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DETERMINACY OF SCHMIDT'S GAME AND OTHER INTERSECTION GAMES

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Abstract. Schmidt's game and other similar intersection games have played an important role in recent years in applications to number theory, dynamics, and Diophantine approximation theory. These games are real games, that is, games in which the players make moves from a complete separable metric space. The determinacy of these games trivially follows from the axiom of determinacy for real games, $\mathsf{AD}_{\mathbb{R}}$, which is a much stronger axiom than that asserting all integer games are determined, AD. One of our main results is a general theorem which under the hypothesis AD implies the determinacy of intersection games which have a property allowing strategies to be simplified. In particular, we show that Schmidt's (α, β, ρ) game on $\mathbb{R}$ is determined from AD alone, but on $\mathbb{R}^n$ for $n \geq 3$ we show that AD does not imply the determinacy of this game. We then give an application of simple strategies and prove that the winning player in Schmidt's (α, β, ρ) game on $\mathbb{R}$ has a winning positional strategy, without appealing to the axiom of choice. We also prove several other results specifically related to the determinacy of Schmidt's game. These results highlight the obstacles in obtaining the determinacy of Schmidt's game from AD.

§1. Introduction. In 1966, Schmidt [13] introduced a two-player game referred to thereafter as Schmidt's game. Schmidt invented the game primarily as a tool for studying certain sets which arise in number theory and Diophantine approximation theory. Schmidt's game and other similar games have since become an important tool in number theory, dynamics, and related areas.

Schmidt's game (defined precisely in Section 2.3) and related games are real games, that is games in which each player plays a "real" (an element of a *Polish space*: a completely metrizable and separable space). Questions regarding which player, if any, has a winning strategy in various games have been systematically studied over the last century. Games in which one of the players has a winning strategy are said to be *determined*. The existence of winning strategies often have implications in both set theory and applications to other areas. In fact, the assumption that certain classes of games are determined can have far-reaching structural consequences. One such assumption is the axiom of determinacy, AD, which is the statement that all integer games are determined. The axiom of determinacy for real games, $\mathsf{AD}_{\mathbb{R}}$, would immediately imply the determinacy of Schmidt's game, but it is significantly stronger than AD (see Section 2.1 for a more thorough discussion). A natural question is what form of determinacy axiom is necessary to obtain the determinacy of Schmidt's game. In particular, can one obtain the determinacy of this game from AD, or does one need the full strength of $\mathsf{AD}_{\mathbb{R}}$?

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Consider the case of the Banach–Mazur game on a Polish space (X, d) with target set $T \subseteq X$. Here the players ***I*** and ***II*** at each turn n play a real which codes a closed ball $B(x_n, \rho_n) = \{y \in X : d(x_n, y) \leq \rho_n\}$. The only "rule" of the game is that the players must play a decreasing sequence of closed balls (that is, the first player to violate this rule loses). If both players follow the rule, then ***II*** wins iff $\bigcap_n B(x_n, \rho_n) \cap T \neq \emptyset$. Although this is a real game, this game is determined for any $T \subseteq X$ just from AD. This follows from the easy fact that the Banach–Mazur game is equivalent to the integer game in which both players play closed balls with "rational centers" (i.e., from a fixed countable dense set) and rational radii.

For Schmidt's game on a Polish space (X, d) with target set $T \subseteq X$, we have in addition fixed parameters $\alpha, \beta \in (0, 1)$. In this game ***I***'s first move is a closed ball $B(x_0, \rho_0)$ as in the Banach–Mazur game. In subsequent moves, the players play a decreasing sequence of closed balls as in the Banach–Mazur game, but with a restriction of the radii. Namely, ***II*** must shrink the previous radius by a factor of α, and ***I*** must shrink the previous radius by β. So, at move $2n$, ***I*** plays a closed ball of radius $\rho_{2n} = (\alpha\beta)^n \rho_0$, and at move $2n + 1$, ***II*** plays a closed ball of radius $\rho_{2n+1} = \alpha(\alpha\beta)^n \rho_0$. As with the Banach–Mazur game, if both players follow these rules, then ***II*** wins iff $x \in T$ where $\{x\} = \bigcap_n B(x_n, \rho_n)$. We call this game the (α, β) Schmidt's game for T. A variation of Schmidt's game, first introduced by Akhunzhanov in [1], has an additional rule that the initial radius $\rho_0 = \rho$ of ***I***'s first move is fixed in advance. We call this the (α, β, ρ) Schmidt's game for T. In all practical applications of the game we are aware of, the difference between these two versions is immaterial. However, in general, these games are not literally equivalent, as the following simple example demonstrates.

EXAMPLE 1.1. Consider $\mathbb{R}$ with the usual metric and let the target set for ***II*** be $T = (-\infty, -1] \cup [1, \infty) \cup \mathbb{Q}$. Notice that this set is dense. It is easy to see that if $\rho \geq 2$ and $\alpha \leq \frac{1}{4}$ then for any β, ***II*** wins the (α, β, ρ)-game, simply by maximizing the distance from the center of her first move to the origin. But if ***I*** is allowed to choose any starting radius and $\beta < \frac{1}{2}$, then he is allowed to play, for instance, $(0, \frac{1}{2})$, and then on subsequent moves, simply avoid each rational one at a time, so that in fact ***I*** wins the (α, β)-game.

In the case of Schmidt's game (either variation) it is not immediately clear that the game is equivalent to an integer game, and thus it is not clear that AD suffices for the determinacy of these games. Our main results have implications regarding the determinacy of Schmidt's game.

Another class of games which is similar in spirit to Schmidt's game are the so-called Banach games whose determinacy has been investigated by Becker [2] and Freiling [3] (with an important result being obtained by Martin). Work of these authors has shown that the determinacy of these games follows from (and is, in fact, equivalent to) AD. Methods similar to those used by Becker, Freiling, and Martin are instrumental in the proofs of our results as well.

In Section 2 we introduce notation and give some relevant background in the theory of games, descriptive set theory, and the history of Schmidt's game in particular.

In Section 3 we prove our main results, including those regarding the determinacy of Schmidt's game. We prove general results, Theorems 3.6 and 3.8, which give some

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conditions under which certain real games are determined under AD alone. Roughly speaking, these results state that "intersection" games which admit strategies which are simple enough to be "coded by a real," in a sense to made precise, are determined from AD. Schmidt's game, Banach–Mazur games, and other similar games are intersection games. The simple strategy condition, however, depends on the specific game. For Schmidt's (α, β, ρ) game on $\mathbb{R}$, we show the simple strategy condition is met, and so this game is determined from AD. Moreover, for the (α, β) Schmidt's game on $\mathbb{R}$, AD implies that either player $\boldsymbol{I}$ has a winning strategy or else for every ρ, $\boldsymbol{II}$ has a winning strategy in the (α, β, ρ) game (this does not immediately give a strategy for $\boldsymbol{II}$ in the (α, β) game from AD, as we are unable in the second case to choose, as a function of ρ, a winning strategy for $\boldsymbol{II}$ in the (α, β, ρ) game). For $\mathbb{R}^n$, $n \geq 2$, the simple strategy condition is not met. In fact, for $n \geq 3$ we show that the determinacy of Schmidt's (α, β, ρ) games does not follow from AD. For $n = 2$, we do not know if AD suffices to get the determinacy of Schmidt's game.

We end Section 3 by giving an interesting application of the simple strategy hypothesis for Schmidt's game on $\mathbb{R}$ to show that whichever player has a winning strategy must have a winning positional strategy i.e., a strategy which needs only the latest move to compute a response. Schmidt [13] proved this fact for general intersection games, but the proof heavily relies on the axiom of choice, which we are able to avoid here using simple strategies. Precise statements are included in the section.

In Section 4 we prove two other results related to the determinacy of Schmidt's game in particular. First, we show assuming AD that in any Polish space (X, d), any $p \in (0, 1)$, and any $T \subseteq X$, there is at most one value of $(\alpha, \beta) \in (0, 1)^2$ with $\alpha\beta = p$ such that the (α, β) Schmidt's game for T is not determined. Second, we show assuming AD that for a general Polish space (X, d) and any target set $T \subseteq X$, the "non-tangent" version of Schmidt's (α, β, ρ) game is determined. This game is just like Schmidt's game except we require each player to play a "non-tangent ball," that is, $d(x_n, x_{n+1}) < \rho_n - \rho_{n+1}$. These results help to illuminate the obstacles in analyzing the determinacy of Schmidt's game.

Finally in Section 5 we list several open questions which are left unanswered by our results. We feel that the results and questions of the current paper show an interesting interplay between determinacy axioms and the combinatorics of Schmidt's game.

§2. Background. In this section we fix the notation we use to describe the games we will be considering, both for general games and specifically for Schmidt's game. We recall some facts about the forms of determinacy we will be considering, some necessary background in descriptive set theory to state and prove our theorems, and we explain some of the history and significance of Schmidt's game.

Throughout we let $\omega = \mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers. We let $\mathbb{R}$ denote the set of real numbers (here we mean the elements of the standard real line, not the Baire space ω^ω as is frequently customary in descriptive set theory).

2.1. Games. Let X be a non-empty set. Let $X^{<\omega}$ and X^ω denote respectively the set of finite and infinite sequences from X. For $s \in X^{<\omega}$ we let $|s|$ denote the length of s. If $s, t \in X^{<\omega}$ we write $s \leq t$ if s is an initial segment of t, that is, $t \restriction |s| = s$. If $s, t \in X^{<\omega}$, we let $s^\frown t$ denote the concatenation of s and t.

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We call $R \subseteq X^{<\omega}$ a *tree on* X if it is closed under initial segments, that is, if $t \in R$ and $s \leq t$, then $s \in R$. We can view R as the set of *rules* for a game. That is, each player must move at each turn so that the finite sequence produced stays in R (the first player to violate this "rule" loses the game). If $\vec{x} = (x_0, x_1, \dots) \in X^\omega$, we say $\vec{x}$ has followed the rules if $\vec{x} \restriction n \in R$ for all n. We let $[R]$ denote the set of all $\vec{x} \in X^\omega$ such that $\vec{x} \restriction n \in R$ for all n (i.e., $\vec{x}$ has followed the rules). We also refer to $[R]$ as the set of *branches* through R. We likewise say $s \in X^{<\omega}$ has followed the rules just to mean $s \in R$.

