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LATTICES OF ZERO-SETS

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Abstract

Characterizations are obtained of those lattices that are isomorphic to the lattice of zero-sets of the following types of Tychonoff spaces: (i) compact (Hausdorff), (ii) Lindelöf, (iii) realcompact and normal, (iv) realcompact.

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In this paper we obtain characterizations of those lattices that are lattice isomorphic to the lattice of zero-sets of a compact Hausdorff, Lindelöf, realcompact and normal, or realcompact topological space. All topological spaces considered are assumed to be Tychonoff spaces (that is completely regular and Hausdorff). T. P. Speed (1973) has independently obtained characterizations of those lattices that are zero-set lattices of compact Hausdorff or realcompact spaces. The characterization obtained by Speed of the zero-set lattice of a realcompact space differs significantly from the one given here.

The Stone–Čech compactification of a completely regular and Hausdorff topological space X is denoted βX. The basic references for compactifications are Gillman and Jerison (1960) and Walker (1974). The basic references for lattice theory that we use are Birkhoff (1948) and Grätzer (1971). We use without further comment the topological and lattice theoretic notation therein. The lattice operations of "meet" and "join" are denoted by $\wedge$ and $\vee$ respectively. N denotes the set of natural numbers.

1. Definition. Let L be a lattice with a minimum (that is, 0 element). An *ultrafilter* on L is a non-empty subset $M \subseteq L$ such that:

(i) if $a, b \in M$ then $a \wedge b \in M$,

(ii) if $a \in M$ and $a \leqslant b$, then $b \in M$,

(iii) $0 \notin M$,

(iv) M is maximal with respect to (i), (ii) and (iii).

Let L be a bounded distributive lattice (that is, a distributive lattice with both a 0 and a 1 element). Let $M(L)$ denote the set of ultrafilters on L. The *Stone topology* on $M(L)$ is the topology generated using the following family as a base

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for the closed sets: $\{h(z): z \in L\}$ where $h(z) = \{M \in M(L): z \in M\}$. With this topology $M(L)$ is a compact (not necessarily Hausdorff) space as in Grätzer (1971), section 11.

We now introduce several lattice conditions which will occur frequently. The symbols on the left will be used to denote the conditions. Let L be a bounded lattice.

(α) If $s, t \in L$ and $s \wedge t = 0$, then there exist elements $a, b \in L$ such that $s \wedge a = b \wedge t = 0$ and $a \vee b = 1$.

(β) If $s > t$ then there is an $a \in L$ such that $a \wedge s > 0$ and $a \wedge t = 0$.

(γ) If $\{z_i: i \in N\} \subseteq L$ then $\bigwedge \{z_i: i \in N\} \in L$.

(δ) If $z \in L$ then there exist sequences $\{z_i: i \in N\}$, $\{w_i: i \in N\} \subseteq L$ such that $z = \bigwedge \{z_i: i \in N\}$, and for all $i \in N$ $z_{i+1} \leqslant z_i$, $z_{i+1} \wedge w_i = 0$, and $z_i \vee w_i = 1$.

2. Lemma. *Let L be a bounded distributive lattice. If L satisfies condition δ then L satisfies condition β.*

Proof. Suppose that L satisfies condition δ and let $s > t$. Let $\{t_i: i \in N\}$ and $\{w_i: i \in N\}$ be sequences in L such that $t = \bigwedge \{t_i: i \in N\}$, and for all $i \in N$ $t_{i+1} \leqslant t_i$, $t_{i+1} \wedge w_i = 0$ and $t_i \vee w_i = 1$. Since $t \leqslant t_i$ for all $i \in N$, $t \wedge w_i = 0$ for all $i \in N$. Then there must be an $i \in N$ such that $s \wedge w_i > 0$. For if $s \wedge w_i = 0$ for all $i \in N$ then

$$s \wedge t_i = (s \wedge t_i) \vee 0 = (s \wedge t_i) \vee (s \wedge w_i) = s \wedge (t_i \vee w_i) = s \wedge 1 = s.$$

Thus $s \leqslant t_i$ for all $i \in N$ and hence $s \leqslant \bigwedge \{t_i: i \in N\} = t$. This contradiction shows that $s \wedge w_i > 0$ for some $i \in N$ and thus condition β is satisfied.

