ROTATION AND STELLAR EVOLUTION *

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* This paper was presented by R. Kippenhahn.

## Introduction

Does rotation influence stellar evolution? Does it cause observational effects other than line broadening? Can rotation be responsible for mixing of chemical elements throughout the star? Do evolved stars have rapidly rotating cores? This, for instance, is of interest if one wants to compute the details of supernova events. We are not sure whether rotation has really important effects on the life of a star. There might be no rapidly rotating cores. If we think that a fossile general magnetic field couples core and envelope of an evolved star, the core will always be slowed down by the big inertial momentum of the outer regions.

Indeed, there is observational evidence that rotation in the very interior of a star cannot be too important. White dwarfs seem to be rather slow rotators which indicates that they were slowly rotating when they still were cores of evolved stars. But we do not know too much about white dwarf rotation. They certainly do not rotate critically which would demand an equatorial velocity of 5000 km/sec. But the white dwarf in nova DQ Her with its rotational period of 142 sec indicates a rather rapid rotation with $\omega \approx 0.049$ $sec^{-1}$ (compared to $\omega_{crit} \approx 0.32$ if we assume that the mass is 0.5 $M_\odot$). This dwarf rotates much faster than it would if it were coupled by a magnetic field to a red giant envelope. Then its angular velocity would only be $\approx 10^{-6}$. But we do not know whether accretion has sped up the rotation since the formation of the white dwarf out of an evolved red giant or a supergiant. But the high angular velocity can only have been obtained by accretion if the white dwarf has increased its mass by about 3%. This, on the other hand, seems to be a rather high amount of mass accreted, and therefore this system might give a hint that magnetic fields in evolved stars cannot couple completely cores and envelopes with respect to their rotation. There is also indication that the crab pulsar after the supernova event rotated faster than one would expect if it was formed out of a core which was in solid body rotation with a red supergiant envelope. Hardorp (1974) discussed the empirical



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facts and came to the conclusion that there is some coupling between core and envelope, but that the rotation of the core is not completely slaved to that of the envelope.

But not only the question of rapidly rotating cores is important. Paczynski (1973) showed that the depletion of C relative to N in some early-type stars can be explained by Eddington-Vogt-circulation which mixes material (partially processed by the CNO cycle in the region near but outside the convective core) into the outer regions. Is the carbon depletion in these stars evidence for the existence of Eddington-Vogt circulation? Cottrell and Norris (1978) tried to explain the Bidelman-MacConnell weak g-band stars by circulation caused by rotation. Sweigart and Mengel (1979) used circulation to explain $^{12}C/^{13}C$ ratios and the weak g-band stars among evolved stars. They find a sufficiently big effect if they assume that $\omega$ is considerably bigger near the bottom of the convective zone than on the top. They give some arguments for that case, but since not very much is known about the rotation of convective regions, it is not clear that the stars these authors investigate really do have enough angular velocity to provide the mixing.

At the present moment, we are far from understanding how the angular velocity distribution of a star changes during its evolution. Even if one starts out with a rather simple angular velocity law at zero age main sequence, for instance assuming solid body rotation hoping that in the earlier Hayashi phase all differential rotation has been washed out, even then the future is rather unknown, even if one neglects magnetic effects. There is general acceptance that regions of varying molecular weight can create barriers which cannot be penetrated by circulation and which therefore insulate different regions in a star with respect to their rotation. But in chemically homogeneous radiative regions we are rather uncertain about the evolution of angular velocity distributions. Early attempts to predict the rotation of the interior of an evolved star have been made by Kippenhahn (1962), Kippenhahn, Thomas, Weigert (1965), Kippenhahn, Meyer-Hofmeister, Thomas (1969). Recently elaborate computations have been made by Endal and Sofia (1976), (1978) to follow the angular velocity distribution into the very advanced stages of stellar evolution. All these computations show that as long as there is no exchange of angular momentum throughout the $\mu$-barriers rapidly rotating cores form which may become rotationally unstable. Whether these rapidly rotating cores really form and whether they are of importance for the fate of the star, is uncertain. But many people, ourselves included, would love to see that nature allows rapidly rotating cores in stars, because rapid rotation is much more interesting than slow rotation. The world would be less exciting without rapidly rotating stellar cores!

In the following we shall give some examples where the physics of rotation is not completely clear, or at least not correctly applied by some authors. We shall concentrate on three topics:

a) Can $\omega$ vary on equipotential surfaces?

b) What is the time scale of Goldreich-Schubert-Fricke instabilities?

c) How fast is circulation in surface-near regions?

We think we know the answer to the first two questions, while the third one might still be considered as open.

