

BOSE–EINSTEIN CONDENSATION FOR PARTICLES WITH REPULSIVE SHORT-RANGE PAIR INTERACTIONS IN A POISSON RANDOM EXTERNAL POTENTIAL IN \mathbb{R}^d

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Abstract

We study Bose gases in $d \geq 2$ dimensions with short-range repulsive pair interactions at positive temperature, in the canonical ensemble and in the thermodynamic limit. We assume the presence of hard Poissonian obstacles and focus on the non-percolation regime. For sufficiently strong interparticle interactions, we show that almost surely there cannot be Bose–Einstein condensation into a sufficiently localized, normalized one-particle state. The results apply to the canonical eigenstates of the underlying one-particle Hamiltonian.

Keywords: Interacting Bose gas; strong repulsive pair interactions; hard Poissonian obstacles; non-percolation regime

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1. Introduction

An important phenomenon in many-body quantum theory is Bose–Einstein condensation (BEC). It refers to a surprising coherent behavior in (possibly interacting) bosonic many-particle systems which occurs below some critical temperature or, equivalently, above some critical particle density. Originally predicted by Einstein to occur in non-interacting Bose gases in three dimensions [7, 10, 11], a rigorous proof of BEC for realistic interacting continuum systems was achieved only some twenty years ago [23, 25]. Since then, BEC has remained a highly active area in mathematical physics. We refer to [1, 4, 5, 12] and references therein for further information on current developments in BEC in a non-random setting.

An important open question regarding BEC is whether it is stable with respect to repulsive short-range interparticle interactions in the classical thermodynamic limit. Recently, we studied this question in a one-dimensional setting, namely, in the so-called Luttinger–Sy model where the external potential is a random (singular) potential generated by a Poisson point process on \mathbb{R} [17]. In the present paper, it is our aim to generalize some of the results obtained there to the higher-dimensional setting. More explicitly, we study BEC in $2 \leq d \in \mathbb{N}$ dimensions in the canonical ensemble at positive temperature and in the presence of hard

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Poissonian obstacles, that is, hard balls of a fixed radius that are distributed according to a Poisson point process on \mathbb{R}^d . We assume the intensity of the Poisson point process to be large enough that no percolation is present. Regarding the interparticle interaction, we explore the ‘hardcore’ case, that is, two bosons experience each other as hard balls, with a radius that can be constant or converge to zero at some speed. We will also consider soft repulsive pair interactions, modeled by a non-negative function with certain properties. In either case, whenever the pair interaction is sufficiently strong, we show that almost surely there cannot be BEC into a sufficiently localized one-particle state. As a consequence, almost surely there cannot be BEC into any canonical one-particle eigenstate of the underlying one-particle Hamiltonian.

We want to stress that hardcore interactions are not only interesting from a mathematical point of view since particles in realistic gases repel each other strongly at very short distances, as famously expressed by potentials of the Lennard–Jones type. In addition, at positive temperature and whenever the particle density is sufficiently large, BEC is expected to occur (in the grand canonical ensemble) in a non-interacting Bose gas placed in a Poisson random potential [15, 16, 24]. Our results then show that such a condensate would be destroyed by the presence of sufficiently repulsive pair interactions. For this reason, it might prove interesting to study generalized BEC in such a scenario (as done, for example, in [18] for the one-dimensional case at zero temperature). In addition, it would be interesting to understand if, for example via the method of enlargement of obstacles (as illustrated, for example, in [32]), some of our results obtained for the non-percolation regime and hard Poissonian obstacles can be carried over to the percolation regime and soft Poissonian obstacles.

The paper is organized as follows. In Section 2 we introduce our model and in Section 3 we discuss the probabilistic properties of our system that are used subsequently. In Section 4 we then present our results regarding BEC; we discuss the case of hardcore interactions in Section 4.1 and the case of soft interactions in Section 4.2.

2. The model

We study interacting Bose gases in \mathbb{R}^d , $2 \leq d \in \mathbb{N}$, and in an external Poisson random potential $V(\omega, x)$. Denoting the underlying probability space by $(\Omega, \Sigma, \mathbb{P})$, on an informal level the external potential reads

$$V(\omega, x) := \sum_j u(\|x - x_j^\omega\|_{\mathbb{R}^d}), \quad x \in \mathbb{R}^d, \omega \in \Omega, \tag{2.1}$$

where $\{x_j^\omega\}_j$ is a set of random points generated by a Poisson point process on \mathbb{R}^d with intensity $\nu > 0$. For more details regarding Poisson point processes, we refer the reader to [20, 22]. Furthermore, we assume that the single-site potential $u : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is given by

$$u(x) := \begin{cases} 0 & \text{if } x > R, \\ \infty & \text{otherwise,} \end{cases}$$

where $R > 0$ is a constant. This means that we place hard balls $\overline{B_R(x_j^\omega)}$ with radius $R > 0$ at each random point x_j^ω . Note that such a random potential appears in well-known models such as the Kac–Luttinger model in the area of BEC [15, 16] and the Poisson Boolean model in stochastic geometry [13].

Also, we will investigate BEC in the thermodynamic limit. In this limit, N bosons are placed in the cube $\Lambda_N := (-L_N/2, +L_N/2)^d \subset \mathbb{R}^d$ of side length $L_N > 0$ such that the particle density,

$$\rho := \frac{N}{L_N^d}, \quad (2.2)$$

remains constant in the limit $N \rightarrow \infty$. The N -particle configuration space in the external random potential (2.1) is given by $(\Lambda_N^\omega)^N$, with

$$\Lambda_N^\omega := (-L_N/2, +L_N/2)^d \setminus \bigcup_j \overline{B_R(x_j^\omega)} \quad (2.3)$$

representing the one-particle configuration space.