If X is a topological space, a *zero-set* in X is a set of the form $f^{-1}(\{0\})$ where f is some continuous real-valued function on X. We denote by $Z(X)$ the family of all zero-sets of a topological space X. Under the operations of intersection and union for meet and join, $Z(X)$ becomes a bounded distributive lattice for any topological space X. The structure of $Z(X)$ is investigated in detail in Gillman and Jerison (1960), Chapters 1–3.

By saying that an ultrafilter M on a lattice L is *closed under countable meets*, we mean that if $\{z_i: i \in N\} \subseteq M$ and $\bigwedge \{z_i: i \in N\}$ exists in L then $\bigwedge \{z_i: i \in N\} \in M$. Such ultrafilters are also referred to as *real ultrafilters*.

The characterization theorems that follow have similar proofs although certain technical details differ for each proof. Therefore we provide only the proof of Theorem 9.

3. Theorem. *Let L be a bounded distributive lattice. Then $L = Z(X)$ for some compact Hausdorff topological space X if and only if L satisfies conditions α and δ and every ultrafilter on L is real.*

A topological space is called *pseudocompact* if every continuous real-valued function on the space is bounded. The Hewitt realcompactification of a topological space X is denoted υX. In Gillman and Jerison (1960), Chapter 8, it is shown that for any topological space X, $Z(X)$ and $Z(\upsilon X)$ are lattice isomorphic, and if X is pseudocompact then $\upsilon X = \beta X$. Since compact spaces are obviously pseudocompact, we obtain the following corollary.

4. COROLLARY. *Let L be a bounded distributive lattice. Then $L = Z(X)$ for some pseudocompact space X if and only if L satisfies conditions γ and δ and every ultrafilter on L is real.*

A *filter* F on a lattice L is a subset of L that satisfies conditions (i), (ii) and (iii) in Definition 1, but not necessarily condition (iv). A filter F on the lattice L is said to have the *countable meet property* if $\{z_i: i \in N\} \subseteq F$ implies that $\bigwedge \{z_i: i \in N\} \neq 0$. Recall that a topological space is called *Lindelöf* if every family of closed subsets of the space with the countable intersection property (that is, every countable subfamily has non-empty intersection) has non-empty intersection.

5. THEOREM. *Let L be a bounded distributive lattice. Then $L = Z(X)$ for some Lindelöf space X if and only if L satisfies conditions α, γ, δ, and every filter on L with the countable meet property is contained in a real ultrafilter.*

6. COROLLARY. *Let X be a topological space. Then υX is Lindelöf if and only if every family of zero-sets of $Z(X)$ with the countable intersection property is contained in a real z-ultrafilter* (*that is, a real ultrafilter on $Z(X)$*).

A topological space X is called *realcompact* if every real ultrafilter on $Z(X)$ has non-empty intersection in X. Recall that X is *normal* if disjoint closed sets in X can be separated by disjoint open sets in X.

7. THEOREM. *Let L be a bounded distributive lattice. Then $L = Z(X)$ for some normal realcompact space X if and only if L satisfies conditions α, γ, δ and the following two conditions:*

(a) *If $z \in L$ and $z \neq 0$, then there is a real ultrafilter M on L such that $z \in M$.*

(b) *If $S = A \cup B \subseteq L$ such that S is not contained in any real ultrafilter on L, then there are elements $s, t \in L$ such that $s \vee t = 1$ and neither of the sets $A \cup \{s\}$ or $B \cup \{t\}$ is contained in any real ultrafilter on L.*

8. COROLLARY. *Let X be a topological space. Then υX is normal if and only if $Z(X)$ satisfies condition* (b) *of Theorem* 7.

The above corollary follows from Theorem 7 since every lattice of zero-sets c topological space always satisfies condition (a).

If L is a bounded lattice, let $Mv(L)$ denote the family of all real ultrafilters on Note that $Mv(L) \subseteq M(L)$.