## Can ω vary on equipotential surfaces?

Endal and Sofia (1978) assume that ω must be constant on equipotential surfaces. Law (1980) in her Yale thesis on differential rotation of low mass stars also assumed it. Papaloizou and Pringle (1979) believe that ω must be constant on equipotential surfaces and use it as an argument against the theory of accretion belts proposed by Kippenhahn and Thomas (1978). As long as one considers only axisymmetric perturbations the situation is clear. Along equipotential surfaces a compressible fluid behaves like an incompressible one because the equipotential surfaces are surfaces of constant density. Any exchange between elements on the same equipotential surface does not require compression or expansion. Therefore, along an equipotential surface the Rayleigh criterion is necessary and sufficient for stability:

$$\frac{ds^2\omega}{ds} \geq 0 \qquad (1)$$

where s is the distance from the axis of rotation. This would indicate that there can be a variation of ω along equipotential surfaces as long as the condition (1) is fulfilled. It is well known that this criterion can easily be derived by computing the work which has been put into centrifugal force if one exchanges two tori of equal mass. Condition (1) then is equivalent with the condition that the net work against centrifugal acceleration is positive; one has to put energy in in order to make the exchange. It therefore does not occur spontaneously. The situation seems to be more complicated if non-axisymmetric perturbations are taken into account. At first sight it does not seem that conservation of angular momentum during the exchange is a good approximation, because then azimuthal pressure gradients occur, and by these angular momentum can, in principle, be transported. Indeed Cowling (1951) postulates "if more general (non axisymmetric) displacements are considered, azimuthal pressure gradients insure that the specific angular momentum does not remain constant". But in an inviscid fluid it is difficult to transport momentum even if there are azimuthal pressure gradients. A typical example is a rigid sphere carried horizontally through an inviscid fluid. It is well known that there is no drag. All the momentum lost on the front side of the sphere to the surrounding liquid is gained back on the rear side. The flow is a potential flow and no momentum is lost from the sphere. If this heuristic argument were correct, then Rayleigh's criterion (1) would also be valid for non-axisymmetric perturbations and ω could vary on equipotential surfaces as long as (1) is fulfilled. The general reluctance of many authors to believe this, might come from the feeling that differential rotation might

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cause shear instabilities. Also Zahn (1975) seems to be influenced by this argument when he tries to transform the Richardson number for plane parallel motion into one which can be used for rotational flow, where he replaced the plane parallel shear $\partial v/\partial z$ by $r d\omega/dr$ and concluded that shear instability (which for motion along equipotential surfaces cannot be stabilized by buoyancy) permits only $\omega$ = constant on equipotential surfaces. Cowling's and Zahn's arguments favour that in stars $\omega$ must be constant on equipotential surfaces and that this condition should be achieved within a dynamical time scale. But, as already mentioned, motions along equipotential surfaces in a stratified gas are very similar to motions in compressible fluids. Indeed Cowling's effect of azimuthal pressure components as well as Zahn's form of defining shear for a rotational flow should also hold for incompressible fluids. What, then do we know about the stability of flow? Chandrasekhar (1961) published a proof that the Rayleigh criterion (1) is necessary and sufficient for stability, even if one includes non-axisymmetric perturbations. This would indicate that an angular velocity distribution with $\omega \sim s^{-2}$ is marginal while $\omega \sim s^{-3/2}$, for instance, would be stable (the latter would be unstable according to Zahn's arguments). In order to make the confusion complete, in the 1970 print of Chandrasekhar's book the old statement is revised and the author finds that instabilities cannot be excluded even if condition (1) is fulfilled.

Fortunately, in the case of incompressible fluids experiment can give the answer. In Fig. 1 one can see that the marginal state is that of $d(s^2\omega)/ds = 0$. If one starts out with an angular velocity distribution which violates (1), then instability followed by a re-distribution of angular momentum will occur until the rotational state is marginally stable.

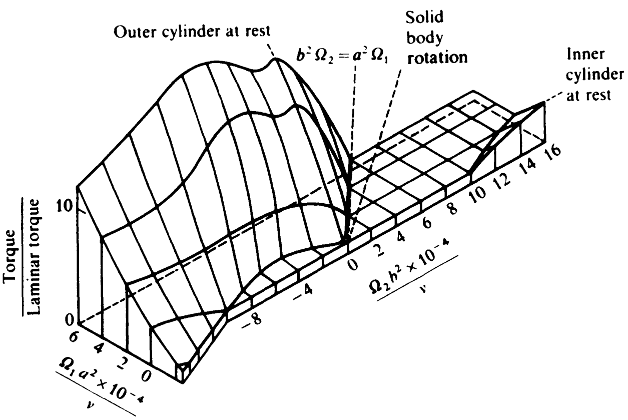


Figure 1 The stability of Couette flow (Joseph, 1976)

The torque in units of the laminar torque between two coaxial cylinders of radii a,b and of angular velocities $\Omega_1$, $\Omega_2$ is plotted. If its value is one the flow is laminar, the motion of the fluid is stable. If it is bigger than one the torque is enhanced by turbulence, the flow is unstable (the graph is based on experiments with rotating fluids). The coordinates in the $\Omega_1$-$\Omega_2$-plane are normalized with respect to the kinematic viscosity $\nu$.

