langle\vec{g}, \xi, 1\rangle \in E_l$, then there can be no $l' < \omega$ for which there is a $\vec{g}'$ that's compatible in $\mathbb{P}_j$ with $\vec{g}$ such that $\langle\vec{g}', \xi, 0\rangle \in E_{l'}$.

This is because if so, letting $\vec{h} \le \mathbb{P}_j\vec{g}, \vec{g}'$ and $m = \max(l, l')$, we could pick $\{\vec{\lambda}\} \in [B]^m$, and we would obtain:

$$(\chi'^\frown\vec{\lambda})[\vec{h}] \le (\chi'^\frown(\vec{\lambda}{\restriction}l))[\vec{g}] \Vdash \check{\xi} \in \dot{b}$$

since $\langle\vec{g}, \xi, 1\rangle \in F(\{\vec{\lambda}{\restriction}l\})$ but also

$$(\chi'^\frown\vec{\lambda})[\vec{h}] \le (\chi'^\frown(\vec{\lambda}{\restriction}l'))[\vec{g}'] \Vdash \check{\xi} \notin \dot{b}$$

since $\langle\vec{g}', \xi, 0\rangle \in F(\{\vec{\lambda}{\restriction}l'\})$. This is a contradiction.

Define $\chi = \langle\vec{\kappa}^{\chi'}, \vec{f}^{\chi'}, B, F^{\chi'}{\restriction}[B]^2\rangle$. The claim is that χ is as wished, that is, χ forces that $\dot{b}$ is in $\mathrm{V}[\dot{\vec{f}}{\restriction}j]$.

Note that whenever $\tilde{G}$ is $\mathbb{P}_j$-generic over V, then the set

$$\{\xi < \mu \mid \exists l < \omega, \vec{g} \in \tilde{G} \quad \langle\vec{g}, \xi, 1\rangle \in E_l\}$$

is in $\mathrm{V}[\tilde{G}]$. By maximality, let $\dot{c} \in \mathrm{V}^{\mathbb{P}_j}$ be a $\mathbb{P}_j$-name for this set, so that letting $\Gamma_{\mathbb{P}_j}$ be the canonical name for the $\mathbb{P}_j$-generic filter,

$$1_{\mathbb{P}_j} \Vdash_{\mathbb{P}_j} \dot{c} \subseteq \check{\mu} \wedge \forall\xi < \check{\mu}(\xi \in \dot{c} \iff \exists l < \omega\exists\vec{g} \in \Gamma_{\mathbb{P}_j}\ \langle\vec{g}, \xi, 1\rangle \in \check{E}_l).$$

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Now let $\tilde{\Gamma}$ be the canonical $\mathbb{M}$-name for the generic filter on $\mathbb{P}_j$. If $\mathbb{P}$ is a forcing notion, τ is a $\mathbb{P}$-name, and $H \subseteq \mathbb{P}$ is a filter, write $\mathtt{val}(\tau, H)$ for the interpretation of τ by H. We claim that

(2) $\chi \Vdash_{\mathbb{M}} \dot{b} = \mathtt{val}(\check{\dot{c}}, \tilde{\Gamma})$

thus completing the proof. To see this, suppose G is $\mathbb{M}$-generic over V with $\chi \in G$. Let $\tilde{G} = \tilde{\Gamma}^G$ be the $\mathbb{P}_j$-generic filter generated by G (which is equivalent to $\dot{\vec{f}}^G{\restriction}j$). Suppose $\xi \in \dot{b}^G$. Let $\pi \leq \chi$, $\pi \in G$ be such that $\pi \Vdash_{\mathbb{M}} \check{\xi} \in \dot{b}$. Then already the j-interpolant of χ' and π forces this. That is, letting $\vec{g} = \vec{f}^\pi{\restriction}j$, $\mathrm{lh}(\chi) + l = \mathrm{lh}(\pi)$ and $\vec{\kappa}^\pi = \vec{\kappa}^\frown\vec{\lambda}$, we have that $\{\vec{\lambda}\} \in [B]^l$ and the condition $(\chi'^\frown\vec{\lambda})[\vec{g}]$ forces $\check{\xi} \in \dot{b}$. This means that $\langle \vec{g}, \xi, 1\rangle \in E_l$, and hence, as $\vec{g} \in \tilde{G}$, that $\xi \in \dot{c}^{\tilde{G}}$.