Hardcore pair interactions are then introduced by further reducing the configuration space $(\Lambda_N^\omega)^N$. For this, we define the set

$$\Lambda_N^{(\text{HC}),\omega} := \{x = (x_j) \in (\Lambda_N^\omega)^N : \|x_i - x_j\|_{\mathbb{R}^d} > a_N, \ i, j = 1, \dots, N, \ i \neq j\},$$

where $(a_N)_{N \in \mathbb{N}} \subset (0, \infty)$ denotes the sequence of radii describing the range of the pair interaction. On a rigorous level, the N -particle Hamiltonian with hardcore pair interactions is the self-adjoint dN -dimensional Dirichlet Laplacian defined on $L_s^2(\Lambda_N^{(\text{HC}),\omega})$; here, the index s refers to the totally symmetric subspace of $L^2(\Lambda_N^{(\text{HC}),\omega})$. On an informal level, the N -particle Hamiltonian with hardcore pair interaction is given by

$$H_N^\omega := \sum_{i=1}^N (-\Delta_i + V(\omega, x_i)) + \sum_{1 \leq i < j \leq N} w_N^{\text{hc}}(\|x_i - x_j\|_{\mathbb{R}^d}), \quad (2.4)$$

where

$$w_N^{\text{hc}}(x) := \begin{cases} 0 & \text{if } x > a_N, \\ \infty & \text{otherwise.} \end{cases}$$

We study hardcore pair interactions in Section 4.1.

In Section 4.2 we also consider a class of soft repulsive pair interactions: for all $N \in \mathbb{N}$, with $w_N \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}, x^{d-1} dx)$ non-negative, we introduce the N -particle Hamiltonian

$$H_N^\omega := \sum_{i=1}^N (-\Delta_i + V(\omega, x_i)) + \sum_{1 \leq i < j \leq N} w_N(\|x_i - x_j\|_{\mathbb{R}^d}) \quad (2.5)$$

on the Hilbert space $L_s^2((\Lambda_N^\omega)^N)$. Finally, we write

$$h_0^\omega := -\Delta + V(\omega, x) \quad (2.6)$$

for the underlying one-particle Hamiltonian on $L^2(\Lambda_N^\omega)$. Note that h_0^ω can be defined rigorously as a direct sum of Dirichlet Laplacians, each defined over a connected component of Λ_N^ω . In this way we obtain a canonical set of eigenfunctions, namely those from each component, continued by zero to the rest of Λ_N^ω .

Remark 2.1. In this paper we will abuse notation slightly to increase readability. To be more precise, for given $\omega \in \Omega$ and $N \in \mathbb{N}$ it might be that $\Lambda_N^\omega = \emptyset$ or $\Lambda_N^{(\text{HC}),\omega} = \emptyset$ (for example, depending on the choice of radii $(a_N)_{N \in \mathbb{N}}$ and the particle density $\rho > 0$). In such a case, the underlying one-particle or N -particle system is not well-defined. But, since we are going to establish statements regarding the absence of BEC only, we choose to formulate statements under the proviso that, for a given $\omega \in \Omega$, all systems are well-defined for all $N \in \mathbb{N}$. In the case of interest, statements can be made more precise by restricting attention to suitable subsequences $(N_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ together with other obvious changes.

3. Probabilistic results

First, we note that the volume of the vacancy set Λ_N^ω , see (2.3), tends to be a constant fraction of Λ_N in the limit $N \rightarrow \infty$. More precisely, for any $\varepsilon > 0$ we have $\lim_{N \rightarrow \infty} \mathbb{P}(|\Lambda_N^\omega|/|\Lambda_N| - e^{-v\omega_d R^d}| < \varepsilon) = 1$, where ω_d is the volume of the unit ball in d dimensions; see, for example, [32, p. 147]. Consequently, for any $0 < c < e^{-v\omega_d R^d}$ there is \mathbb{P} -almost surely a subsequence $(N_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ such that $|\Lambda_{N_j}^\omega| > c|\Lambda_{N_j}|$ for all but finitely many $j \in \mathbb{N}$. Also, Λ_N^ω is possibly divided into components (regions), but \mathbb{P} -almost surely has only finitely many components for each $N \in \mathbb{N}$ [28, Proposition 4.1]. We denote the component of Λ_N^ω with the largest volume by $\Lambda_{N,>}^{(1),\omega}$ and its volume by $|\Lambda_{N,>}^{(1),\omega}|$.

Next, we estimate the volume of the largest component of the vacancy set Λ_N^ω . Note that, \mathbb{P} -almost surely, a ball free of Poisson points with radius $(d/(v\omega_d))^{1/d}(\ln L_N)^{1/d} - c$ (for an arbitrary constant $c > 0$) occurs within Λ_N for all but finitely many $N \in \mathbb{N}$ and for dimensions $d \geq 2$; recall that ω_d is the volume of the unit ball in d dimensions. This has been shown in [32, Proof of Proposition 4.4.3] but, for the convenience of the reader, we provide more details on this fact. We set $\hat{R} := (d/(v\omega_d))^{1/d}$ and $R_N := \hat{R}(\ln L_N)^{1/d} - c$ for a constant $c > 0$. For each $N \in \mathbb{N}$, we place $\hat{c}L_N^d / \ln L_N$ disjoint boxes with side length $2R_N$ in the box Λ_N , where $\hat{c} = \hat{c}(v, d) > 0$ is a constant independent of N . The probability that, in any of these smaller boxes, the centered ball with radius R_N is free of Poisson points is given by $e^{-v\omega_d R_N^d}$. Thus, the probability that none of the $\hat{c}L_N^d / \ln L_N$ disjoint boxes has such a centered ball free of Poisson points is (using the inequality $0 \leq 1 - x \leq e^{-x}$ for $0 \leq x \leq 1$)