Fix a set $B \subseteq X^\omega$, which we call the *target set*, and let $R \subseteq X^{<\omega}$ be a rule set (i.e., a tree on X). The game $G(B, R)$ on the set X is defined as follows. ***I*** and ***II*** alternate playing elements $x_i \in X$. So, ***I*** plays $x_0, x_2, \dots$, while ***II*** plays $x_1, x_3, \dots$. This produces the *run* of the game $\vec{x} = (x_0, x_1, \dots)$. The first player, if any, to violate the rules R loses the run $\vec{x}$ of the game. If both players follow the rules (i.e., $\vec{x} \in [R]$), then we declare ***I*** to have won the run iff $\vec{x} \in B$ (otherwise we say ***II*** has won the run). Oftentimes, in defining a game the set of rules R is defined implicitly by giving requirements on each players' moves. If there are no rules, i.e., $R = X^{<\omega}$, then we write $G(B)$ for $G(B, R)$. Also, it is frequently convenient to define the game by describing the payoff set for ***II*** instead of ***I***. This, of course, is formally just replacing B with $X^\omega - B$.

A *strategy* for ***I*** in a game on the set X is a function $\sigma \colon \bigcup_{n\in\omega} X^{2n} \to X$. A strategy for ***II*** is a function $\tau \colon \bigcup_{n\in\omega} X^{2n+1} \to X$. We say σ follows the rule set R is whenever $s \in R$ of even length, than $s^\frown\sigma(s) \in R$. We likewise define the notion of a strategy τ for ***II*** to follow the rules. We say $\vec{x} \in X^\omega$ follows the strategy σ for ***I*** if for all $n \in \omega$, $x_{2n} = \sigma(\vec{x} \restriction 2n)$, and similarly define the notion of $\vec{x}$ following the strategy τ for ***II***. We also extend this terminology in the obvious way to say an $s \in X^{<\omega}$ has followed σ (or τ). Finally, we say a strategy σ for ***I*** is a *winning strategy* for ***I*** in the game $G(B, R)$ if σ follows the rules R and for all $\vec{x} \in [R]$ which follows σ we have $\vec{x} \in B$, that is, player ***I*** has won the run $\vec{x}$. We likewise define the notion of τ being a winning strategy for ***II***.

If σ is a strategy for ***I***, and $\vec{z} = (x_1, x_3, \dots)$ is a sequence of moves for ***II***, we write $\sigma * \vec{z}$ to denote the corresponding run $(x_0, x_1, x_2, x_3, \dots)$ where $x_{2n} = \sigma(x \restriction 2n)$. We likewise define $\tau * \vec{z}$ for τ a strategy for ***II*** and $\vec{z} = (x_0, x_2, \dots)$ a sequence of moves for ***I***. If σ, τ are strategies for ***I*** and ***II*** respectively, then we let $\sigma * \tau$ denote the run $(x_0, x_1, \dots)$ where $x_{2n} = \sigma(x \restriction 2n)$ and $x_{2n+1} = \tau(x \restriction 2n + 1)$ for all n.

We say the game $G(B, R)$ on X is *determined* if one of the players has a winning strategy. The *axiom of determinacy* for games on X, denoted AD_X is the assertion that all games on the set X are determined. Axioms of this kind were first introduced by Mycielski and Steinhaus. We let AD denote AD_ω, that is, the assertion all two-player integer games are determined. Also important for the current paper is the axiom $\mathsf{AD}_\mathbb{R}$, the assertion that all real games are determined. Both AD and $\mathsf{AD}_\mathbb{R}$ play an important role in modern descriptive set theory. Although both axioms contradict the axiom of choice, AC, and thus are not adopted as axioms for the true universe V of set theory, they play a critical role in developing the theory of natural models such as $L(\mathbb{R})$ containing "definable" sets of reals. It is known that $\mathsf{AD}_\mathbb{R}$ is a much stronger assertion than AD (see Theorem 4.4 of [14]).

Sitting between AD and $\mathsf{AD}_\mathbb{R}$ is the determinacy of another class of games called $\frac{1}{2}\mathbb{R}$ games, in which one of the players plays reals and the other plays integers.

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The proof of one of our theorems will require the use of $\frac{1}{2}\mathbb{R}$ games. The axiom $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ that all $\frac{1}{2}\mathbb{R}$ games are determined is known to be equivalent to $\mathsf{AD}_{\mathbb{R}}$ ($\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ immediately implies Uniformization; see Theorem 2.3). However, AD suffices to obtain the determinacy of $\frac{1}{2}\mathbb{R}$ games with Suslin, co-Suslin payoff (a result of Woodin; see [4]). We define these terms more precisely in Section 3. As in [2], this fact will play an important role in one of our theorems.

One of the central results in the theory of games is the result of Martin [6] that all Borel games on any set X are determined in ZFC. By "Borel" here we are referring to the topology on X^ω given by the product of the discrete topologies on X. In fact, in just ZF we have that all Borel games (on any set X) are *quasi-determined* (see [12] for the definition of quasi-strategy and proof of the extension of Martin's result to quasi-strategies in ZF, which is due to Hurkens and Neeman).

THEOREM 2.1 (Martin, Hurkens, and Neeman for quasi-strategies). *Let X be a nonempty set, and let $B \subseteq X^\omega$ be a Borel set, and $R \subseteq X^{<\omega}$ a rule set R (a tree). Then the game $G(B, R)$ is determined (assuming* ZFC, *or quasi-determined just assuming* ZF).

As we mentioned above, AD contradicts AC. In fact, games played for particular types of "pathological" sets constructed using AC are frequently not determined. For example, the following result is well-known (e.g., [5, p. 137, paragraph 8]):

PROPOSITION 2.2. *Let $B \subseteq \omega^\omega$ be a Bernstein set (i.e., neither the set nor its complement contains a perfect set). Then the game $G(B)$ is not determined.*

2.2. Determinacy and pointclasses. We briefly review some of the terminology and results related to the determinacy of games and some associated notions concerning pointclasses which we will need for the proofs of some of our results.

We have introduced above the axioms AD, $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, and $\mathsf{AD}_{\mathbb{R}}$ which assert the determinacy of integer games, half-real games, and real games respectively. We trivially have $\mathsf{AD}_{\mathbb{R}} \Rightarrow \mathsf{AD}_{\frac{1}{2}\mathbb{R}} \Rightarrow \mathsf{AD}$. All three of these axioms contradict AC, the axiom of choice. They are consistent, however, with DC, the axiom of dependent choice, which asserts that if T is a non-empty *pruned* tree (i.e., if $(x_0, \dots, x_n) \in T$ then $\exists x_{n+1}\ (x_0, \dots, x_n, x_{n+1}) \in T$) then there is a branch f through T (i.e., $\forall n\ (f(0), \dots, f(n)) \in T$). DC is a slight strengthening of the axiom of countable choice. On the one hand, DC holds in the minimal model $L(\mathbb{R})$ of AD, while on the other hand even $\mathsf{AD}_{\mathbb{R}}$ does not imply DC. Throughout this paper, our background theory is ZF + DC.

The axiom $\mathsf{AD}_{\mathbb{R}}$ is strictly stronger than AD (see [14]), and in fact it is known that $\mathsf{AD}_{\mathbb{R}}$ is equivalent to AD + Unif, where Unif is the axiom that every $R \subseteq \mathbb{R} \times \mathbb{R}$ has a *uniformization*, that is, a function $f : \text{dom}(R) \to \mathbb{R}$ such that $(x, f(x)) \in R$ for all $x \in \text{dom}(R)$ (see Theorem 2.3). This equivalence will be important for our argument in Theorem 3.11 that AD does not suffice for the determinacy of Schmidt's game in $\mathbb{R}^n$ for $n \geq 3$. The notion of uniformization is closely connected with the descriptive set theoretic notion of a *scale*. If a set $R \subseteq X \times Y$ (where X, Y are Polish spaces) has a scale, then it has a uniformization. The only property of scales which we use is the existence of uniformizations, so we will not give the definition, which is

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rather technical, here (though they are equivalent to Suslin representations, defined below).

A (boldface) *pointclass* $\boldsymbol{\Gamma}$ is a collection of subsets of Polish spaces closed under continuous preimages, that is, if $f: X \to Y$ is continuous and $A \subseteq Y$ is in $\boldsymbol{\Gamma}$, then $f^{-1}(A)$ is also in $\boldsymbol{\Gamma}$. We say $\boldsymbol{\Gamma}$ is self-dual if $\boldsymbol{\Gamma} = \check{\boldsymbol{\Gamma}}$ where $\check{\boldsymbol{\Gamma}} = \{X \setminus A : A \in \boldsymbol{\Gamma}\}$ is the dual pointclass of $\boldsymbol{\Gamma}$. We say $\boldsymbol{\Gamma}$ is non-self-dual if $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$. A set $U \subseteq \omega^\omega \times X$ is *universal* for the $\boldsymbol{\Gamma}$ subsets of X if $U \in \boldsymbol{\Gamma}$ and for every $A \subseteq X$ with $A \in \boldsymbol{\Gamma}$ there is an $x \in \omega^\omega$ with $A = U_x = \{y : (x, y) \in U\}$. A standard fact is that for any Polish space X, the usual non-self-dual Borel levels $\boldsymbol{\Sigma}^0_\alpha \restriction X$, $\boldsymbol{\Pi}^0_\alpha \restriction X$ are pointclasses and have universal sets, as are the projective levels $\boldsymbol{\Sigma}^1_n \restriction X$, $\boldsymbol{\Pi}^1_n \restriction X$. In the case $X = \omega^\omega$, there is a complete analysis of pointclasses under AD. The pointclasses $\boldsymbol{\Gamma} \restriction \omega^\omega$ fall into a natural well-ordered hierarchy (modulo considering $\boldsymbol{\Gamma}$ and its dual class $\check{\boldsymbol{\Gamma}}$ at the same level) by Wadge's lemma. Furthermore, every non-self-dual $\boldsymbol{\Gamma} \restriction \omega^\omega$ has a universal set, an easy consequence of Wadge's lemma (see Theorem 7D.3 of [12]).