9. THEOREM. *Let L be a bounded distributive lattice. Then $L = Z(X)$ for some re compact space X if and only if L satisfies conditions α, γ, δ and the following t conditions.*

(a) *Every non-zero element of L is contained in a real ultrafilter on L.*

(b) *$\beta(Mv(L)) = M(L)$ (that is, if $Mv(L)$ is regarded as a subspace of $M(L)$ tl $M(L)$ is the Stone–Čech compactification of $Mv(L)$).*

PROOF. *Necessity.* Let X be a realcompact topological space. Let $L = Z(X)$. is well known (see Gillman and Jerison (1960), Chapter 1) that L satisfies conditio α, γ and δ. In Gillman and Jerison (1960), Chapter 8 it is shown that L satisf condition (a). By the results of Gillman and Jerison (1960), Chapters 6 and 8, ' know that $M(L) = \beta X$, $Mv(L) = vX$ and $\beta(vX) = \beta X$. Hence condition (b) satisfied.

Sufficiency. Suppose L satisfies the above conditions. We regard $Mv(L)$ as subspace of $M(L)$. Recall that the family $\{h(z): z \in L\}$ is taken as a base for tl closed sets in $M(L)$ where $h(z) = \{M \in M(L): z \in M\}$ for each $z \in L$. Thus tl family $\{k(z): z \in L\}$ may be taken as a base for the closed subsets of $Mv(L)$ whe $k(z) = h(z) \cap Mv(L) = \{M \in Mv(L): z \in M\}$. The family of closed subsets of topological space X is denoted $K(X)$ and becomes a bounded distributive latti under the operations of intersection and union for meet and join. Since ultrafilte in a bounded distributive lattice are prime, the mappings $h: L \to K(M(L))$ ar $k: L \to K(Mv(L))$ defined above are homomorphisms onto a base for the close sets in $M(L)$ and $Mv(L)$ respectively. The homomorphism k is one-to one (as is h For let $s, t \in L$ such that $s \neq t$. Then either $s \vee t > s$ or $s \vee t > t$. Without loss (generality let us assume that $s \vee t > s$. By hypothesis, L satisfies condition δ. Henc by Lemma 2, L satisfies condition β. That is to say there is an element $a \in L$ suc that $a \wedge (s \vee t) > 0$ while $a \wedge s = 0$. Since

$$a \wedge (s \vee t) = (a \wedge s) \vee (a \wedge t) = 0 \vee (a \wedge t) = a \wedge t > 0,$$

by condition (a) there is an $M \in Mv(L)$ such that $a \wedge t \in M$. Clearly $M \in k(a \wedge t$ hence $M \in k(t)$ and $M \in k(a)$. But if $M \in k(s)$ then

$$M \in k(s) \cap k(a) = k(s \wedge a) = k(0) = \varnothing$$

which is false. Thus $M \in k(t)$ but $M \notin k(s)$ and k is one-to-one.

We show that k actually maps onto $Z(Mv(L))$ and that $Mv(L)$ is a realcompac space so that k is an isomorphism between L and $Z(X)$ for the realcompact spac

$Mv(L)$. Let $z \in L$. We first show that $k(z) \in Z(Mv(L))$. Let $\{z_i: i \in N\}, \{w_i: i \in N\} \subseteq L$ such that $z = \bigwedge\{z_i: i \in N\}$ and for all $i \in N z_{i+1} \leqslant z_i$, $z_{i+1} \wedge w_i = 0$, and $z_i \vee w_i = 1$. It is easily verified that $M(L)$ is a compact Hausdorff space. In particular, $M(L)$ is normal. Thus by Urysohn's Lemma any two disjoint closed sets in $M(L)$ can be separated by disjoint zero-sets in $M(L)$. Since $z_{i+1} \wedge w_i = 0$, $h(z_{i+1}) \cap h(w_i) = \varnothing$ hence for each $i \in N$ there is a zero-set $Z_{i+1} \in Z(M(L))$ such that

$$h(z_{i+1}) \subseteq Z_{i+1} \subseteq M(L) - h(w_i).$$

For $i \in N$ let $T_{i+1} = Z_{i+1} \cap Mv(L)$. Then $T_{i+1} \in Z(Mv(L))$ and

$$\begin{aligned} k(z_{i+1}) &= h(z_{i+1}) \cap Mv(L) \subseteq Z_{i+1} \cap Mv(L) = T_{i+1} \subseteq (M(L) - h(w_i)) \cap Mv(L) \\ &= Mv(L) - k(w_i). \end{aligned}$$