$$\begin{aligned} (1 - e^{-v\omega_d R_N^d})^{\hat{c}L_N^d / \ln L_N} &\leq \exp[-\hat{c}(L_N^d / \ln L_N)e^{-v\omega_d R_N^d}] \\ &\leq \exp[-\hat{c}(L_N^d / \ln L_N)e^{-v\omega_d \hat{R}^d \ln(L_N)(1 - (\text{const.})(\ln L_N)^{-1/d})}] \\ &\leq \exp[-\hat{c}(1 / \ln L_N)e^{d(\text{const.})(\ln L_N)^{1/2}}] \\ &\leq e^{-(\ln L_N)^2} \leq L_N^{-2} = \rho^2 N^{-2} \end{aligned}$$

for all but finitely many $N \in \mathbb{N}$. The statement then follows with the Borel–Cantelli lemma. On the other hand, we have the following result.

Theorem 3.1. *Let $2 \leq d \in \mathbb{N}$ be given. For any radius $R > 0$ of the hard Poissonian obstacles there is a $\tilde{v} > 0$ such that, for all intensities $v > \tilde{v}$ of the Poisson random potential, the following holds: there is a $\tilde{C} > 0$ such that, for the number A_N^ω of disjoint boxes $[sj_1 - \frac{s}{2}, sj_1 + \frac{s}{2}] \times [sj_2 - \frac{s}{2}, sj_2 + \frac{s}{2}] \times \dots \times [sj_d - \frac{s}{2}, sj_d + \frac{s}{2}]$ where $j = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$ and $s := R/\sqrt{d}$ that intersect any one component of the vacancy set within Λ_N , we have $\lim_{N \rightarrow \infty} \mathbb{P}(A_N^\omega \leq C \ln(L_N)) = 1$ as well as $A_N^\omega \leq C \ln(L_N)$ \mathbb{P} -almost surely for all but finitely many $N \in \mathbb{N}$, and for all $C > \tilde{C}$.*

Proof. We partition \mathbb{R}^d into the boxes $[sj_1 - \frac{s}{2}, sj_1 + \frac{s}{2}) \times [sj_2 - \frac{s}{2}, sj_2 + \frac{s}{2}) \times \dots \times [sj_d - \frac{s}{2}, sj_d + \frac{s}{2})$, where $j = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$ and $s := R/\sqrt{d}$. The centers of the boxes are then given by the points $(sj_1, sj_2, \dots, sj_d)$, and we shall call them *vertices*. Vertices with a Euclidean distance of s are called *adjacent* and consequently we obtain a discrete graph \mathcal{G} . A sequence $(v_i)_{i=1}^J$, $J \in \{1, 2, \dots, \infty\}$, of vertices in \mathcal{G} such that v_i and v_{i+1} are adjacent for all $i \in \{1, 2, \dots, J-1\}$ is called a *path* in \mathcal{G} . A path in \mathcal{G} is called *finite* if $J < \infty$ and *infinite* whenever $J = \infty$.

It is important to note that if a Poisson point x_j^ω is contained in such a box, then the box is not contained in Λ_N^ω (informally, this is equivalent to saying that the external potential is infinitely high across the box). We call a vertex *vacant* if the corresponding box does not contain any Poisson point, and *occupied* if the corresponding box contains at least one Poisson point. In the same way we call a path in \mathcal{G} *vacant* if the path contains only vacant vertices.

Furthermore, we shall assume that the intensity of the Poisson point process $\nu > 0$ is larger than the critical intensity $\nu_c := \inf\{\nu > 0 : \theta^0(\nu) = 0\}$ where $\theta^0(\nu) = \mathbb{P}(\text{there exists an infinite, self-avoiding, vacant path starting at } 0)$. Note that $0 < \nu_c < \infty$, due to a Peierls argument and since the graph \mathcal{G} is of finite degree; see [14], [19, p. 349].

Now, let $W^\omega(v)$, $v \in s\mathbb{Z}^d = (sj_1, sj_2, \dots, sj_d)$, $j = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$, be the union of all vertices that can be reached by a vacant path on \mathcal{G} from v , and let $\#W^\omega(v)$ denote the number of vertices in $W^\omega(v)$. Due to [19, Theorem 2], [2] and [29], there are constants $0 < C_1, C_2 < \infty$ such that, for any $n \in \mathbb{N}$,

$$\mathbb{P}(\#W^\omega(0) \geq n) \leq C_1 e^{-C_2 n}. \tag{3.1}$$

We choose a $C > 2C_2^{-1}$ and set $n = C \ln((L_N + 2)^d)$. Using inequality (3.1), we conclude that, for any $N \in \mathbb{N}$, the probability that the number of boxes $[sj_1 - \frac{s}{2}, sj_1 + \frac{s}{2}) \times [sj_2 - \frac{s}{2}, sj_2 + \frac{s}{2}) \times \dots \times [sj_d - \frac{s}{2}, sj_d + \frac{s}{2})$ intersecting any component of the vacancy set $\Lambda_{N, >}^{(1), \omega}$ is equal to or larger than n is bounded from above by