In this paper we will be looking at more general pointclasses in more general Polish spaces (in particular in $\mathbb{R}^n$). We note that the general definition of pointclass given above allows in this context some unnatural examples. For example, consider $\boldsymbol{\Gamma}$ defined by: $A \in \boldsymbol{\Gamma} \restriction X$ iff $A \in \boldsymbol{\Pi}^0_{10} \restriction X$ and $A \cap C$ is closed for every connected component C of the space X (here $\boldsymbol{\Pi}^0_{10}$ can be replaced by any pointclass). Then it is easy to check that $\boldsymbol{\Gamma}$ is a pointclass, but $\boldsymbol{\Gamma} \restriction \omega^\omega = \boldsymbol{\Pi}^0_{10} \restriction \omega^\omega$ and $\boldsymbol{\Gamma} \restriction \mathbb{R} = \boldsymbol{\Pi}^0_1 \restriction \mathbb{R}$. Nevertheless, an arbitrary pointclass $\boldsymbol{\Gamma}$ in the Baire space (i.e., $\boldsymbol{\Gamma}$ is closed under continuous preimages by functions $f: \omega^\omega \to \omega^\omega$) can be extended to general Polish spaces in a natural way as follows. Given $\boldsymbol{\Gamma} \restriction \omega^\omega$, say $A \in \boldsymbol{\Gamma}' \restriction X$ if for any continuous $f: \omega^\omega \to X$ we have that $f^{-1}(A) \in \boldsymbol{\Gamma} \restriction \omega^\omega$. Extended this way, it is easy to check that $\boldsymbol{\Gamma}'$ is a pointclass and $\boldsymbol{\Gamma}' \restriction \omega^\omega = \boldsymbol{\Gamma} \restriction \omega^\omega$. We will henceforth just write $\boldsymbol{\Gamma} \restriction X$ instead of $\boldsymbol{\Gamma}' \restriction X$ for this extension. Note that if $\boldsymbol{\Gamma}$ is a general pointclass (closed under inverse images by continuous functions) then if we consider $\boldsymbol{\Gamma} \restriction \omega^\omega$ then the extension $(\boldsymbol{\Gamma} \restriction \omega^\omega)'$ of $\boldsymbol{\Gamma} \restriction \omega^\omega$ to all Polish spaces contains $\boldsymbol{\Gamma}$.

Suppose $\boldsymbol{\Gamma}$ is a pointclass in ω^ω which is non-self-dual, and closed under inverse images by $\boldsymbol{\Sigma}^0_2$-measurable functions (recall $f: X \to Y$ is $\boldsymbol{\Sigma}^0_2$-measurable if $f^{-1}(U) \in \boldsymbol{\Sigma}^0_2 \restriction X$ for every U open in Y). Then $\boldsymbol{\Gamma} \restriction X$ has universal sets for all Polish spaces X. This includes all *Levy* classes, that is pointclasses $\boldsymbol{\Gamma}$ closed under $\wedge$, $\vee$, and either $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$. To see this, first note that since $\boldsymbol{\Gamma}$ is a non-self-dual pointclass in ω^ω, it has a universal set $U \subseteq \omega^\omega \times \omega^\omega$. Let $\varphi: F \to X$ be one-to-one, onto, continuous, and with φ^{-1} being $\boldsymbol{\Sigma}^0_2$-measurable, where $F \subseteq \omega^\omega$ is closed (see Theorem 1G.2 of [12] or Theorem 7.9 of [5]). Let $\tilde{U} \subseteq \omega^\omega \times X$ be defined by $\tilde{U}(x, y) \leftrightarrow U(x, \varphi^{-1}(y))$. It is straightforward to check that every $\boldsymbol{\Gamma} \restriction X$ set occurs as a section of $\tilde{U}$ [If $A \subseteq X$ is in $\boldsymbol{\Gamma}$, let $r: \omega^\omega \to F$ be a continuous retraction (i.e., f is continuous, onto, and $r \restriction F$ is the identity). Then $(\varphi \circ r)^{-1}(A) \in \boldsymbol{\Gamma} \restriction \omega^\omega$. Let x be such that $U_x = (\varphi \circ r)^{-1}(A)$. Then $A = \tilde{U}_x$.] Also, $\tilde{U} \in \boldsymbol{\Gamma} \restriction \omega^\omega \times X$ since if $\psi: \omega^\omega \to \omega^\omega \times X$ is continuous, then $\psi^{-1}(\tilde{U})(z) \leftrightarrow U(\psi(z)_0, \varphi^{-1} \circ \psi(z)_1)$ and the function $z \mapsto (\psi(z)_0, \varphi^{-1} \circ \psi(z)_1)$ from ω^ω to $\omega^\omega \times \omega^\omega$ is $\boldsymbol{\Sigma}^0_2$-measurable (here $\psi(z) = (\psi(z)_0, \psi(z)_1)$).

For κ an ordinal number we say a set $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$, where $p[T] = \{x \in \omega^\omega : \exists f \in \kappa^\omega\ (x, f) \in [T]\}$ denotes the projection of the body of the tree T. We say A is Suslin if it is κ-Suslin for some κ. We say A is co-Suslin if $\omega^\omega \setminus A$ is Suslin. For a general Polish space X, we say

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$A \subseteq X$ is Suslin if for some continuous surjection $\varphi\colon \omega^\omega \to X$ we have that $\varphi^{-1}(A)$ is Suslin (this does not depend on the choice of φ). Scales are essentially the same thing as Suslin representations, in particular a set $A \subseteq Y$ is Suslin iff it has a scale, thus relations which are Suslin have uniformizations. If $\boldsymbol{\Gamma}$ is a pointclass, then we say a set A is *projective over* $\boldsymbol{\Gamma}$ if it is in the smallest pointclass $\boldsymbol{\Gamma}'$ containing $\boldsymbol{\Gamma}$ and closed under complements and existential and universal quantification over $\mathbb{R}$. Assuming AD, if $\boldsymbol{\Gamma}$ is contained in the class of Suslin, co-Suslin sets, then every set projective over $\boldsymbol{\Gamma}$ is also Suslin and co-Suslin. For this result, more background on these general concepts, as well as the precise definitions of scale and the scale property, the reader can refer to [12].

Results of Martin and Woodin (see [7, 9]) show that assuming $\mathsf{AD} + \mathsf{DC}$, the axioms $\mathsf{AD}_{\mathbb{R}}$, Unif, and scales are all equivalent. More precisely we have the following.

THEOREM 2.3 (Martin and Woodin). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. *Then the following are equivalent*:

(1) $\mathsf{AD}_{\mathbb{R}}$,
(2) Unif,
(3) *Every* $A \subseteq \mathbb{R}$ *has a scale.*

Scales and Suslin representations are also important as it follows from AD that ordinal games where the payoff set is Suslin and co-Suslin (the notion of Suslin extends naturally to sets $A \subseteq \lambda^\omega$ for λ an ordinal number) are determined. One proof of this is due to Moschovakis, Theorem 2.2 of [11], another due to Steel can be found in the proof of Theorem 2 of [8]. We will not need this result for the current paper.

A strengthening of AD, due to Woodin, is the axiom AD^+. This axiom has been very useful as it allows the development of a structural theory which has been used to obtain a number of results. It is not currently known if AD^+ is strictly stronger than AD, but it holds in all the natural models of AD obtained from large cardinal axioms. In particular, if the model $L(\mathbb{R})$ satisfies AD (which it does if there is any inner model containing the reals which satisfies AD), then it satisfies AD^+. Also, if $L(\mathbb{R})$ satisfies AD, then $L(\mathbb{R})$ does not satisfy uniformization, and so $L(\mathbb{R})$ does not satisfy $\mathsf{AD}_{\mathbb{R}}$. So, AD^+ is strictly weaker than $\mathsf{AD}_{\mathbb{R}}$. In our Theorem 3.11 we in fact show that AD^+ does not suffice to get the determinacy of Schmidt's (α, β, ρ) game in $\mathbb{R}^n$ for $n \geq 3$.

2.3. Schmidt's game. As mentioned in the introduction, Schmidt invented the game primarily as a tool for studying certain sets which arise in number theory and Diophantine approximation theory. These sets are often exceptional with respect to both measure and category, i.e., Lebesgue null and meager. One of the most significant examples is the following. Let $\mathbb{Q}$ denote the set of rational numbers. A real number x is said to be *badly approximable* if there exists a positive constant $c = c(\alpha)$ such that $\left|x - \frac{p}{q}\right| > \frac{c}{q^2}$ for all $\frac{p}{q} \in \mathbb{Q}$. We denote the set of badly approximable numbers by BA. This set plays a major role in Diophantine approximation theory,

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and is well known to be both Lebesgue null and meager. Nonetheless, using his game, Schmidt was able to prove the following remarkable result:

THEOREM 2.4 (Schmidt [13]). *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of* $\mathcal{C}^1$ *diffeomorphisms of* $\mathbb{R}$. *Then the Hausdorff dimension of the set* $\bigcap_{n=1}^{\infty} f_n^{-1}(\mathbf{BA})$ *is* 1. *In particular,* $\bigcap_{n=1}^{\infty} f_n^{-1}(\mathbf{BA})$ *is uncountable.*

Yet another example of the strength of the game is the following. Let $b \geq 2$ be an integer. A real number x is said to be normal to base b if, for every $n \in \mathbb{N}$, every block of n digits from $\{0, 1, \dots, b-1\}$ occurs in the base-b expansion of x with asymptotic frequency $1/b^n$. It is readily seen that the set of numbers normal to no base is both Lebesgue null and meager. Nevertheless, Schmidt used his game to prove:

THEOREM 2.5 (Schmidt [13]). *The Hausdorff dimension of the set of numbers normal to no base is* 1.

2.3.1. The game's description For the (α, β) Schmidt's game on the complete metric space (X, d) with target set $T \subseteq X$, ***I*** and ***II*** each play pairs (x_i, ρ_i) in $Y = X \times \mathbb{R}^{>0}$. The $R \subseteq Y^{<\omega}$ of rules is defined by the conditions that $\rho_{i+1} + d(x_i, x_{i+1}) \leq \rho_i$ and $\rho_{i+1} = \begin{cases} \alpha\rho_i, & \text{if } i \text{ is even,} \\ \beta\rho_i, & \text{if } i \text{ is odd.} \end{cases}$ The rules guarantee that the closed balls $B(x_i, \rho_i) = \{x \in \mathbb{R}^n \colon d(x, x_i) \leq \rho_i\}$ are nested. Since the $\rho_i \to 0$, there is a unique point $z \in X$ such that $\{z\} = \bigcap_i B(x_i, \rho_i)$. For $\vec{x} \in [R]$, a run of the game following the rules, we let $f(\vec{x})$ be this corresponding point z. The payoff set $B \subseteq Y^{\omega}$ for player ***I*** is $\{\vec{x} \in Y^{\omega} \cap [R] \colon f(\vec{x}) \notin T\}$. Formally, when we refer to the (α, β) Schmidt's game with target set T, we are referring to the game $G(B, R)$ with these sets B and R just described. The formal definition of Schmidt's (α, β, ρ) game with target set T and initial radius ρ (i.e., $\rho_0 = \rho$) is defined in the obvious analogous manner.