Now suppose $\xi \notin \dot{b}^G$. Then we can find $\pi \leq \chi$ as above, but $\pi \Vdash \check{\xi} \notin \dot{b}$. Defining $\vec{g}$ and l as before it then follows that $\langle \vec{g}, \xi, 0\rangle \in E_l$. But by claim (1), there can then be no $\vec{h} \in \tilde{G}$ and $l' < \omega$ such that $\langle \vec{h}, \xi, 1\rangle \in E_{l'}$ since $\vec{g}$ and $\vec{h}$ would be compatible. Thus, $\xi \notin \dot{c}^{\tilde{G}}$. □

Corollary 9.12. *Suppose G is $\mathbb{M}$-generic over* V*, and let $\vec{f} = \vec{f}^G$. Then for every bounded subset b of κ in* V$[G]$*, there is a $j < \omega$ such that $a \in$* V$[f{\restriction}j]$.

Proof. Let $\gamma < \kappa$ and $\dot{b} \in \mathrm{V}^{\mathbb{M}}$ be such that $b = \dot{b}^G \subseteq \gamma$. By genericity, there is an even $j < \omega$ such that $\kappa_j \geq \gamma$ (where $\vec{\kappa} = \vec{\kappa}^G$). Pick a condition $\pi \in G$ with $\mathrm{lh}(\pi) > j$ such that $\pi \Vdash \dot{b} \subseteq \check{\gamma}$. By Theorem 9.11, π forces that $\dot{b} \in \mathrm{V}[\dot{\vec{f}}{\restriction}j]$, so, since $\pi \in G$, $b \in \mathrm{V}[\vec{f}{\restriction}j]$. □

As a consequence, if G is $\mathbb{M}$-generic and $\vec{\kappa} = \vec{\kappa}^G$, $\vec{f} = \vec{f}^G$, then the infinite cardinals below κ in the sense of V$[G]$ are precisely the following (in increasing order):

$$\omega = \kappa_0, \kappa_0^+, \kappa_2, \kappa_2^+, \kappa_4, \kappa_4^+, \ldots$$

where for even i, the calculation of κ_i^+ gives the same result in V and in V$[G]$. In particular, $\kappa = \aleph_\omega^{\mathrm{V}[G]}$, and κ is a strong limit cardinal in V$[G]$, since given $\alpha < \kappa$, letting $\kappa_n > \alpha$, we have that $\mathcal{P}(\alpha)^{\mathrm{V}[G]} \subseteq \mathcal{P}(\alpha)^{\mathrm{V}[\vec{f}{\restriction}n]}$, where $\vec{f}{\restriction}n$ is generic for a forcing of size less than κ, and κ is inaccessible.

9.2. **Perturbing the odd part of the Příkrý sequence.** In this subsection, simply put, we will see how to change the $\vec{\kappa}$-sequence of a generic filter for $\mathbb{M}$ without changing its $\vec{f}$-sequence.

Lemma 9.13. *Let $\pi \in \mathbb{M}$, and let $\alpha \in A^\pi$. Then there is a set $B \subseteq A^\pi \setminus (\alpha + 1)$ of cardinality κ such that for all $\beta, \gamma \in B$, $F^\pi(\alpha, \beta)$ and $F^\pi(\alpha, \gamma)$ are compatible in* $\mathrm{Col}(\alpha^+, {<}\kappa)$.

Proof. Fix π and α. Recall that $\mathrm{Col}(\alpha^+, {<}\kappa)$ consists of all functions c with $\mathrm{dom}(c) \subseteq \{\langle i, \xi\rangle \mid i \in (\alpha^+, \kappa) \text{ and } \xi < \alpha^+\}$, such that $\mathsf{card}(\mathrm{dom}(c)) < \alpha^+$ and for all $\langle i, \xi\rangle \in \mathrm{dom}(c)$, $c(i, \xi) < i$.

Let $\Delta = \{\mathrm{dom}(F^\pi(\alpha, \beta)) \mid \beta \in A^\pi\}$.

If Δ has cardinality less than κ, then by a simple pigeonhole principle, there is a subset $\bar{B}$ of A^π on which $\beta \mapsto \mathrm{dom}(F^\pi(\alpha, \beta))$ is constant, and since there are fewer than κ conditions in $\mathrm{Col}(\alpha^+, {<}\kappa)$ with this domain, $\bar{B}$ can be further shrunk to B, so that the map $\beta \mapsto F^\pi(\alpha, \beta)$ is constant on B, and we are done.