In addition, for each $i \in N$ $z_i \vee w_i = 1$ hence $k(z_i) \cup k(w_i) = Mv(L)$ and thus $Mv(L) - k(w_i) \subseteq k(z_i)$. Thus for each $i \in N$ $k(z_{i+1}) \subseteq T_{i+1} \subseteq Mv(L) - k(w_i) \subseteq k(z_i)$. Now, $z = \bigwedge\{z_i: i \in N\}$ and thus since we are dealing in $Mv(L)$ with real ultrafilters, $k(z) = k(\bigwedge\{z_i: i \in N\}) = \bigcap\{k(z_i): i \in N\}$. Therefore $k(z) = \bigcap\{T_{i+1}: i \in N\}$, and since any countable intersection of zero-sets in a topological space is again a zero-set, this shows that $k(z)$ is a zero-set in $Mv(L)$, that is, $k(z) \in Z(Mv(L))$.

We now show that k maps L onto $Z(Mv(L))$. Let $Z \in Z(Mv(L))$. Then there is a continuous function $f: Mv(L) \to [0, 1]$ (where $[0, 1]$ denotes the closed unit interval) such that $Z = f^{-1}(\{0\})$. For each $i \in N$ let $V_i = f^{-1}([(1/i), 1]) \in Z(Mv(L))$. Then $Z \cap V_i = \varnothing$ for all $i \in N$ and $Z = \bigcap\{(Mv(L) - V_i): i \in N\}$. By condition (b), $\beta(Mv(L)) = M(L)$ and hence Z and V_i have disjoint closures in $M(L)$ for all $i \in N$. Since $\{h(z): z \in L\}$ is a base for the closed subsets of $M(L)$, there is a subset $A \subseteq L$ such that $\mathrm{cl}_{M(L)} Z = \bigcap\{h(z): z \in A\}$. Thus for each $i \in N$,

$$\mathrm{cl}_{M(L)} Z = \bigcap\{h(z): z \in A\} \subseteq M(L) - \mathrm{cl}_{M(L)} V_i.$$

Since $M(L)$ is compact there is a finite subset $A_i \subseteq A$ for each $i \in N$ such that $\mathrm{cl}_{M(L)} Z \subseteq \bigcap\{h(z): z \in A_i\} \subseteq M(L) - \mathrm{cl}_{M(L)} V_i$. Thus

$$\begin{aligned} Z &= \mathrm{cl}_{M(L)} Z \cap Mv(L) \subseteq \bigcap\{h(z): z \in A_i\} \cap Mv(L) \\ &= \bigcap\{k(z): z \in A_i\} \subseteq (M(L) - \mathrm{cl}_{M(L)} V_i) \cap Mv(L) = Mv(L) - V_i. \end{aligned}$$

Let $z_i = \bigwedge\{z: z \in A_i\}$ for $i \in N$ (which we can do since A_i is finite). Then for each $i \in N$, $Z \subseteq k(z_i) \subseteq Mv(L) - V_i$ and since $Z = \bigcap\{(Mv(L) - V_i): i \in N\}$ we have $Z = \bigcap\{k(z_i): i \in N\} = k(\bigwedge\{z_i: i \in N\}) \in Z(Mv(L))$ and k maps L onto $Z(Mv(L))$. Thus k is an isomorphism between L and $Z(Mv(L))$.

Finally, we show that $Mv(L)$ is a realcompact space. Let U be a real ultrafilter on $Z(Mv(L))$. We must show that $\bigcap U \neq \varnothing$. Let $M = k^{-1}(U)$. Then M is a real ultrafilter on L since k is an isomorphism. Let $Z \in U$. Then there is an element $z \in L$ such that $Z = k(z)$. Thus $z \in M$ so $M \in k(z) = Z$. Therefore $M \in \bigcap U$.

In view of the fact that for any topological space X the lattices $Z(X)$ and $Z(\upsilon X$ are isomorphic, Theorem 9 characterizes the lattice of zero-sets of any topologic space.

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