§3. Main results. We next prove a general result which states that certain real games are equivalent to $\frac{1}{2}\mathbb{R}$ games. The essential point is that real games which are intersection games (i.e., games where the payoff only depends on the intersection of sets coded by the moves the players make) with the property that if one of the players has a winning strategy in the real game, then that player has a strategy "coded by a real" (in a precise sense defined below), then the game is equivalent to a $\frac{1}{2}\mathbb{R}$ game. In [2] a result attributed to Martin is presented which showed that the determinacy of a certain class of real games, called Banach games, follows from $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, the axiom which asserts the determinacy of $\frac{1}{2}\mathbb{R}$ games (that is, games in which one player plays reals, and the other plays integers). In Theorem 3.6 we use ideas similar to Martin's to prove a general result which applies to intersection games satisfying a "simple strategy" hypothesis. Since many games with applications to number theory and dynamics are intersection games, it seems that in practice the simple strategy hypothesis is the more significant requirement.

DEFINITION 3.1. Let $\mathbf{\Gamma}$ be a pointclass. A simple one-round $\mathbf{\Gamma}$ strategy s for the Polish space X is a sequence $s = (A_n, y_n)_{n\in\omega}$ where $y_n \in X$, $A_n \in \mathbf{\Gamma}$, and the A_n are a partition of X. A simple $\mathbf{\Gamma}$ strategy τ for player ***II*** is a collection $\{s_u\}_{u\in\omega^{<\omega}}$ of simple one-round $\mathbf{\Gamma}$ strategies s_u. A simple $\mathbf{\Gamma}$ strategy σ for player ***I*** is a pair $\sigma = (\bar{y}, \tau)$ where $\bar{y} \in X$ is the first move and τ is a simple $\mathbf{\Gamma}$ strategy for player ***II***.

The idea for a simple one-round strategy is that if the opponent moves in the set A_n, then the strategy will respond with y_n. Thus there is only "countably much" information in the strategy; it is coded by a real in a simple manner. If $s = (A_n, y_n)$ is a simple one-round strategy, we will write $s(n) = y_n$ and also $s(x) = y_n$ for any $x \in A_n$. A general simple strategy produces after each round a new simple one-round strategy to follow in the next round. For example, suppose σ is a simple strategy for ***I***. σ gives a first move $x_0 = \bar{y}$ and a simple one-round strategy $s_\emptyset$. If ***II*** plays x_1, then $x_2 = \sigma(x_0, x_1) = s_\emptyset(x_1)$ =the unique y_{n_0} such that $x_1 \in A_{n_0}$ where $s_\emptyset = (A_n, y_n)$. If ***II*** then plays x_3, then σ responds with $s_{n_0}(x_3)$. The play by σ continues in this manner. Formally, a general simple strategy is a sequence $(s_u)_{u \in \omega^{<\omega}}$ of simple one-round strategies, indexed by $u \in \omega^{<\omega}$.

If $\boldsymbol{\Gamma}$ is a pointclass with a universal set $U \subseteq \omega^\omega \times X$, then we may use U to code simple one-round $\boldsymbol{\Gamma}$ strategies. Namely, the simple one-round $\boldsymbol{\Gamma}$ strategy $s = (A_n, y_n)$ is coded by $z \in \omega^\omega$ if z codes a sequence $(z)_n \in \omega^\omega$ and $U_{(z)_{2n}} = A_n$ and $(z)_{2n+1}$ codes the response $y_n \in X$ in some reasonable manner (e.g., via a continuous surjection from ω^ω to X, the exact details are unimportant).

Remark 3.2. For the remainder of this section, X and Y will denote Polish spaces.

Definition 3.3. Let $R \subseteq X^{<\omega}$ be a tree on X which we identify as a *set of rules* for a game on X. We say a simple one-round $\boldsymbol{\Gamma}$ strategy *s follows the rules R* at position $p \in R$ if for any $x \in X$, if $p^\frown x \in R$, then $p^\frown x^\frown s(x) \in R$.

Definition 3.4. Let $R \subseteq X^{<\omega}$ be a set of rules for a real game. Suppose $p \in X^{<\omega}$ is a position in R. Suppose $f : X \to X$ is such that for all $x \in X$, if $p^\frown x \in R$, then $p^\frown x^\frown f(x) \in R$ (i.e., f is a one-round strategy which follows the rules at p). A *simplification* of f at p is simple one-round strategy $s = (A_n, y_n)$ such that:

(1) For every x in any A_n, if $p^\frown x \in R$, then $p^\frown x^\frown y_n \in R$.
(2) For every n, if there is an $x \in A_n$ such that $p^\frown x \in R$, then there is an $x' \in A_n$ with $p^\frown x' \in R$ and $f(x') = y_n$.

We say s is a $\boldsymbol{\Gamma}$ simplification of f if all of the sets A_n are in $\boldsymbol{\Gamma}$.

Definition 3.5. We say a tree $R \subseteq X^{<\omega}$ is *positional* if for all $p, q \in R$ of the same length and $x \in X$, if $p^\frown x$, $q^\frown x$ are both in R then for all $r \in X^{<\omega}$, $p^\frown x^\frown r \in R$ iff $q^\frown x^\frown r \in R$.

Theorem 3.6 (ZF + DC). *Let $\boldsymbol{\Gamma}$ be a pointclass with a universal set with $\boldsymbol{\Gamma}$ contained within the Suslin, co-Suslin sets. Suppose $B \subseteq X^\omega$ and $R \subseteq X^{<\omega}$ is a positional tree, and suppose both B and R are in $\boldsymbol{\Gamma}$. Let $G = G(B, R)$ be the real game on X with payoff B and rules R. Suppose the following two conditions on G hold*:

(1) (*intersection condition*) *For any $\vec{x}, \vec{y} \in [R]$, if $x(2k) = y(2k)$ for all k, then $\vec{x} \in B$ iff $\vec{y} \in B$.*
(2) (*simple one-round strategy condition*) *If $p \in R$ has odd length, and $f : X \to X$ is a rule following one-round strategy at p, then there is a $\boldsymbol{\Gamma}$-simplification of f at p.*

*Then G is equivalent to a Suslin, co-Suslin $\frac{1}{2}\mathbb{R}$ game G^* in the sense that if **I** (or **II**) has a winning strategy in G^*, then **I** (or **II**) has a winning strategy in G.*

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Proof. Consider the game G^* where $\boldsymbol{I}$ plays pairs (x_{2k}, s_{2k}) and $\boldsymbol{II}$ plays integers n_{2k+1}. The rules R^* of G^* are that $\boldsymbol{I}$ must play at each round a real coding s_{2k} which is a simple one-round $\boldsymbol{\Gamma}$ strategy which follows the rules R relative to a position $p^\frown x_{2k}$ for any p of length $2k$ (this does not depend on the particular choice of p as R is positional). $\boldsymbol{I}$ must also play such that $x_{2k} = s_{2k-2}(n_{2k-1})$. $\boldsymbol{II}$ must play each n_{2k+1} so that there is a legal move $x_{2k+1} \in A^{s_{2k}}_{n_{2k+1}}$ with $p^\frown x_{2k}{}^\frown x_{2k+1} \in R$ (for any p of length $2k$).

If $\boldsymbol{I}$ and $\boldsymbol{II}$ have followed the rules, to produce x_{2k}, s_{2k} and n_{2k+1}, the payoff condition for G^* is as follows. Since $\boldsymbol{II}$ has followed the rules, there is a sequence x_{2k+1} such that the play $(x_0, x_1, \dots) \in [R]$. $\boldsymbol{I}$ then wins the run of G^* iff $(x_0, x_1, \dots) \in B$. Note that by the intersection condition, this is independent of the particular choice of the x_{2k+1}.

From the definition, G^* is a Suslin, co-Suslin game.

We show that G^* is equivalent to G. Suppose first that $\boldsymbol{I}$ wins G^* by σ^*. Then σ^* easily gives a strategy Σ for G. For example, let $\sigma^*(\emptyset) = (x_0, s_0)$. Then $\Sigma(\emptyset) = x_0$. If $\boldsymbol{II}$ plays x_1, then let n_1 be such that $x_1 \in A^{s_0}_{n_1}$. Then $\Sigma(x_0, x_1) = s_0(n_1)$. Continuing in this manner defines Σ. If $(x_0, x_1, \dots)$ is a run of Σ, then there is a corresponding run $((x_0, s_0), n_1, \dots)$ of σ^*. As each s_{2k} follows the rules R, then as long as $\boldsymbol{II}$'s moves follow the rules R, $\boldsymbol{I}$'s moves by Σ also follow the rules R. If $\boldsymbol{II}$ has followed the rules R in the run of G, then the run $((x_0, s_0), n_1, \dots)$ of σ^* has followed the rules for G^* ($\boldsymbol{II}$ has followed the rules of G^* since for each n_{2k+1}, x_{2k+1} witnesses that n_{2k+1} is a legal move). Since σ^* is winning for G^*, the sequence $(x_0, x'_1, x_2, x'_3, \dots) \in B \cap [R]$ for some x'_{2k+1}. By the intersection condition, $(x_0, x_1, x_2, x_3, \dots) \in B$.

Assume now that $\boldsymbol{II}$ has winning strategy τ' in G^*. We first note that there is winning strategy τ^* for $\boldsymbol{II}$ in G^* such that τ^* is projective over $\boldsymbol{\Gamma}$. To see this, first note that the payoff set for G^* is projective over $\boldsymbol{\Gamma}$ as both B and R are in $\boldsymbol{\Gamma}$. Also, there is a scaled pointclass $\boldsymbol{\Gamma}'$, projective over $\boldsymbol{\Gamma}$, which contains the payoff set for $\boldsymbol{II}$ in G^*. By a result of Woodin in [4] (since $\boldsymbol{II}$ is playing the integer moves in G^*) there is a winning strategy τ^* which is projective over $\boldsymbol{\Gamma}'$, and thus projective over $\boldsymbol{\Gamma}$. For the rest of the proof we fix a winning strategy τ^* for $\boldsymbol{II}$ in G^* which is projective over $\boldsymbol{\Gamma}$.

We define a strategy Σ for $\boldsymbol{II}$ in G. Consider the first round of G. Suppose $\boldsymbol{I}$ moves with x_0 in G. We may assume that $(x_0) \in R$.