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Otherwise, Δ is a set of cardinality κ all of whose members have cardinality less than $\alpha^+ < \kappa$. Since κ is inaccessible, an application of the Δ-system lemma shows that there is a $\bar{B} \subseteq A$ such that $\bar{\Delta} = \{\mathrm{dom}(F^\pi(\alpha,\beta)) \mid \beta \in \bar{B}\}$ forms a Δ-system, say with root r, meaning that for all $\beta,\gamma \in \bar{B}$ with $\alpha < \beta < \gamma$, $\mathrm{dom}(F^\pi(\alpha,\beta)) \cap \mathrm{dom}(F^\pi(\alpha,\gamma)) = r$. Clearly, there are fewer than κ members of $\mathrm{Col}(\alpha^+,{<}\kappa)$ with domain r, so there is a $B \subseteq \bar{B}$ such that for all $\beta,\gamma \in B$ with $\alpha < \beta < \gamma$, $F^\pi(\alpha,\beta){\restriction}r = F^\pi(\alpha,\gamma){\restriction}r$. It follows that B is as wished. □

Definition 9.14. Given V-generic filters $G, H \subseteq \mathbb{M}$, H is *close to* G if $\mathrm{V}[G] = \mathrm{V}[H]$, $\vec{f}^G = \vec{f}^H$ and there is an odd natural number m such that for all $n < \omega$, if $n \neq m$, then $\kappa_n^G = \kappa_n^H$.

Lemma 9.15. *Let $G \subseteq \mathbb{M}$ be* V*-generic, and let $\pi \in G$. Then, in* $\mathrm{V}[G]$*, there are κ many* V*-generic filters $H \subseteq \mathbb{M}$ which are close to G and contain π.*

Proof. Working in $\mathrm{V}[G]$, since κ is a limit cardinal, it suffices to show that for any cardinal $\theta < \kappa$, there are at least θ many V-generic filters close to G and containing π. Fix such a θ, and let $\vec{\kappa} = \vec{\kappa}^G$.

For any condition $\sigma \in \mathbb{M}$, say that $\sigma' \leq \sigma$ is a *θ-nice extension* of σ if $\mathrm{lh}(\sigma') = \mathrm{lh}(\sigma) + 4$ and, letting $\vec{\kappa}^{\sigma'} = \vec{\kappa}^\sigma {}^\frown \langle \lambda_0, \lambda_1, \lambda_2, \lambda_3 \rangle$, we have that

- $\lambda_0 \geq \theta$.
- There is a set $B \subseteq (\lambda_1, \lambda_2) \cap A^\sigma$ of cardinality at least θ such that for all $\beta \in B$, $F^\sigma(\lambda_0,\beta) \subseteq f^{\sigma'}_{\mathrm{lh}(\sigma)}$.

Every $\sigma \in \mathbb{M}$ has a θ-nice extension: let $\lambda_0 \in A^\sigma \setminus \theta$, and by Lemma 9.13, let $\bar{B} \subseteq A^\sigma \setminus (\lambda_0 + 1)$ have cardinality κ, such that for all $\beta, \gamma \in \bar{B}$, with $\beta < \gamma$, $F^\sigma(\lambda_0,\beta)$ and $F^\sigma(\lambda_0,\gamma)$ are compatible. Let $\lambda_1 = \min(\bar{B})$, let λ_2 be the θ-th element of $\bar{B}$, $B = \bar{B} \cap \lambda_2$, and $\lambda_3 \in A^\sigma \setminus (\lambda_2 + 1)$. We can then define σ' to be like $\sigma^\frown\langle \lambda_0, \lambda_1, \lambda_2, \lambda_3 \rangle$, except that we set $f^{\sigma'}_{\mathrm{lh}(\sigma)} = \bigcup_{\beta \in B} F^\sigma(\lambda_0, \beta)$. The latter is a condition in $\mathrm{Col}(\lambda_0^+,{<}\lambda_2)$ because it is a union of pairwise compatible elements in that poset, of cardinality $\theta < \lambda_0^+$, and the latter is the closure of the poset.

In particular, the set D of conditions which are a θ-nice extension of some $\sigma \leq \pi$ is dense below π: given $\pi' \leq \pi$, π' has a θ-nice extension σ', and that σ' is in D, as witnessed by π'.

Since $\pi \in G$, we can pick $\tilde{\pi}' \in G \cap D$. Let $\tilde{\pi} \leq \pi$ be such that $\tilde{\pi}'$ is a θ-nice extension of $\tilde{\pi}$. In particular, $\tilde{\pi} \in G$ as well.