$$\begin{aligned} \sum_{v \in s\mathbb{Z}^d \cap (-\lceil L_N/2 \rceil, \lceil L_N/2 \rceil)^d} \mathbb{P}(\#W^\omega(v) \geq n) &\leq s^{-d} (L_N + 2)^d \mathbb{P}(\#W^\omega(0) \geq n) \\ &\leq C_1 s^{-d} [(L_N + 2)^d]^{1 - CC_2}, \end{aligned}$$

which converges to zero in the limit $N \rightarrow \infty$. In addition, using the Borel–Cantelli lemma, we conclude that for \mathbb{P} -almost all $\omega \in \Omega$ there exists an $\tilde{N} \in \mathbb{N}$ such that, for all $N \geq \tilde{N}$, the number of these boxes intersecting any component of the vacancy set within Λ_N is smaller than $C \ln((L_N + 2)^d)$. □

Remark 3.1. This theorem implies the following. Suppose that the intensity of the Poisson random potential is sufficiently large. Then the probability that the volume of the largest component $\Lambda_{N, >}^{(1), \omega}$ is bounded by $C \ln(L_N)$, for a sufficiently large constant $C > 0$ converges to 1; i.e. there is a $\tilde{C} > 0$ such that, for all $C > \tilde{C}$,

$$\lim_{N \rightarrow \infty} \mathbb{P}(|\Lambda_{N, >}^{(1), \omega}| < C \ln(L_N)) = 1.$$

In addition, there is a $\tilde{C} > 0$ such that, for all $C > \tilde{C}$ and for \mathbb{P} -almost all $\omega \in \Omega$, there is an $\tilde{N} \in \mathbb{N}$ such that, for all $N \geq \tilde{N}$, $|\Lambda_{N, >}^{(1), \omega}| < C \ln(L_N)$.

For the proof of Lemma 4.1 in Section 4.2, we need the following lemma. It is a statement about the number of disjoint balls with a given constant radius within Λ_N that are free of Poisson points.

Lemma 3.1. *Let $d \geq 2$ and $\nu > 0$ be given. Also, let $(c_N)_{N \in \mathbb{N}}$ be a sequence that goes to infinity. Then, for \mathbb{P} -almost all $\omega \in \Omega$, there exists an $\tilde{N} \in \mathbb{N}$ such that, for all $N \geq \tilde{N}$, the number B_N^ω of disjoint balls with diameter $\hat{R} > 0$ that are completely within Λ_N and are free of Poisson points is at least $L_N^d / (c_N \ln(N))$, that is,*

$$B_N^\omega \geq \frac{L_N^d}{c_N \ln(N)}. \tag{3.2}$$

Proof. We shall put $(2\lfloor((L_N/2) - (\hat{R}/2))/\hat{R}\rfloor + 1)^d$ disjoint balls, each with diameter $\hat{R} > 0$, in Λ_N . More specifically, the balls shall have the centers $(\hat{R}j_1, \hat{R}j_2, \dots, \hat{R}j_d)$ with $(j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$ and $j_i \in [-\lfloor L_N/(2\hat{R}) - \frac{1}{2} \rfloor, \lfloor L_N/(2\hat{R}) - \frac{1}{2} \rfloor], i = 1, \dots, d$.

Next, we derive an upper bound on the probability that less than $\lfloor L_N^d / (c_N \ln(L_N)) \rfloor$ of these balls are free of Poisson points. We denote the probability that one given ball is free of Poisson points by c . Notice that $0 < c < 1$. Furthermore,

$$\begin{aligned} & \sum_{i=0}^{\lfloor L_N^d / (c_N \ln(L_N)) \rfloor - 1} \binom{(2\lfloor L_N/(2\hat{R}) - \frac{1}{2} \rfloor + 1)^d}{i} c^i (1-c)^{(2\lfloor L_N/(2\hat{R}) - \frac{1}{2} \rfloor + 1)^d - i} \\ & \leq \frac{L_N^d}{c_N \ln(L_N)} \left(\frac{L_N}{\hat{R}}\right)^{\frac{dL_N^d}{c_N \ln(L_N)}} (1-c)^{(2\lfloor L_N/(2\hat{R}) - \frac{1}{2} \rfloor + 1)^d - \frac{L_N^d}{c_N \ln(L_N)}} \\ & \leq \exp \left\{ d \ln(L_N) + \frac{dL_N^d}{c_N} \left(1 - \frac{\ln(\hat{R})}{\ln(L_N)}\right) \right. \\ & \quad \left. + \left[\left(2\left\lfloor \frac{L_N}{2\hat{R}} - \frac{1}{2} \right\rfloor + 1\right)^d - \frac{L_N^d}{c_N \ln(L_N)} \right] \ln(1-c) \right\} \\ & \leq e^{3-d} L_N^d \hat{R}^{-d} \ln(1-c) \end{aligned}$$

for all but finitely many $N \in \mathbb{N}$. Since, using relation (2.2),

$$\sum_{N \in \mathbb{N}} e^{3-d} L_N^d \hat{R}^{-d} \ln(1-c) = \sum_{N \in \mathbb{N}} ((1-c)^{3-d} \rho^{-1} \hat{R}^{-d})^N \leq \frac{1}{1 - (1-c)^{3-d} \rho^{-1} \hat{R}^{-d}} < \infty,$$

the claim follows with the Borel–Cantelli lemma. □

We would like to comment on the main difference between Lemma 3.1 and the corresponding one-dimensional result [17, Lemma A.1]. In the one-dimensional case, the lengths of the intervals that are introduced by a Poisson point process on the real line are independent, identically distributed random variables with exponential distribution. This fact was used in the proof of [17, Lemma A.1]. In higher dimensions, however, we know less about the distribution of the volume of the components of the vacancy set. To offset this, we require that the denominator in (3.2) converges to infinity at a sufficient speed, and needed to use a different strategy here compared to the corresponding one-dimensional case.