Claim 3.7. *There is an x_1 with $(x_0, x_1) \in R$ such that for all x_2 with $(x_0, x_1, x_2) \in R$, there is a simple one-round $\boldsymbol{\Gamma}$ strategy s_0 which follows the rules R from position x_0 (so (x_0, s_0) is a legal move for $\boldsymbol{I}$ in G^*) such that if $n_1 = \tau^*(x_0, s_0)$ then $x_1 \in A^{s_0}_{n_1}$ and $x_2 = s_0(x_1)$.*

Proof. Suppose not, then for every x_1 with $(x_0, x_1) \in R$ there is an x_2 with $(x_0, x_1, x_2) \in R$ which witnesses the failure of the claim. Define the relation $S(x_1, x_2)$ to hold iff $(x_0, x_1) \notin R$ or $(x_0, x_1, x_2) \in R$ and the claim fails, that is, for every simple one-round $\boldsymbol{\Gamma}$ strategy s which follows R, if we let $n_1 = \tau^*(x_0, s)$, then either $x_1 \notin A^s_{n_1}$ or $x_2 \neq s(x_1)$. Since τ^*, B, R are projective over $\boldsymbol{\Gamma}$, so is the relation S. By assumption, $\mathrm{dom}(S) = \mathbb{R}$. Since S is projective over $\boldsymbol{\Gamma}$, it is within the scaled pointclasses, and thus there is a uniformization f for S. Note that f follows the rules R. By the simple one-round strategy hypothesis of Theorem 3.6, there is a $\boldsymbol{\Gamma}$-simplification s_0 of f. Let $n_1 = \tau^*(x_0, s_0)$. Since τ^* follows the rules R^* for $\boldsymbol{II}$,

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there is an $x_1 \in A^{s_0}_{n_1}$ such that $(x_0, x_1) \in R$. Since s_0 is a simplification of f, there is an x_1' with $(x_0, x_1') \in R$ and $f(x_1') = s_0(n_1)$. Let $x_2 = f(x_1')$. From the definition of S we have that $(x_0, x_1', x_2) \in R$. Since $S(x_1', x_2)$, there does not exist an s (following the rules) such that $(x_1' \in A^s_{n_1}$ and $x_2 = s(x_1'))$ where $n_1 = \tau^*(x_0, s)$. But on the other hand, the s_0 we have produced does have this property. This proves the claim. ⊣

Now that we've proved this claim, we can attempt to define the strategy Σ. We would like to have $\Sigma(x_0)$ be any x_1 as in the claim. Now since the relation $A(x_0, x_1)$ which says that x_1 satisfies the claim relative to x_0 is projective over $\boldsymbol{\Gamma}$, we can uniformize it to produce the first round $x_1(x_0)$ of the strategy Σ.

Suppose ***I*** now moves x_2 in G. For each such x_2 such that $(x_0, x_1, x_2) \in R$, there is a rule-following simple one-round $\boldsymbol{\Gamma}$ strategy s_0 as in the claim for x_1 and x_2. The relation $A'(x_0, x_2, s_0)$ which says that s_0 satisfies the claim for $x_1 = x_1(x_0)$, x_2 is projective over $\boldsymbol{\Gamma}$ and so has a uniformization $g(x_0, x_2)$. In the G^* game we have ***I*** play $(x_0, g(x_0, x_2))$. Note that $n_1 = \tau^*(x_0, s_0)$ is such that $x_1 \in A^{s_0}_{n_1}$, and $x_2 = s_0(x_1)$.

This completes the definition of the first round of Σ, and the proof that a one-round play according to Σ has a one-round simulation according to τ^*, which will guarantee that Σ wins. The definition of Σ for the general round is defined in exactly the same way, using DC to continue. The above argument also shows that a run of G following Σ has a corresponding run of G^* following τ^*. If ***I*** has followed the rules of G, then ***I*** has followed the rules of G^* in the associated run. Since τ^* is winning for ***II*** in G^*, there is no sequence x'_{2k+1} of moves for ***II*** such that $(x_0, x_1', x_2, x_3', \dots) \in B \cap [R]$. In particular, $(x_0, x_1, x_2, x_3, \dots) \notin B$ (since $(x_0, x_1, \dots) \in [R]$). Thus, ***II*** has won the run of G following Σ. ⊣

If G is a real game on the Polish space X with rule set R, we say that G is an *intersection game* if it satisfies the intersection condition of Theorem 3.6. This is equivalent to saying that there is a function $f : X^\omega \to Y$ for some Polish space Y such that $f(\vec{x}) = f(\vec{y})$ if $x(2k) = y(2k)$ for all k, and the payoff set for G is of the form $f^{-1}(T)$ for some $T \subseteq Y$. In many examples, the rules R require the players to play decreasing closed sets with diameters going to 0 in some Polish space, and the function f is simply giving the unique point of intersection of these sets. If we have a fixed rule set R and a fixed function f, the *class of games* $G_{R,f}$ associated with R and f is the collection of games with rules R and payoffs of the form $f^{-1}(T)$ for $T \subseteq Y$. Thus, we allow the payoff set T to vary, but the set of rules R and the "intersection function" f are fixed. In practice, R and f are usually simple, such as Borel relations/functions.

Theorem 3.8 (AD). *Suppose $\boldsymbol{\Gamma}$ is a pointclass within the Suslin, co-Suslin sets and $G_{R,f}$ is a class of intersection games on the Polish space X with R, $f \in \boldsymbol{\Gamma}$, and R is positional (as above $f : X^\omega \to Y$, where Y is a Polish space). Suppose that for every $T \subseteq Y$ which is Suslin and co-Suslin, if player **I** or **II** has a winning strategy in $G_{R,f}(T)$, then that player has a winning simple $\boldsymbol{\Gamma}$-strategy. Then for every $T \subseteq Y$, the game $G_{R,f}(T)$ is determined.*

Proof. First consider $\boldsymbol{\Gamma} \restriction \omega^\omega$. Considering this as a pointclass in the Baire space, there is a larger pointclass $\boldsymbol{\Gamma}' \restriction \omega^\omega$ which is non-self-dual and closed under $\wedge, \vee$, and $\exists^{\omega^\omega}$ and is still within the Suslin and co-Suslin sets. We now extend $\boldsymbol{\Gamma}' \restriction \omega^\omega$ to all Polish spaces to get $\boldsymbol{\Gamma}' \restriction X$ as defined in the introduction, and as noted

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there this extension contains the original $\mathbf{\Gamma} \restriction X$. The closure properties of $\mathbf{\Gamma}'$ ensure that it is closed under substitutions by Borel functions and so (as discussed in the introduction) $\mathbf{\Gamma}'$ has universal sets. So, without loss of generality we may assume that $\mathbf{\Gamma}$ has universal sets.

Fix the rule set R and function f in $\mathbf{\Gamma}$. Let $T \subseteq Y$, and we show the real game $G_{R,f}(T)$ is determined. Following Becker, we consider the integer game G where ***I*** and ***II*** play out reals x and y which code trees (indexed by $\omega^{<\omega}$) of simple one-round $\mathbf{\Gamma}$ strategies. The winning condition for ***II*** is as follows. If exactly one of x, y fails to be a simple $\mathbf{\Gamma}$-strategy, then that player loses. If both fail to code simple $\mathbf{\Gamma}$-strategies, then ***II*** wins. If x codes a simple $\mathbf{\Gamma}$-strategy σ_x and y codes a simple $\mathbf{\Gamma}$-strategy τ_y, then ***II*** wins iff $\sigma_x * \tau_y \in G_{R,f}(T)$, where $\sigma * \tau$ denotes the unique sequence of reals obtained by playing σ and τ against each other. From AD, the game G is determined. Without loss of generality we may assume that ***II*** has a winning strategy w for G. Let $S_1 \subseteq \omega^\omega$ be the set of z such that z codes a simple $\mathbf{\Gamma}$-strategy for player ***I*** which follows the rules R. Likewise, S_2 is the set of z coding rule following $\mathbf{\Gamma}$-strategies τ_z for ***II***. Note that S_1, S_2 are projective over $\mathbf{\Gamma}$. Let

$$A = \{\vec{y} \in X^\omega : \exists z \in S_1\ \vec{y} = \sigma_z * \tau_{w(z)}\}.$$

Since w is a winning strategy for ***II*** in G, $A \subseteq X^\omega \setminus G_{R,f}(T)$, so $f(A) \subseteq Y \setminus T$. Note that A is projective over $\mathbf{\Gamma}$ by the complexity assumption on R and the fact that S_1 is also projective over $\mathbf{\Gamma}$. We claim that it suffices to show that ***II*** wins the real game $G_{R,f}(Y \setminus f(A))$. This is because if ***II*** wins $G_{R,f}(Y \setminus f(A))$ with run $\vec{y}$, i.e., $\vec{y} \notin G_{R,f}(Y \setminus f(A))$, then $f(\vec{y}) \in f(A) \subseteq Y \setminus T$, so $\vec{y} \notin G_{R,f}(T)$, thus $\vec{y}$ is a winning run for ***II*** in $G_{R,f}(T)$.

We see that $Y \setminus f(A)$ is projective over $\mathbf{\Gamma}$, and thus $G_{R,f}(Y \setminus f(A))$ is equivalent to a Suslin, co-Suslin $\frac{1}{2}\mathbb{R}$ game by Theorem 3.6 which is determined (see [4]), and so $G_{R,f}(Y \setminus f(A))$ is determined. Now it suffices to show that ***I*** doesn't have a winning strategy in $G_{R,f}(Y \setminus f(A))$.

Suppose ***I*** had a winning strategy for $G_{R,f}(Y \setminus f(A))$. By hypothesis, ***I*** has a winning simple $\mathbf{\Gamma}$-strategy coded by some $z \in \omega^\omega$. Let $\vec{y} = \sigma_z * \tau_{w(z)}$ (note that $z \in S_1$ and so $w(z) \in S_2$). Since σ_z is a winning strategy for ***I*** in $G_{R,f}(Y \setminus f(A))$, we have $f(\vec{y}) \in Y \setminus f(A)$. On the other hand, from the definition of A from w we have that $f(\vec{y}) \in f(A)$, a contradiction. ⊣

We next apply Theorem 3.8 to deduce the determinacy of Schmidt's (α, β, ρ) games in $\mathbb{R}$ from AD.

THEOREM 3.9 (AD). *For any $\alpha, \beta \in (0, 1)$, any $\rho \in \mathbb{R}_{>0}$, and any $T \subseteq \mathbb{R}$, the (α, β, ρ) Schmidt's game with target set T is determined.*

PROOF. Let $\mathbf{\Gamma}$ be the pointclass $\mathbf{\Pi}^1_1$ of co-analytic sets. Let R be the tree described by the rules of the (α, β, ρ) Schmidt's game. R is clearly a closed set and is positional. The function f of Theorem 3.8 is given by $\{f((x_i, \rho_i)_i)\} = \bigcap_i B(x_i, \rho_i)$. This clearly satisfies the intersection condition, that is, $G_{R,f}$ is a class of intersection games. Also, f is continuous, so $f \in \mathbf{\Gamma}$.