Let $l = \mathrm{lh}(\tilde{\pi})$. It follows that

$$\mathrm{lh}(\tilde{\pi}') = l + 4 \text{ and } \vec{\kappa}^{\tilde{\pi}'} = \vec{\kappa}{\restriction}(l+4) = \vec{\kappa}^{\tilde{\pi}}{}^\frown\langle \kappa_l, \kappa_{l+1}, \kappa_{l+2}, \kappa_{l+3} \rangle.$$

Let B witness that $\tilde{\pi}'$ is a θ-nice extension of $\tilde{\pi}$, and fix $\beta \in B$. Let $\tilde{\pi}_\beta$ be like $\tilde{\pi}'$, except that $\kappa_{l+1}^{\tilde{\pi}_\beta} = \beta$. Note that $\tilde{\pi}_\beta \leq \tilde{\pi}$ (this is the point)! Then an isomorphism $I_\beta : \mathbb{M}_{\leq \tilde{\pi}'} \longrightarrow \mathbb{M}_{\leq \tilde{\pi}_\beta}$ can be defined by letting $I_\beta(\sigma)$ be like σ, except that $\kappa_{l+1}^{I_\beta(\sigma)} = \beta$.

Now $I_\beta[G]$ generates an $\mathbb{M}$-generic filter G_β over V. Clearly, $\mathrm{V}[G] = \mathrm{V}[G_\beta]$ (since G_β can be defined from G using I_β, and vice versa). In fact, $\vec{f}^G = \vec{f}^{G_\beta}$ and $\kappa_n^G = \kappa_n^{G_\beta}$ for all $n \in \omega \setminus \{l+1\}$. So G_β is close to G. And since $\pi_\beta \in G_\beta$ and $\pi_\beta \leq \pi$, we have that $\pi \in G_\beta$.

Thus, we have found θ many V-generic filters which are close to G and contain π, as wished. □

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9.3. **Putting everything together.** Now suppose $\mathrm{V} = L[U]$, where U is a normal ultrafilter on κ, and let $\mathbb{M}$ be defined relative to U. Let G be $\mathbb{M}$-generic over V. Let $\vec{f} = \vec{f}^G$, $\vec{\kappa} = \vec{\kappa}^G$. In $\mathrm{V}[G]$, let $\mathbb{C}$ be the forcing to code $\vec{f}$ into the continuum function above $\aleph_{\omega+2}$, say; see [7, Theorem 66] as an example for this method. Let H be $\mathbb{C}$-generic over $\mathrm{V}[G]$.

Theorem 9.16. *In $\mathrm{V}[G][H]$, $\kappa = \aleph_\omega$ is a strong limit cardinal, and the least leap is $\aleph_{\omega+1}$.*

Proof. We have already seen that $\kappa = \aleph_\omega$ is a strong limit cardinal in $\mathrm{V}[G]$. Since $\mathbb{C}$ is sufficiently closed, these facts are preserved to $\mathrm{V}[G][H]$. Let's break the remainder of the argument down.

(1) $\vec{\kappa} \notin {<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G][H]}$.

Proof of (1). Suppose that $\vec{\kappa}$ belongs to some OD set $A = \{x \mid \varphi(x, \rho)\}$ in $\mathrm{V}[G][H]$. The idea is that there are κ many finite variations of $\vec{\kappa}$ that also belong to A, because we can vary G by changing its $\vec{\kappa}$ sequence (by modifying one odd coordinate) without changing its $\vec{f}$ sequence in κ many ways, so that the modified $\vec{\kappa}$ sequence still belongs to A. In detail, since $\mathbb{C}$ is homogeneous, its trivial condition forces over $\mathrm{V}[G]$ that $\varphi(\check{\vec{\kappa}}, \check{\rho})$ holds. Let $\dot{\mathbb{C}}$ be the canonical $\mathbb{M}$-name for $\mathbb{C}$. We have some $\pi \in G$ forcing "$\varphi(\dot{\vec{\kappa}}, \check{\rho})$ holds after forcing with $\dot{\mathbb{C}}$". But by Lemma 9.15, we can find κ many V-generic filters G' which are close to G and contain π. For each such G', we have that $\mathrm{V}[G] = \mathrm{V}[G']$ and $\vec{f}^G = \vec{f}^{G'}$. The point is that $\dot{\mathbb{C}}^{\mathrm{V}[G']} = \dot{\mathbb{C}}^{\mathrm{V}[G]}$, because the definition of $\mathbb{C}$ depends only on $\vec{f}$. So $\varphi(\dot{\vec{\kappa}}^{G'}, \rho)$ holds in $\mathrm{V}[G'][H] = \mathrm{V}[G][H]$. Letting G' range over κ many possibilities, we obtain κ many variations $\vec{\kappa}^{G'}$ of $\vec{\kappa}$ which all belong to A. □

(2) $\vec{\kappa} \in {<}\kappa^+\text{-}\mathsf{HOD}^{\mathrm{V}[G][H]}$.