4. Results on Bose–Einstein condensation

In this section we apply the probabilistic results derived in Section 3 in order to say something about BEC in a system of interacting bosons placed in a random environment. In fact, we consider two kinds of pair interactions: hardcore interactions, where each particle has to keep a certain distance to all other particles, and soft interactions. Physically, hardcore interactions are described by the informal Hamiltonian (2.4) and soft interactions by the Hamiltonian (2.5). As mentioned before, based on methods presented in [3], our aim is to generalize the results from [17] to the higher-dimensional setting.

In the canonical ensemble, the N -particle state of the system (the density matrix) at inverse temperature $\beta = 1/T \in (0, \infty)$ is given by $\varrho_N^{\beta, \omega} = e^{-\beta H_N^\omega} / \text{Tr}(e^{-\beta H_N^\omega})$. Here, $\text{Tr}(\cdot)$ refers to the trace of a (trace-class) operator on the associated N -particle Hilbert space. Regarding $e^{-\beta H_N^\omega}$ being trace-class, we note that $\text{Tr}(e^{-\beta H_N^\omega}) \leq \sum_{j \in \mathbb{N}_0} e^{-\beta E_N^j}$, where $(E_N^j)_{j \in \mathbb{N}_0}$ are the eigenvalues of the Dirichlet Laplacian on Λ_N^ω . The latter series is finite due to Weyl's law. Moreover, let $\varrho_N^{\beta, \omega}(\cdot, \cdot)$ denote the kernel of $\varrho_N^{\beta, \omega}$. In order to calculate the density of particles in a given one-particle state, we use the reduced one-particle density matrix which acts as a trace-class operator on the underlying one-particle Hilbert space $L^2(\Lambda_N^\omega)$; see [30, Chapter 4]. The kernel of the corresponding reduced one-particle density matrix is then obtained as

$$\varrho_N^{\beta, (1), \omega}(x, y) = N \int_{\Lambda_N^\omega} dz_1 \cdots \int_{\Lambda_N^\omega} dz_{N-1} \varrho_N^{\beta, \omega}(x, z_1, \dots, z_{N-1}, y, z_1, \dots, z_{N-1}), \quad (4.1)$$

with $x, y \in \Lambda_N^\omega$. Here, with a slight abuse of notation and whenever we consider hardcore interactions, we understand the kernel $\varrho_N^{\beta, \omega}(\cdot, \cdot)$ to be extended by zero such that the integration in (4.1) makes sense. The average particle density in a one-particle state $\varphi \in L^2(\Lambda_N^\omega)$ can be calculated as

$$\rho_N^{\beta, \omega}(\varphi) := \frac{1}{L_N^d} \int_{\Lambda_N^\omega} \int_{\Lambda_N^\omega} \overline{\varphi(x)} \varrho_N^{\beta, (1), \omega}(x, y) \varphi(y) dy dx;$$

see, for example, [3] or [30, Chapter 4]. This leads to the following definition.

Definition 4.1. Let $\omega \in \Omega$ and $\varphi_N^\omega \in L^2(\Lambda_N^\omega)$ be a normalized one-particle state, $N \in \mathbb{N}$. We call $(\varphi_N^\omega)_{N \in \mathbb{N}}$ macroscopically occupied at inverse temperature $\beta \in (0, \infty)$ if $\limsup_{N \rightarrow \infty} \rho_N^{\beta, \omega}(\varphi_N^\omega) > 0$. In this case, we say that BEC into $(\varphi_N^\omega)_{N \in \mathbb{N}}$ is present.

For related definitions of BEC we refer to [9, 30].

4.1. Hardcore interactions

We first consider the N -particle Hamiltonian with hardcore pair interaction,

$$H_N^\omega = \sum_{i=1}^N (-\Delta_i + V(\omega, x_i)) + \sum_{1 \leq i < j \leq N} w_N^{\text{hc}}(\|x_i - x_j\|_{\mathbb{R}^d}),$$

where

$$w_N^{\text{hc}}(x) := \begin{cases} 0 & \text{if } x > a_N, \\ \infty & \text{otherwise.} \end{cases}$$

We decompose \mathbb{R}^d into the boxes $\Lambda_N^{(n)} := \{x \in \mathbb{R}^d : r_N n_j \leq x_j < r_N(n_j + 1), j = 1, \dots, d\}$, where $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$ and $N \in \mathbb{N}$. If the side length of these boxes satisfies

$r_N \leq a_N/\sqrt{d}$, then any box $\Lambda_N^{(n)}$ can be occupied by at most one particle. Consequently, for a normalized one-particle state $\varphi_N^\omega \in L^2(\Lambda_N^\omega)$, $N \in \mathbb{N}$, $\omega \in \Omega$, we have that, \mathbb{P} -almost surely, for all $\beta \in (0, \infty)$, and for all but finitely many $N \in \mathbb{N}$,

$$\rho_N^{\beta, \omega}(\varphi_N^\omega) \leq \frac{1}{L_N^d} \left(\sum_{n \in \mathbb{Z}^d} \left(\int_{\Lambda_N^{(n)}} |\varphi_N^\omega(x)|^2 dx \right)^{1/2} \right)^2; \tag{4.2}$$

see [3, Lemma 2]. Note that each φ_N^ω in (4.2) is understood to be extended by zero to all of \mathbb{R}^d . We can now give and prove the main statement of this subsection.