It remains to verify the simple strategy condition of Theorem 3.8. The argument is essentially symmetric in the players, so we consider the case of player ***II***. In fact we show that for any $T \subseteq \mathbb{R}$, if ***II*** has a winning strategy for the (α, β, ρ)

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Schmidt's game, then ***II*** has a simple Borel strategy. Fix a winning strategy Σ for ***II*** in this (real) game. Consider Σ restricted to the first round of the game. For every $z_0 \in \mathbb{R}$, there is a half-open interval I_{z_0} of the form $[z_0, z_0 + \varepsilon)$ or $(z_0 - \varepsilon, z_0]$ such that for any $x_0 \in I_{z_0}$, we have that $((x_0, \rho_0), \Sigma(z_0, \rho_0)) \in R$. That is, for any $x_0 \in I_{z_0}$ we have that Σ's response to (z_0, ρ_0) is still a legal response to the play (x_0, ρ_0). Note that we need to consider half-open intervals because Σ may play an interval tangent to ***I***'s move. Consider the collection $\mathcal{C}$ of all intervals $I = [z, z + \varepsilon)$ or $I = (z - \varepsilon, z]$ having this property. So, $\mathcal{C}$ is a cover of $\mathbb{R}$ by half-open intervals. There is a countable subcollection $\mathcal{C}' \subseteq \mathcal{C}$ which covers $\mathbb{R}$. To see this, first get a countable $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\cup\mathcal{C}_0 \supseteq \bigcup_{I\in\mathcal{C}} \operatorname{int}(I)$. The set $\mathbb{R} \setminus \bigcup_{I\in\mathcal{C}} \operatorname{int}(I)$ must be countable, and so adding countably many sets of $\mathcal{C}$ to $\mathcal{C}_0$ will get $\mathcal{C}'$ as desired. Let $\mathcal{C}' = \{I_{z_n}\}_{n\in\omega}$. The first round of the simple Borel strategy τ is given by (A_n, y_n) where $A_n = \{(x_0, \rho_0) \colon x_0 \in I_{z_n} \setminus \bigcup_{m<n} I_{z_m}\}$ and $y_n = \Sigma(z_n, \rho_0)$. Clearly (A_n, y_n) is a simple one-round Borel strategy which follows the rules R of the (α, β, ρ) Schmidt's game. This defines the first round of τ. Using DC, we continue inductively to define each subsequent round of τ in a similar manner.

To see that τ is a winning strategy for ***II***, simply note that for any run of τ following the rules there is a run of Σ producing the same point of intersection. ⊣

This theorem immediately implies the following corollary about Schmidt's original (α, β) game.

Corollary 3.10 (AD). *For any $\alpha, \beta \in (0, 1)$, and any $T \subseteq \mathbb{R}$, exactly one of the following holds.*

(1) *Player **I** has a winning strategy in Schmidt's (α, β) game.*
(2) *For every $\rho \in \mathbb{R}_{>0}$, player **II** has a winning strategy in Schmidt's (α, β, ρ) game.*

In contrast to these results, the situation is dramatically different for $\mathbb{R}^n$, $n \geq 3$.

Theorem 3.11. AD^+ *does not imply that the (α, β, ρ) Schmidt's game for $T \subseteq \mathbb{R}^n$, $n \geq 3$ is determined.*

Proof. We will show that the determinacy of these games in $\mathbb{R}^3$ implies that all relations $R \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized. It is known that AD^+ does not suffice to imply this. The proof for larger n is identical.

Let $R \subseteq \mathbb{R} \times [0, 2\pi)$ such that $\forall x \in \mathbb{R}\ \exists\theta \in [0, 2\pi)\ (x, \theta) \in R$. Let $r = \rho - 2\rho\alpha(1 - \beta) \sum_{n=0}^{\infty} (\alpha\beta)^n$. Let the target set for player ***II*** be $T = \{(x, r\cos\theta, r\sin\theta) \colon (x, \theta) \in R\} \cup \{(x, y, z) \colon y^2 + z^2 > r\}$. The value r is the distance from the x-axis that is obtained if ***I*** makes a first move $B((x_0, 0, 0), \rho)$ centered on the x-axis, and at each subsequent turn ***II*** moves to maximize the distance from the x-axis and ***I*** moves to minimize it (note that these moves all have centers having the same x-coordinate x_0). The target set T codes the relation R to be uniformized along the boundary of the cylinder of radius r centered along the x-axis.

We claim that ***I*** cannot win the (α, β, ρ) Schmidt's game for T. First note that if ***I*** plays his center not on the x-axis, then ***II*** can easily win in finitely many moves by simply playing to maximize distance to the x-axis. (This will win the game by the definition of r.) So suppose ***I*** plays $(x, 0, 0)$ as the center of his first move. Fix θ so that $R(x, \theta)$ holds. Then ***II*** can win by always playing tangent towards the direction $(0, \cos\theta, \sin\theta)$ maximizing distance to the x-axis. If ***I*** resists and minimizes distance

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to the x-axis, then the limit point will be in $\{(x, r\cos\theta, r\sin\theta) : (x,\theta) \in R\}$. If ***I*** ever deviates from this, then again ***II*** can win after finitely many moves by maximizing distance to the x-axis.

This shows that ***I*** does not have a winning strategy, so by the assumption that these games are determined, ***II*** has a winning strategy τ. By similar arguments to those above, τ must maximize distance from the x-axis in response to optimal play by ***I***. But one can take advantage of this to easily define a uniformization f of R from τ by the following:

$$f(x) = \theta \iff \tau\big(B\big((x,0,0),\rho\big)\big) = B\big(\big(x,(\rho-\alpha\rho)\cos\theta,(\rho-\alpha\rho)\sin\theta\big),\alpha\rho\big). \qquad \dashv$$

The fact that strategies can be simplified actually gives a fairly useful result about so-called *positional winning strategies* in Schmidt's game.

Definition 3.12. A *positional strategy* for a game on a set X is a function $f : X \to X$. A positional strategy is *winning* for a player if every run which follows the strategy on every move, i.e., $x_{n+1} = f(x_n)$ is a win for that player. For player ***I*** a positional strategy must include a special first move, separate from the instructions for responses.

This definition is useful mostly in the context of intersection games, in which the last move is generally the intersection of all moves up to that point. Many games trivially cannot satisfy this definition. This definition can be made more general by specifying exactly to what extent information can be ignored. For instance, when considering certain classes of games, it may be more appropriate to call a strategy positional if it considers only the latest move *and* what round of the game it is.

We can use the technology of simple strategies to give us the following theorem regarding the existence of positional strategies in Schmidt's game on $\mathbb{R}$.

Theorem 3.13 ($\mathsf{ZF} + \mathsf{DC}$). *Let $T \subseteq \mathbb{R}$ and $\alpha,\beta \in (0,1)$ and $\rho > 0$. Whichever player has a winning strategy in Schmidt's (α,β,ρ)-game with target set T has a winning positional strategy. If player **I** has a winning strategy in Schmidt's (α,β)-game, then player **I** has a positional winning strategy. If player **II** has a winning strategy in Schmidt's (α,β)-game and if any of the following holds*:

- AC,
- AD *and T is Suslin*,
- *T is Borel*,

*then player **II** has a winning positional strategy.*

Remark 3.14. We note that the argument we are about to give is not particular to Schmidt's game, and the only use of DC, as opposed to countable choice, is to guarantee that a simple strategy exists for the winning player (see the proof of Theorem 3.9). The argument below works for any intersection game with positional rules which satisfies the simple strategy hypothesis.

Proof. We will first prove the portion regarding the (α,β,ρ)-game. We've already proven that whichever player wins has a winning Borel simple strategy. It is worth noting that one can use the complexity of the simple strategy in the proof below to get a complexity bound on the positional strategies in all cases except for

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player $\boldsymbol{II}$ winning in the (α, β)-game, but we will not concern ourselves with that here.

Let $\tau = \{(A_{s^\frown i}), (x_{s^\frown i}, r_{s^\frown i})\}_{s\in\omega^{<\omega}, i\in\omega}$ be a simple winning strategy for player $\boldsymbol{II}$ in the (α, β, ρ)-game. We first define a simple choice function we will need in the proof. Let $s^\frown i \in \omega^{n+1}$. If $s = \emptyset$, let $z_i = z_{s^\frown i} \in A_i$. If $s \neq \emptyset$, let $z_{s^\frown i} \in A_{s^\frown i}$ and a legal response to $\boldsymbol{II}$'s play of x_s, if such a legal response exists. Otherwise let $z_{s^\frown i} \in A_{s^\frown i}$ be arbitrary. This makes sense, as the rules of the game are positional (see Definition 3.5). We only use countable choice to define the $z_{s^\frown i}$.

We will define a positional strategy $\hat{\tau}$. Let (x, r) be some potential move by $\boldsymbol{I}$ for which we need to define a response. If r is not of the form $(\alpha\beta)^n\rho$ for some n, then this move is illegal, and so we may play anything, say $\hat{\tau}(x, r) = (x, \alpha r)$. Now if r is of the form $(\alpha\beta)^n\rho$ for some n, then let $s^\frown i \in \omega^{n+1}$ be lexicographically least so that $x \in A_{s^\frown i}$ and so that the sequence $\left(z_{(s^\frown i)\restriction 1}, z_{(s^\frown i)\restriction 2}, \dots, z_{(s^\frown i)\restriction n}, x\right)$ of centers for moves by $\boldsymbol{I}$ is a legal sequence to play against τ. If no such $s^\frown i$ exists, again we play arbitrarily, say $\hat{\tau}(x, r) = (x, \alpha r)$. If we have such an $s^\frown i$, then τ's responses to both $(z_{s\restriction 1}, z_{s\restriction 2}, \dots, z_{s\restriction n}, x)$ and $(z_{s\restriction 1}, z_{s\restriction 2}, \dots, z_{s\restriction n}, z_{s^\frown i})$ of centers will be the same, and so we play

$$\hat{\tau}(x, r) = \tau\left(\left\{\left(z_{(s^\frown i)\restriction j}, (\alpha\beta)^j\rho\right)\right\}_{1\leq j\leq n+1}\right).$$

This completes the definition of $\hat{\tau}$. To see that it wins, let $\{y_j, r_j\}_{j\in\omega}$ be a run following $\hat{\tau}$. It is important to note that the only case in which we could have played arbitrarily is if our opponent broke the rules. To see this, assume by induction that for $(y(0), y(1), \dots, y(2k))$ there is a sequence $(z_{s\restriction 1}, z_{s\restriction 2}, \dots, z_{s\restriction k+1})$ of legal moves for $\boldsymbol{I}$ such that the responses of τ will be $(y(1), y(3), \dots, y(2k+1))$ (which will also be legal as τ is a winning strategy). There is an i so that $y(2k+2) \in A_{s^\frown i}$ (as the $A_{s^\frown j}$ partition all possible next moves). As $y(2k+2)$ is also legal, we have that $z_{s^\frown i}$ is defined and legal. Thus we have such a z sequence for $(y(0), y(1), \dots, y(2k+2))$.