Proof of (2). Arguing in $\mathrm{V}[G][H]$, by the Dodd-Jensen Covering Lemma for $L[U]$, Theorem 4.5, since $0^\dagger$ does not exist and covering fails, there is a maximal Příkrý sequence C over $L[U]$. Since every Příkrý sequence is determined by C, a subset of ω and a finite subset of κ, and since the continuum is less than κ, there are altogether no more than κ Příkrý sequences over $L[U]$. One of them is $\vec{\kappa}$. So $\vec{\kappa}$ is ${<}\kappa^+$-HOD in $\mathrm{V}[G][H]$. □

Claims (1) and (2) together immediately imply:

(3) κ^+ is a leap in $\mathrm{V}[G][H]$.

It remains to show that no cardinal below κ is a leap in $\mathrm{V}[G][H]$. Assume, towards a contradiction:

(4) $\bar{\kappa}$, the least leap of $\mathrm{V}[G][H]$, is less than κ^+.

Since the least leap cannot be a limit cardinal, but is uncountable, it follows that

(5) $\omega < \bar{\kappa} < \kappa$.

Applying Corollary 3.7 in $\mathrm{V}[G][H]$ to $\mathsf{HOD} \subsetneq {<}\bar{\kappa}\text{-}\mathsf{HOD}$, there are sets a and b such that $a \in {<}\bar{\kappa}\text{-}\mathsf{HOD} \setminus \mathsf{HOD}$, $a \subseteq b \in \mathsf{HOD}$ and $\mathsf{card}(b) < \bar{\kappa}$.

Clearly, there are an ordinal β and an $\mathsf{OD}^{\mathrm{V}[G][H]}$ bijection $i : \beta \longrightarrow b$ – i can be taken to be the enumeration of b according to the canonical well-ordering of $\mathsf{HOD}^{\mathrm{V}[G][H]}$ in $\mathrm{V}[G][H]$. Since $\mathsf{card}(b) < \bar{\kappa}$, it follows that $\beta < \kappa$.

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Letting $\bar{a} = i^{-1}[a]$, we have that $\bar{a}$ is $<\bar{\kappa}$-$\mathsf{HOD}^{V[G][H]}$, since $a, i \in <\bar{\kappa}$-$\mathsf{HOD}^{V[G][H]}$. And $\bar{a}$ is not in $\mathsf{HOD}^{V[G][H]}$, because if it were, then so would $a = i[\bar{a}]$, as $b \in \mathsf{HOD}^{V[G][H]}$. And of course, $\bar{a} \subseteq \beta < \kappa$.

Because $\mathbb{C}$ is more than κ^+-closed and $\bar{a}$ is a bounded subset of κ, it follows that $\bar{a} \in V[G]$. But then, by Corollary 9.12, it follows that $\bar{a} \in V[\vec{f}\restriction j] = L[U][\vec{f}\restriction j]$, for some $j < \omega$. But $\vec{f}$ is coded by H and is hence $\mathsf{OD}^{V[G][H]}$, and U is $\mathsf{OD}^{V[G][H]}$, so $\bar{a} \in \mathsf{HOD}^{V[G][H]}$, a contradiction!

This contradiction shows that (4) fails, that is, that the least leap in $V[G][H]$ must be κ^+. $\square$

10. Other limit cardinals of countable cofinality

In this section, I will sketch how to produce models in which the least leap is any strong limit cardinal of countable cofinality wished. More precisely, let us work in $L[U]$, where κ is the measurable cardinal, and let $\lambda < \kappa$ be a limit ordinal of countable cofinality.

Recall that if G is $\mathbb{M}$-generic and $\vec{\kappa} = \vec{\kappa}^G$, $\vec{f} = \vec{f}^G$, then the infinite cardinals below κ in the sense of $V[G]$ are given by:

$$\omega = \kappa_0, \kappa_0^+, \kappa_2, \kappa_2^+, \kappa_4, \kappa_4^+, \ldots$$

Thus, in addition to the κ_n, where n is even, their successors also remain cardinals. There was a technical need to leave this extra space, but there is no reason why we couldn't leave more space. The idea is to leave exactly the space needed in order to arrange that in the forcing extension, $\kappa = \aleph_\lambda$.