Theorem 4.1 (Absence of BEC I.) *Let $\beta, \rho > 0$ be arbitrarily given. We assume that $R > 0$ and $v > 0$ are such that Theorem 3.1 holds. Suppose that the bounded sequence of radii $(a_N)_{N \in \mathbb{N}}$ is such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{\ln(N)}{a_N^d} \right)^2 = 0.$$

Then, for \mathbb{P} -almost $\omega \in \Omega$, if $(\varphi_N^\omega)_{N \in \mathbb{N}}$, $\varphi_N^\omega \in L^2(\Lambda_N^\omega)$ for all $N \in \mathbb{N}$, is a sequence of normalized one-particle states for which the number A_N^ω of components of Λ_N^ω intersecting $\text{supp}(\varphi_N^\omega)$ satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{A_N^\omega \ln(N)}{a_N^d} \right)^2 = 0,$$

then $(\varphi_N^\omega)_{N \in \mathbb{N}}$ is not macroscopically occupied, that is, there cannot exist a subsequence $(N_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\lim_{j \rightarrow \infty} \rho_{N_j}^{\beta, \omega}(\varphi_{N_j}^\omega) > 0$.

Proof. The proof is obtained from a suitable adaptation of the proof of [17, Theorem 3.3]. Using inequality (4.2) and Theorem 3.1, we obtain, for a constant $C > 0$ and \mathbb{P} -almost surely,

$$\begin{aligned} \lim_{N \rightarrow \infty} \rho_N^{\beta, \omega}(\varphi_N^\omega) &\leq \lim_{N \rightarrow \infty} \frac{1}{L_N^d} \left(\sum_{n \in \mathbb{Z}^d} \left(\int_{\Lambda_N^{(n)}} |\varphi_N^\omega(x)|^2 dx \right)^{1/2} \right)^2 \\ &\leq \lim_{N \rightarrow \infty} \frac{1}{L_N^d} \left(\sum_{n \in \mathbb{Z}^d: \text{supp}(\varphi_N^\omega) \cap \Lambda_N^{(n)} \neq \emptyset} 1 \right)^2 \\ &\leq C \lim_{N \rightarrow \infty} \frac{1}{L_N^d} \left(\frac{A_N^\omega \ln(N)}{a_N^d} \right)^2 = 0. \end{aligned}$$

Note here that $\Lambda_N^{(n)}$ is the grid defined with $r_N = a_N/\sqrt{d}$. Hence, comparing this N -dependent grid with the fixed grid used in Theorem 3.1 eventually leads to the factor a_N^{-d} in the third line. □

In particular, Theorem 4.1 shows that \mathbb{P} -almost surely all canonical eigenstates of the underlying one-particle Hamiltonian (2.6) are not macroscopically occupied if the interaction strength is large enough since they are supported on one component only. In this context, it would be desirable to know if the ground state (or all eigenstates for that matter) is \mathbb{P} -almost surely simple for all $N \in \mathbb{N}$ large enough. Although this seems to be true, to the best of our knowledge it has not been proved for the Poissonian model with hard obstacles so far; we refer to [21] for related results on the Poissonian model with soft obstacles.

4.2. Soft interactions

Finally, we study the case when a soft pair interaction is present. Namely, we now consider the N -particle Hamiltonian (2.5), that is,

$$H_N^\omega := \sum_{i=1}^N (-\Delta_i + V(\omega, x_i)) + \sum_{1 \leq i < j \leq N} w_N(\|x_i - x_j\|_{\mathbb{R}^d}),$$

where $w_N \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}, x^{d-1} dx)$ is a non-negative function. We furthermore assume that for every $N \in \mathbb{N}$ there exist two numbers $a_N, b_N > 0$ (recall that $(a_N)_{N \in \mathbb{N}}$ is assumed to be bounded) such that

$$w_N(x) \geq b_N \quad \text{for almost every } x \in [-a_N, +a_N]. \quad (4.3)$$

Note that similar ‘volume-dependent’ interactions were also considered in [6]. The main result of this section is the following theorem, which says that any normalized one-particle state that is sufficiently localized, such as any canonical eigenstate of the corresponding one-particle Hamiltonian (2.6), is \mathbb{P} -almost surely not macroscopically occupied, given that the pair interactions are sufficiently strong.

Theorem 4.2. (Absence of BEC II) *Let $v > 0$ and $R > 0$ be given such that Theorem 3.1 holds. Let $\varphi_N^\omega \in L^2(\Lambda_N^\omega)$ be a normalized one-particle state and A_N^ω the number of components of Λ_N^ω with non-empty intersection with $\text{supp}(\varphi_N^\omega)$, $N \in \mathbb{N}$, $\omega \in \Omega$. Furthermore, suppose that*

$$\lim_{N \rightarrow \infty} \frac{b_N a_N^{3d} N}{(A_N^\omega)^3 \ln^3(N)} = \infty, \quad \lim_{N \rightarrow \infty} \frac{a_N^{3d} N}{(A_N^\omega)^3 \ln^3(N)} = \infty,$$

and $\lim_{N \rightarrow \infty} \ln^2(N) \|w_N(\|\cdot\|_{\mathbb{R}^d})\|_{L^1(\mathbb{R}^d)} = 0$. Then, for all $\beta \in (0, \infty)$, $(\varphi_N^\omega)_{N \in \mathbb{N}}$ is \mathbb{P} -almost surely not macroscopically occupied.