By the definition of $\hat{\tau}$ we have, for each y_{2k+1}, some lexicographically least $s_k{}^\frown i_k$ and the corresponding sequence of centers $\{z_{(s_k{}^\frown i_k)\restriction j}\}_{1\leq j\leq k}$ so that $(y_{2k+1}, r_{2k+1}) = \tau(\{(z_{(s_k{}^\frown i_k)\restriction j}, (\alpha\beta)^j\rho)\}_{1\leq j\leq k+1})$. By the lexicographical minimality of each $s_k{}^\frown i_k$, it must be that $s_{k+1}(0) \leq s_k(0)$ for all k, and this digit can only decrease finitely often, and so must stabilize. This means $z_{(s_k{}^\frown i_k)\restriction 1}$ and $\tau((z_{(s_k{}^\frown i_k)\restriction 1}, r_0))$ are both eventually constant. For any k large enough so that this has occurred, we must also have $s_{k+1}(1) \leq s_k(1)$, and so $z_{(s_k{}^\frown i_k)\restriction 2}$ and $\tau((z_{(s_k{}^\frown i_k)\restriction 1}, r_0), (z_{s_k{}^\frown i_k)\restriction 2}, r_2))$ are also eventually constant. Continuing, we have that both $z_{(s_k{}^\frown i_k)\restriction m}$ and $\tau(\{(z_{(s_k{}^\frown i_k)\restriction j}, (\alpha\beta)^j\rho)\}_{1\leq j\leq m})$ are eventually constant (as $k \to \infty$) for every fixed $m \geq 1$.

The eventual constant values for these moves give us a sequence of positions which converges to a full run $\{z_0, z_1, \dots\}$ which is consistent with τ, but this run may disagree with the original run $\{y_0, y_1, \dots\}$. However, using a legality argument, it is not hard to show that the centers of these moves converge to the same limit point: Let $y_\infty = \lim_{n\to\infty} y_n$, and let $\varepsilon > 0$. Let N_0 be large enough so that $(\alpha\beta)^{N_0}\rho < \varepsilon/2$, and let M_0 be large enough so that for any $k \geq M_0$, $z_{(s_k{}^\frown i_k)\restriction j+1} = z_{2j}$ for all $j \leq N_0$.

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Then we have for any $k \geq M_0$ that

$$y_k = \tau\Big((z_0, r_0), (z_2, r_2), \ldots (z_{2N_0}, (\alpha\beta)^{N_0}\rho), \\ (z_{(s_k \frown i_k)\restriction N_0+2}, r_{N_0+2}), \ldots (z_{(s_k\frown i_k)\restriction k+1}, r_{k+1})\Big).$$

Now since all the moves made are legal (by the choice of $z_{s\frown i}$ and since τ is winning, and thus rule-following), we can conclude that for any $j \geq 2N_0$, both z_j and y_k are legal moves extending a position in which z_{2N_0} was played, and so they both must be in an interval around z_{2N_0} of radius $(\alpha\beta)^{N_0}\rho$. Thus

$$|z_j - y_k| \leq \left|z_j - z_{2N_0}\right| + \left|z_{2N_0} - y_k\right| < 2(\alpha\beta)^{N_0}\rho < \varepsilon.$$

And so we have $z_n \to y_\infty$ as well. Thus since τ is winning for player ***II***, $y_\infty \in T$ and so the run $\{y_j, r_j\}_{j\in\omega}$ is a win for ***II*** as well.

The case for player ***I*** is identical, as we must only include the first move as an extra instruction. Easily then, we also have that the (α, β)-game for player ***I*** is positional, since he is able to decide which ρ to play.

To see that player ***II*** has a positional strategy in the (α, β)-game we must consider the several cases. If AC holds, then the game is positional by an argument included in [13], which resembles the argument we gave above, but well-orders all possible moves.

If AD holds and T is Suslin, then since we assume that player ***II*** has a winning strategy in the (α, β) game, player ***II*** has a winning strategy in the (α, β, ρ)-game for each fixed ρ, which means player ***II*** has a simple Borel winning strategy, which can be thought of as coded by a real using a $\mathbf{\Pi}^1_1$ universal set. We consider the relation on pairs (ρ, τ) which says that τ codes a winning simple strategy in the (α, β, ρ) game. Since T is Suslin, this relation is also Suslin (assuming AD, the Suslin sets are closed under quantification over reals), and thus we can uniformize it to pick a simple winning strategy for each ρ. Now we will mimic the operation above to produce a positional strategy. We must now consider potential moves by ***I*** of the form (x, r) where r is arbitrary, and must choose some ρ uniformly from it in order to use the simple strategy corresponding to ρ in the above argument. We simply pretend as though we are playing using the largest possible ρ which is less than or equal to some fixed constant (say 1 for instance) so that $r = (\alpha\beta)^n\rho$ for some n, if such a ρ exists. If not, we play arbitrarily, but legally, say by copying our opponent's center. Note that at some point in the game after enough legal moves, there must eventually be ρ satisfying $r = (\alpha\beta)^n\rho$ for some n and ρ less than this constant. Once such a ρ exists, we choose the lexicographically least $s\frown i$ corresponding to the simple strategy assigned to ρ as before, and define our positional strategy exactly as in the first half of this proof relative to this simple strategy. To see that this wins, we simply observe that our choice of ρ can only increase finitely often, and so must be eventually constant, at which point the argument that we win reduces to the one given above.

In the case that we don't have AD but T is Borel, we note that the relation we uniformized above in this case is $\mathbf{\Pi}^1_1$, and so we can uniformize it with no extra hypotheses. It is important here that the simple strategies are Borel simple strategies. The rest of the argument is the same as in the case of AD and T is Suslin. ⊣

§4. Further results regarding Schmidt's game. In Section 3 we showed that AD suffices to get the determinacy of the (α, β, ρ) Schmidt's game for any target set $T \subseteq \mathbb{R}$, but that for $T \subseteq \mathbb{R}^n$, $n \geq 3$, AD (or AD^+) is not sufficient. The proof for the positive result in $\mathbb{R}$ used a reduction of Schmidt's (α, β, ρ) game to a certain $\frac{1}{2}\mathbb{R}$ game. The fact that AD does not suffice for $T \subseteq \mathbb{R}^n$, $n \geq 3$, shows that in general the (α, β, ρ) Schmidt's game is not equivalent to an integer game (for $T \subseteq \mathbb{R}$ it still seems possible the game is equivalent to an integer game). A natural question is to what extent we can reduce Schmidt's game to an integer game. In this section we prove two results concerning this question.

In the proof of Theorem 3.11 it is important that the value $r = r(\alpha, \beta)$ was calibrated to the particular values of α, β. In other words, if we change the values of α, β to α', β', using the same target set, so that $r(\alpha'\beta') \neq r(\alpha, \beta)$, then the game is easily determined. In Theorem 4.3 we prove a general result related to this phenomenon. Namely, we show, assuming AD, that for T (in any Polish space) and each value of $p \in (0, 1)$ there is at most one value of α, β with $\alpha\beta = p$ such that the (α, β) Schmidt's game with target set T is not determined. Thus the values of α, β must be tuned precisely to have a possibility of the game being not determined from AD.

The proof of Theorem 3.11 also uses critically the ability of each player to play a ball tangent to the previous ball. In Theorem 4.5, we make this precise by showing that the modification of Schmidt's (α, β, ρ) game where the players are required to make non-tangent moves is determined from AD alone. Thus, the ability of the players to play tangent at each move is a key obstacle in reducing Schmidt's game to an integer game.

In the Banach–Mazur game, the rational modification of the game is fairly straightforward, i.e., the allowed moves for the players are just representatives of balls with centers from some fixed countable dense subset of X and the radii are positive rationals, in Schmidt's game there is a slight difference, again due to the restriction on the players' radii.

Definition 4.1. For a Polish (X, d) and a fixed countable dense subset $D \subseteq X$ we define the *rational Schmidt* (α, β) game by modifying Schmidt's (α, β)-game by restricting the set of allowed moves for both players to balls $B(x_i, \rho_i)$ where $x_i \in D$ and $\rho_i \in \left(\bigcup_{n,m \in \mathbb{N}} \alpha^n \beta^m \mathbb{Q}_{>0}\right)$.

Theorem 4.2. *Let* (X, d) *be a Polish space. Let* $0 < \alpha < \alpha' < 1$, $0 < \beta' < \beta < 1$, *and* $\alpha\beta = \alpha'\beta'$.

(1) *If* ***II*** *wins the rational Schmidt's* (α', β') *game for target set* T *then* ***II*** *wins Schmidt's* (α, β) *game for* T.
(2) *If* ***I*** *wins the rational Schmidt's* (α, β) *game for target set* T *then* ***I*** *wins Schmidt's* (α', β') *game for* T.

Proof. We will prove the first statement, and the proof of the second is similar. Fix the target set $T \subseteq X$. Let τ be a winning strategy for ***II*** in the rational Schmidt's (α', β') game. We will construct a strategy for ***II*** in Schmidt's (α, β) game by using τ.

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Suppose $\boldsymbol{I}$ plays (x_0, ρ_0) as his first move in the (α, β) game. Let $\rho = \rho_0$ to conserve notation. Let $\rho' \in \left(\bigcup_{n,m\in\mathbb{N}} \alpha^n \beta^m \mathbb{Q}_{>0}\right)$ with

$$\rho \frac{\alpha}{\alpha'} \frac{1-\beta}{1-\beta'} < \rho' < \rho \frac{1-\alpha}{1-\alpha'}. \tag{4.1}$$

This is possible since $\frac{\alpha}{\alpha'} \frac{1-\beta}{1-\beta'} < 1$ and $\frac{1-\alpha}{1-\alpha'} > 1$ and $\bigcup_{n,m\in\mathbb{N}} \alpha^n \beta^m \mathbb{Q}_{>0}$ is dense in $\mathbb{R}^{>0}$.

Let $\varepsilon_n \overset{\text{def}}{=} \min\left\{(\alpha\beta)^n(\rho(1-\alpha) - \rho'(1-\alpha')), (\alpha\beta)^{n-1}(\alpha'\rho'(1-\beta') - \alpha\rho(1-\beta))\right\}$.

Notice that $\varepsilon_n > 0$ by inequality (4.1). Now let $(x_1', \alpha'\rho') = \tau(x_0', \rho')$ where $x_0' \in D \cap B(x_0, \varepsilon_0)$. Let $x_1 = x_1'$. By the definition of ε_0 and (4.1), $B(x_1, \alpha\rho) \subseteq B(x_0, \rho)$, and thus $(x_1, \alpha\rho)$ is a valid response to (x_0, ρ) in Schmidt's (α, β) game.