To this end, let $c : \omega \longrightarrow \lambda$ be increasing and cofinal. Let $\kappa_0 = c(0)$, and let $s : \{m < \omega \mid m \text{ is even}\} \longrightarrow \lambda$ be defined by

$$s(2n) = c(n+1) - (c(n) + 1)$$

so that $c(n) + 1 + s(2n) = c(n+1)$ for all $n < \omega$. The idea is to define our poset $\mathbb{M}^s$ in such a way that when G, $\vec{\kappa}$ and $\vec{f}$ are as above, then in $V[G]$, the cardinals below κ are:

$$\omega, \omega_1, \ldots, \kappa_0 = c(0), \kappa_0^+, \ldots, \kappa_0^{+s(0)}, \kappa_2, \kappa_2^+, \ldots, \kappa_2^{+s(2)}, \kappa_4, \ldots$$

so that in total, the order type of the infinite cardinals less than κ is

$$c(0) + 1 + s(0) + 1 + s(2) + 1 + \ldots = \lambda$$

making $\kappa = \aleph_\lambda^{V[G]}$ as wished.

Accordingly, the conditions in $\mathbb{M}^s$ should have the form $\pi = \langle \vec{\kappa}, \vec{f}, A, F\rangle$ as before, except that for even $i < \mathrm{lh}(\pi)$, $f_i \in \mathrm{Col}(\kappa_i^{+s(i)}, <\kappa)$, and if $i + 2 < \mathrm{lh}(\pi)$, $f_i \in \mathrm{Col}(\kappa_i^{+s(i)}, <\kappa_{i+2})$. To make it work, F needs to be defined in such a way that it provides meaningful upper bounds, i.e., $F(m, \{\alpha, \beta\})$ is defined whenever $m \geq \mathrm{lh}(\pi)$ is even and $\{\alpha, \beta\} \in [A]^2$, and in this case, $F(m, \alpha, \beta) \in \mathrm{Col}(\alpha^{+s(m)}, <\kappa)$. A condition $\pi' \in \mathbb{M}^s$ extends π if $\mathrm{lh}(\pi') \geq \mathrm{lh}(\pi)$, $\vec{\kappa}^\pi$ is an initial segment of $\vec{\kappa}^{\pi'}$, for every even $i < \mathrm{lh}(\pi)$, $f_i^\pi \subseteq f_i^{\pi'}$, $A^{\pi'} \subseteq A^\pi$, for every $i \in [\mathrm{lh}(\pi), \mathrm{lh}(\pi'))$, $\kappa_i^{\pi'} \in A^\pi$, for every even $i \in [\mathrm{lh}(\pi), \mathrm{lh}(\pi'))$, and $F^\pi(i, \{\kappa_i^{\pi'}, \kappa_{i+1}^{\pi'}\}) \subseteq f_i^{\pi'}$, and for all even $j \in [\mathrm{lh}(\pi'), \omega)$ and $\{\alpha, \beta\} \in [A^{\pi'}]^2$, $F^\pi(j, \{\alpha, \beta\}) \subseteq F^{\pi'}(j, \{\alpha, \beta\})$.

Working over $L[U]$, this forcing is ordinal definable, as $L[U] \models V = \mathsf{HOD}$. All the relevant arguments go through. We obtain:

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Theorem 10.1. *Let* $\mathrm{V} = L[U]$*, and let* κ *be the measurable cardinal. If* $\lambda < \kappa$ *is an ordinal of countable cofinality, then there is a forcing extension in which the statement "*$\Lambda_0 = \aleph_{\lambda+1}$*" (using* λ *as a parameter) holds.*

11. Questions

There are many open questions in this area, but here are a two that I find particularly interesting.

(1) Does the statement "λ is a strong limit and a limit of leaps, and λ^+ is a leap" have the consistency strength of a measurable cardinal?
(2) What is the consistency strength of the statement "Λ_0 is the successor of a singular cardinal of uncountable cofinality"?

So far, I have found a promising lower bound for 2, but an upper bound is missing.

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The College of Staten Island (CUNY), 2800 Victory Blvd., Staten Island, NY 10314

The Graduate Center (CUNY), 365 5th Avenue, New York, NY 10016
Email address: gunter.fuchs@csi.cuny.edu
URL: www.math.csi.cuny.edu/~fuchs