In the remainder of this work we prove Theorem 4.2. We proceed similarly to [17]. For the convenience of the reader and to be able to point out the differences between the higher-dimensional case discussed here and the one-dimensional case discussed in [17], we present the main steps of the proof.

In the first step, we show that the expected energy density with respect to the canonical ensemble is bounded in the thermodynamic limit. However, in contrast to the one-dimensional setting in [17], we have to assume that the pair interaction is weak enough in a suitable sense. This is due to the fact that, unlike in [17, Lemma A.1], we require the denominator in (3.2) to converge to infinity at a certain speed.

Lemma 4.1. (Bound energy density) *Let $\beta \in (0, \infty)$ be arbitrarily given. Assume that $\lim_{N \rightarrow \infty} \ln^2(N) \|w_N(\|\cdot\|_{\mathbb{R}^d})\|_{L^1(\mathbb{R}^d)} = 0$. Then there exists a constant $C > 0$ such that, \mathbb{P} -almost surely,*

$$\limsup_{N \rightarrow \infty} \frac{\langle H_N^\omega \rangle_{\varphi_N^{\beta, \omega}}}{L_N^d} < C. \quad (4.4)$$

Proof. Let a typical $\omega \in \Omega$ be given. As in [17], we prove (4.4) by showing that the right-hand side of the inequality

$$\frac{\beta}{2} \frac{\langle H_N^\omega \rangle_{\varphi_N^{\beta, \omega}}}{L_N^d} \leq \frac{\ln(\text{Tr}(e^{-(\beta/2)H_N^\omega}))}{L_N^d} - \frac{\ln(\text{Tr}(e^{-\beta H_N^\omega}))}{L_N^d} \quad (4.5)$$

is bounded by a constant in the limit $N \rightarrow \infty$. We note that the inequality (4.5) holds because $\langle H_N^\omega \rangle_{\rho_N^{\beta,\omega}} = -\frac{d}{d\beta} \ln(\text{Tr}(e^{-\beta H_N^\omega}))$ and $\ln(\text{Tr}(e^{-\beta H_N^\omega}))$ is a convex function in β .

Regarding the first term in (4.5), we compare the eigenvalues of H_N^ω with the eigenvalues $(E_N^j)_{j \in \mathbb{N}_0}$ of the Dirichlet Laplacian $-\Delta$ on Λ_N^N and conclude that $\text{Tr}(e^{-(\beta/2)H_N^\omega}) \leq \sum_{j \in \mathbb{N}_0} e^{-(\beta/2)E_N^j}$. Then we use the fact that there is a constant $C_1 = C_1(\beta) > 0$ such that $\lim_{N \rightarrow \infty} L_N^{-d} \ln(\sum_{j \in \mathbb{N}_0} e^{-(\beta/2)E_N^j}) = C_1$; see, for example, [31, Theorem 3.5.8]. Thus, there is a constant $C_1 > 0$ such that, \mathbb{P} -almost surely,

$$\limsup_{N \rightarrow \infty} \frac{\ln(\text{Tr}(e^{-(\beta/2)H_N^\omega}))}{L_N^d} \leq C_1.$$

Next, we show that the second term in (4.5), including the minus sign, is also bounded from above by a constant in the limit $N \rightarrow \infty$. We use the inequality [26, Lemma 14.1 and Remark 14.2]

$$-\ln(\text{Tr}(e^{-\beta H_N^\omega})) \leq -\ln(e^{-\beta \|\Psi_N^\omega\|_{\mathbb{R}^d}^{-2} \langle \Psi_N^\omega, H_N^\omega \Psi_N^\omega \rangle}) = \beta \|\Psi_N^\omega\|_{\mathbb{R}^d}^{-2} \langle \Psi_N^\omega, H_N^\omega \Psi_N^\omega \rangle,$$

$N \in \mathbb{N}$, for a state Ψ_N^ω in the domain of H_N^ω . We choose the N -particle state Ψ_N^ω to be a product state $\prod_{j=1}^N \psi_N^\omega(x_j)$. The one-particle state $\psi_N^\omega(x)$, $x \in \mathbb{R}^d$, is constructed as follows. We consider a rotational symmetric function $f(\|x\|_{\mathbb{R}^d})$ that is equal to one for $\|x\|_{\mathbb{R}^d} \leq 1/4$ and smoothly decreases to zero for $1/4 \leq \|x\|_{\mathbb{R}^d} \leq 1/2$. Then ψ_N^ω is the sum of all such functions placed at the center of each disjoint ball with diameter $(1 + 2R)$ that are within Λ_N and free of Poisson points. As in Lemma 3.1, we denote the number of such disjoint balls by B_N^ω . Then,

$$\begin{aligned} \|\Psi_N^\omega\|_{\mathbb{R}^d}^{-2} \langle \Psi_N^\omega, H_N^\omega \Psi_N^\omega \rangle &= N \|\psi_N^\omega\|_{L^2(\mathbb{R}^d)}^{-2} \int_{\Lambda_N} |\nabla \psi_N^\omega(x)|^2 dx \\ &+ \binom{N}{2} \|\psi_N^\omega\|_{L^2(\mathbb{R}^d)}^{-4} \int_{\Lambda_N} \int_{\Lambda_N} w_N(\|x - y\|_{\mathbb{R}^d}) |\psi_N^\omega(x)|^2 |\psi_N^\omega(y)|^2 dy dx. \end{aligned}$$