Now given a partial play with centers $\{x_k : k \le 2n\}$, continue by induction to generate x_{2n+1} by considering $(x_{2n+1}', (\alpha'\beta')^n\alpha'\rho') = \tau\left(\{(x_k', r_k) : k \le 2n\}\right)$ where for each $1 \le k \le n$, x_{2k-1}' is given by τ and $x_{2k}' \in D \cap B(x_{2k}, \varepsilon_k)$. Again by the definition of ε_n and (4.1), $B(x_{2n+1}, (\alpha\beta)^n\alpha\rho) \subseteq B(x_{2n}, (\alpha\beta)^n\rho)$.

We have defined a strategy for $\boldsymbol{II}$ in Schmidt's (α, β) game which has the property that if a run is compatible with this strategy with centers $\{x_k : k \in \omega\}$ then there is a corresponding run compatible with τ with centers $\left\{x_k' : k \in \omega\right\}$ such that for all k, $x_{2k+1} = x_{2k+1}'$, so that $\lim_{n\to\infty} x_n' = \lim_{n\to\infty} x_n$ and so since τ is a winning strategy in the rational Schmidt's (α', β') game, $\lim_{n\to\infty} x_n \in T$. So the strategy we have constructed is winning in Schmidt's (α, β) game. ⊣

As a consequence we have the following theorem.

THEOREM 4.3 (AD). *Let (X, d) be a Polish space. Let $T \subseteq X$. Let $p \in (0, 1)$, then there is at most one point $(\alpha, \beta) \in (0, 1)^2$ with $\alpha\beta = p$ at which Schmidt's (α, β) game for T is not determined.*

PROOF. Suppose that Schmidt's (α, β) game is not determined with $\alpha\beta = p$. Let $\alpha_1 < \alpha < \alpha_2$ and $\beta_1 > \beta > \beta_2$ with $\alpha_1\beta_1 = \alpha\beta = \alpha_2\beta_2$. Note that by part (1) of Theorem 4.2, $\boldsymbol{II}$ cannot have a winning strategy in the rational Schmidt's (α_2, β_2) game, since $\boldsymbol{II}$ does not have a winning strategy in Schmidt's (α, β) game by assumption. This means that $\boldsymbol{I}$ must have a winning strategy in the rational Schmidt's (α_2, β_2) game for any such (α_2, β_2) (by AD) and thus by part (2) of Theorem 4.2, $\boldsymbol{I}$ wins Schmidt's (γ, δ) game for any $(\gamma, \delta) \in (0, 1)^2$ with $\gamma\delta = p$ and $\alpha < \gamma$. By a symmetric argument, $\boldsymbol{I}$ has no winning strategy in the rational Schmidt's (α_1, β_1) game, so $\boldsymbol{II}$ must have a winning strategy in Schmidt's (γ, δ) game for any $(\gamma, \delta) \in (0, 1)^2$ with $\gamma\delta = p$ and $\gamma < \alpha$. ⊣

We next consider the variation of Schmidt's game where we restrict the players to making non-tangent moves. We consider a general Polish space (X, d).

DEFINITION 4.4. We say the ball $B(x_{n+1}, \rho_{n+1})$ is *tangent* to the ball $B(x_n, \rho_n)$ if $\rho_{n+1} + d(x_n, x_{n+1}) = \rho_n$.

In the *non-tangent* Schmidt's (α, β, ρ) game with target set $T \subseteq X$, a rule of the game is that each player must play a nested ball of the appropriate radius, as in Schmidt's game, but that ball must not be tangent to the previous ball. Note that the non-tangent variation of Schmidt's game is still an intersection game, and the rule set R is still Borel. We will show that the "simple strategy" condition of Theorem

3.8 is also satisfied, and so the non-tangent Schmidt's game is determined from AD. The proof of this theorem is similar to that of Theorem 3.9. It is clear that the rules of this game are positional, so it will suffice to check the other hypotheses of Theorem 3.8.

THEOREM 4.5 (AD). *Let (X, d) be a Polish space, let $\alpha, \beta \in (0, 1)$, $\rho \in \mathbb{R}_{>0}$, and $T \subseteq X$, and the non-tangent (α, β, ρ) Schmidt's game with target set T is determined.*

PROOF. We will show that if $\boldsymbol{I}$ (or $\boldsymbol{II}$) has a winning strategy in the non-tangent (α, β, ρ) Schmidt game, then $\boldsymbol{I}$ (or $\boldsymbol{II}$) has a simple Borel winning strategy (in the sense of Definition 3.1), and thus by Theorem 3.8, the result follows.

Without loss of generality, say $\boldsymbol{II}$ has a winning strategy Σ in the non-tangent (α, β, ρ) Schmidt's game. We will define a simple Borel strategy τ for $\boldsymbol{II}$ from Σ. Suppose $\boldsymbol{I}$ makes first move $B(x_0, \rho)$, and Σ responds with $B(x_1, \alpha\rho)$, which is not tangent to $B(x_0, \rho)$. Let $\varepsilon = \rho(1 - \alpha) - d(x_0, x_1) > 0$. If $d(x_0', x_0) < \varepsilon$, then if $\boldsymbol{I}$ plays $B(x_0', \rho)$, then $B(x_1, \alpha\rho)$ is still a valid response for $\boldsymbol{II}$. In other words, for each x_0, there is an open ball U of some radius, for which any $x_0' \in U$ has the property that the response by Σ to (x_0, ρ) is also a legal response to (x_0', ρ). Let $\mathcal{C}$ be the collection of all such open balls U. Then $\mathcal{C}$ is an open cover of X, and since X is Polish, it is Lindelöf, and thus $\mathcal{C}$ has a countable subcover $\mathcal{C}' = \{U_{z_n}\}_{n\in\omega}$. The first round of the simple Borel strategy τ is given by (A_n, y_n) where $A_n = \{(x_0, \rho) \colon x_0 \in U_{z_n} \setminus \bigcup_{m<n} U_{z_m}\}$ and $y_n = \Sigma(z_n, \rho)$. Clearly (A_n, y_n) is a simple one-round Borel strategy which follows the rules R of the non-tangent (α, β, ρ) Schmidt's game. This defines the first round of τ. Using DC, we continue inductively to define each subsequent round of τ in a similar manner.

To see that τ is a winning strategy for $\boldsymbol{II}$, simply note that for any run of τ following the rules there is a run of Σ producing the same point of intersection. ⊣

§5. Questions. In Theorem 3.9 we showed that AD suffices to the determinacy of Schmidt's (α, β, ρ) game on $\mathbb{R}$. In Theorem 3.11 we showed that AD^+ does not suffice to prove the determinacy of Schmidt's (α, β, ρ) game on $\mathbb{R}^n$ for $n \geq 3$. In view of these results several natural questions arise.

First, for $n = 2$ our arguments do not seem to resolve the question of the strength of Schmidt game determinacy in either case of the (α, β, ρ) or the (α, β) game. The proof of Theorem 3.9 does not immediately apply as $\mathbb{R}^2$ does not have the "Lindelöf-like" property we used for $\mathbb{R}$. On the other hand, the proof of Theorem 3.11 also does not seem to apply as we don't seem to have enough freedom in $\mathbb{R}^2$ to code an arbitrary instance of uniformization as we did in $\mathbb{R}^3$. In fact, the method of proof of Theorem 3.9 of using "simple strategies" cannot show the determinacy of Schmidt games in $\mathbb{R}^2$ from AD. This is because while we cannot seem to code an arbitrary uniformization problem into the game, we can code the characteristic function of an arbitrary set $A \subseteq \mathbb{R}$ in a way similar to the proof of Theorem 3.11. We could then choose a set A not projective over the pointclass $\boldsymbol{\Gamma}$ (as in the statement of Theorem 3.8). Then the "simple strategy" hypothesis of Theorem 3.8 will fail for this instance of the game. So we ask:

QUESTION 5.1. *Does* AD *suffice to get the determinacy of either the Schmidt's (α, β, ρ) or (α, β) games on $\mathbb{R}^2$?*

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Although the distinction between Schmidt's (α, β, ρ) game and Schmidt's (α, β) game seem immaterial in practical applications, our main theorems apply to the (α, β, ρ) games only. So we ask:

QUESTION 5.2. *Does* AD *suffice to prove the determinacy of Schmidt's* (α, β) *game on* $\mathbb{R}^n$?

Also interesting is the converse question of whether the determinacy of Schmidt's game (either variation) implies determinacy axioms. In [3] it is shown that the determinacy of Banach games (which are similar in spirit to Schmidt games) implies AD. Here we do not have a corresponding result for $\mathbb{R}^n$. We note though that if $\alpha = \beta = \frac{1}{2}$ and $\rho = \frac{1}{2}$, then the determinacy of Schmidt's (α, β, ρ) game on $X = \omega^\omega$ with the standard metric $d(x, y) = \frac{1}{2^{n+1}}$ where n is least so that $x(n) \neq y(n)$, gives AD. So we ask:

QUESTION 5.3. *Does the determinacy of Schmidt's* (α, β, ρ) *(or* (α, β)*) game on* $\mathbb{R}^n$ *imply* AD? *If* $n \geq 3$, *does Schmidt determinacy imply* $\mathsf{AD}_{\mathbb{R}}$?

A related line of questioning is to ask what hypotheses are needed to get the determinacy of Schmidt's game for restricted classes of target sets. For example, while the determinacy of the Banach–Mazur game for $\mathbf{\Sigma}^1_1$ (that is, analytic) target sets is a theorem of just ZF, the corresponding situation for Schmidt's game is not clear. so we ask:

QUESTION 5.4. *Does* ZF + DC *suffice to prove the determinacy of Schmidt's game in* $\mathbb{R}^n$ *for* $\mathbf{\Sigma}^1_1$ *target sets*?

In view of the results of this paper, it is possible that the answer to Question 5.4 depends on n. We can extend the class of target sets from the analytic sets to the more general class of Suslin, co-Suslin sets. So we ask:

QUESTION 5.5. *Does* AD *suffice to prove the determinacy of Schmidt's game in* $\mathbb{R}^n$ *for Suslin, co-Suslin target sets*?

Again, it is possible that the answer to Question 5.5 depends on n.

Finally, it is reasonable to ask the same questions of this paper for other real games which also have practical application to number theory and related areas. Important examples include McMullen's "strong" and "absolute" variations of Schmidt's game [10]. These are also clearly intersection games, so the question is whether the simple strategy hypothesis of Theorem 3.8 applies.

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