As one can see there is a line of marginal stability with $b^2\Omega_2 = a^2\Omega_1$, this is the line of constant specific angular momentum. The case of solid body rotation is stable. Between the two broken straight lines which indicate solid body rotation and constant specific angular momentum differential rotation is stable. There is also a wide area of differential rotation where the angular velocity increases outwards and the flow is stable. Only if the angular velocity increases outwards too rapidly instability sets in although the Rayleigh criterion predicts stability.

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Solid body rotation, as one can see from Fig. 1, is not marginal, it is safe in the stable region. Only if condition (1) is fulfilled and the angular velocity gradient is very high, does a new instability set in. This instability can be compared with shear instability in the plane parallel case. Indeed, if the characteristic length scale for the variation of $\omega$ is small compared to the radius of curvature, the fluid behaves just like it does in the plane parallel case. But the instability on the left in Fig. 1 is well separated from that on the right by a large region of stability. The instability on the left can also be compared with the plane parallel shear instability, but one has to keep in mind that, for rotation, shear does not mean deviation from $\omega$ = constant but from constant specific angular momentum!

We therefore conclude that if one wants to compute the evolution of the angular velocity distribution in a star, one cannot put $\omega$ = const. on equipotential surfaces. The $\omega$-distribution on equipotential surfaces depends on the history of the star. There is no effect which smoothes out differences in $\omega$ along equipotential surfaces.

Furthermore we want to emphasize that in the case of accretion belts (Kippenhahn, Thomas (1978)) there is no reason to assume that shear instabilities distribute the rapidly rotating accreted material from the equatorial region over the whole surface of the accreting star. Shear instability is sometimes claimed to be responsible for turbulence, and, therefore, for turbulent friction in accretion disks. But Kepler's law $\omega \sim s^{-3/2}$ is stable; there is no shear instability in accretion disks!

## What is the time scale of the Goldreich-Schubert-Fricke instability?

When at least one of the conditions

$$\partial(s^2\omega)/\partial s \geq 0 \ , \quad \partial\omega/\partial z = 0 \qquad (2)$$

is violated ($s$, $\theta$, $z$ being cylindrical polar coordinates about the rotation axis), the angular velocity distribution in a star is

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secularly unstable (Goldreich & Schubert, 1967, Fricke 1968). Goldreich and Schubert originally estimated the time scale in which this instability can redistribute the angular velocity distribution within the sun to be of the order of ten years. Later Colgate (1968) and Kippenhahn (1969) using different physical arguments estimated the time scale to be at least of the order of the Kelvin-Helmholtz time scale of the star, which, for the sun, is about $10^7$ years. There are some hints that the Kelvin-Helmholtz time scale $\tau_{KH}$ is not very good as a lower bound. Since the instability must be driven by rotation, the time scale should depend on the angular velocity. Quickly rotating stars, being more unstable, should redistribute their angular momentum in a shorter time scale than slowly rotating stars. The simplest time scale which fulfills this condition is $\tau_{EV} \approx \tau_{KH}/\chi$, the Eddington-Vogt time scale which is the time scale in which meridional circulation caused by rotation moves throughout the star (it is about $10^{13}$ years for the sun). Here $\chi$ is a mean value over the star for the ratio of the absolute values of centrifugal to gravitational acceleration. Indeed, James and Kahn (1970, 1971) argued in favour of a time scale comparable to the Eddington-Vogt time scale of the star. But their theory has never been fully developed and it seems that it is based on different physical arguments than the estimate which has been given by Kippenhahn, Ruschenplatt, Thomas (1980a) and which we will now discuss in more detail.