Now, by construction there are positive constants $c_1, c_2 > 0$ independent of N such that $\int_{\Lambda_N} |\nabla \psi_N^\omega(x)|^2 dx \leq c_1 B_N^\omega$, $\|\psi_N^\omega\|_{L^2(\mathbb{R}^d)}^2 \geq c_2 B_N^\omega$, and $|\psi_N^\omega(x)| \leq 1$ for all $x \in \mathbb{R}^d$. Employing Lemma 3.1 in combination with our condition on $\|w_N(\|\cdot\|_{\mathbb{R}^d})\|_{L^1(\mathbb{R}^d)}$, we finally conclude that there is a constant $C_2 > 0$ such that, \mathbb{P} -almost surely,

$$\limsup_{N \rightarrow \infty} \frac{-\ln(\text{Tr}(e^{-\beta H_N^\omega}))}{L_N^d} \leq \beta C_2. \quad \square$$

Proof of Theorem 4.2. Suppose there was a set $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) > 0$ and such that, for all $\omega \in \tilde{\Omega}$, there was a sequence of normalized one-particle states $(\varphi_N^\omega)_{N \in \mathbb{N}} \in L^2(\Lambda_N^\omega)$ that are macroscopically occupied and that fulfill the properties described in Theorem 4.2. Then, for all such $\omega \in \tilde{\Omega}$, we show that the expected energy density with respect to the canonical ensemble would diverge in the thermodynamic limit. However, since this is in contradiction to Lemma 4.1, Theorem 4.2 follows.

It remains to prove divergence of the energy density (recall that the proof is a suitable adaptation of the corresponding ones from [8, 6, 17]). Using second quantization (see, for

example, [27] for an introduction), an equivalent definition for a sequence of one-particle states to be macroscopically occupied reads

$$\limsup_{N \rightarrow \infty} \frac{\langle a^*(\varphi_N^\omega) a(\varphi_N^\omega) \rangle_{\varrho_N^{\beta, \omega}}}{L_N^d} > 0.$$

Here, the creation operator $a^*(\varphi_N^\omega) = \int_{\Lambda_N^\omega} \varphi_N^\omega(x) a^*(x) dx$ and the annihilation operator $a(\varphi_N^\omega) = \int_{\Lambda_N^\omega} \varphi_N^\omega(x) a(x) dx$ fulfill the canonical commutation relations $[a(x), a^*(y)] = \delta(x - y)$ and $[a(x), a(y)] = [a^*(x), a^*(y)] = 0$, $x, y \in \Lambda_N^\omega$. Also, $\delta(\cdot)$ is the Dirac δ distribution.

The key ingredient now is a lower bound for the expected energy density. Let $\{G_j\}$ be a partition of \mathbb{R}^d into boxes of side length $a_N/\sqrt{d} > 0$ similar to that constructed in Theorem 3.1. By K_N^ω we denote the number of boxes in $\{G_j\}$ with a non-empty intersection with $\text{supp}(\varphi_N^\omega)$. Employing (4.3), we now obtain

$$\frac{\langle H_N^\omega \rangle_{\varrho_N^{\beta, \omega}}}{L_N^d} \geq \frac{b_N}{2L_N^d} \sum_j^{K_N^\omega} \int_{G_j} dx \int_{G_j} dy \langle a^*(x) a^*(y) a(x) a(y) \rangle_{\varrho_N^{\beta, \omega}} := \frac{b_N}{2L_N^d} \sum_j^{K_N^\omega} C_N^{(j)},$$

where the summation is over all the boxes in $\{G_j\}$ that have a non-empty intersection with $\text{supp}(\varphi_N^\omega)$. Introducing the functions $\varphi_N^{(j), \omega} := \varphi_N^\omega \mathbf{1}_{G_j}$, $j = 1, \dots, K_N^\omega$, as well as using [8, (14)–(16b)], we get

$$\sum_{i,j}^{K_N^\omega} \left(\langle a^*(\varphi_N^{(i), \omega}) a(\varphi_N^{(j), \omega}) \rangle_{\varrho_N^{\beta, \omega}} \right)^4 \leq \left(\sum_j^{K_N^\omega} C_N^{(j)} + \rho L_N^d \right)^2.$$

From this, using the inequality $|\sum_{i=1}^n x_i|^2 \leq n \sum_{i=1}^n |x_i|^2$, we obtain

$$\langle a^*(\varphi_N^\omega) a(\varphi_N^\omega) \rangle_{\varrho_N^{\beta, \omega}}^4 = \left(\sum_{i,j}^{K_N^\omega} \langle a^*(\varphi_N^{(i), \omega}) a(\varphi_N^{(j), \omega}) \rangle_{\varrho_N^{\beta, \omega}} \right)^4 \leq (K_N^\omega)^6 \left(\sum_j^{K_N^\omega} C_N^{(j)} + \rho L_N^d \right)^2,$$

and therefore

$$\frac{\langle H_N^\omega \rangle_{\varrho_N^{\beta, \omega}}}{L_N^d} \geq \frac{b_N}{2L_N^d (K_N^\omega)^3} \langle a^*(\varphi_N^\omega) a(\varphi_N^\omega) \rangle_{\varrho_N^{\beta, \omega}}^2 - \frac{b_N \rho}{2}.$$

Finally, by Theorem 3.1, \mathbb{P} -almost surely and for all but finitely many $N \in \mathbb{N}$ we have $K_N^\omega \leq CA_N^\omega a_N^{-d} \ln(N)$ for some constant $C > 0$. \square

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