In the paper mentioned above we give new arguments which favour a time scale comparable to that of the Eddington-Vogt circulation. Our estimates derived from following the motions due to the instability into the non-linear domain. Our arguments are similar to those which we used to estimate the time scale of a secularly unstable distribution of molecular weight in a star (Kippenhahn et al. 1980b). The simplest form of the argument is the case in which the condition $d(s^2\omega)/ds \geq 0$ is violated. Then any torus of matter which has expanded its big radius $s_o$ by $\ell$ has - compared to its surroundings - an excess angular velocity

$$D\omega = -\frac{1}{s^2}\,\ell\,\frac{d\Theta}{ds}, \quad \Theta = s^2\omega . \tag{3}$$

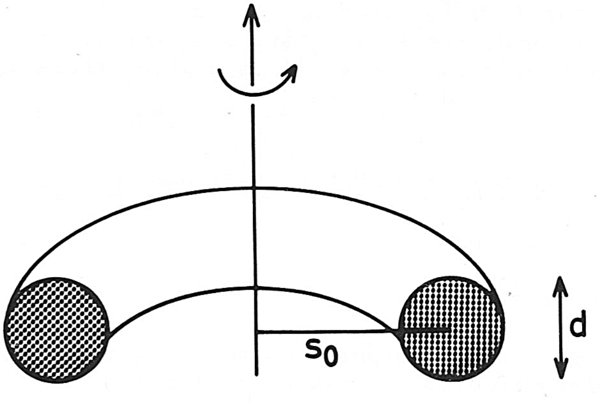


Figure 2 A torus of "big" radius $s_o$ and "small" diameter d around the axis of rotation.

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In order to have hydrostatic equilibrium during the expansion of the big radius it had to change its smaller diameter d in such a way that the density in the expanded torus is bigger so that the gravitational force and the centrifugal force compensate each other. Since there must be pressure equilibrium with the surrounding, the matter in the torus must have a temperature higher by DT from that of the surrounding:

$$\frac{DT}{T} = -2\,\chi\,\frac{D\omega}{\omega} < 0 \tag{4}$$

The torus, therefore, is not thermally adjusted. With the time scale $\tau^*$ of thermal adjustment it tries to heat up to the temperature of the surroundings. Consequently, it cannot remain steadily in hydrostatic equilibrium but must slowly increase its big radius again. An estimate of the velocity by which the big radius expands can be given (Kippenhahn 1969)

$$v_\omega = \frac{H_p}{(\nabla_{ad}-\nabla)\tau^*}\quad 2\,\chi\,\frac{D\omega}{\omega}\,, \quad \tau^* = \frac{3c_p\kappa\rho^2\zeta d^2}{8acT^3}\,, \tag{5}$$

where $H_p$ is the pressure scale height. The other quantities have the usual meaning. Motions of this type by which torus-like mass elements expand their big radius and transport angular momentum constitute the mechanism by which the angular momentum is redistributed in order to get a stable or at least marginally stable angular velocity distribution. The question is, how effective this mixing is. Its effectivity depends on the velocity of the mass elements and their mean free path. The velocity as given by equ. (5) depends on the distance $\ell$ from the region where the mass element originated, as can be seen from equ. (3) and it therefore depends also on the mean free path. Whereas Kippenhahn, in his earlier estimate, assumed that the mean free path is limited by shear instabilities, we have now found a mechanism which destroys the mass element sooner and, therefore, determines the mean free path. This mechanism can be easily understood. While the torus is moving outwards, it is always cooler than the surrounding, its excess temperature follows from equs. (3), (4)

$$\frac{DT}{T} = 2\,\frac{\chi\ell}{\omega s^2}\,\frac{d\Theta}{ds} < 0\,, \tag{6}$$

and the temperature difference becomes bigger the further the mass element has moved from its original position. During its motion it is continuously receiving radiation from the neighbouring region.

It therefore acts as a heat sink for its surrounding and creates a small circulation pattern in its neighbourhood. The topology of the pattern is such that it tries to mix the matter of the mass elements with the surrounding (see Fig. 3).

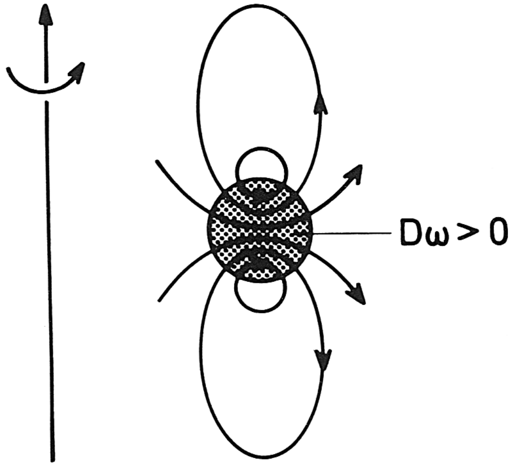


Figure 3 The velocity field in the neighbourhood of a torus with an excess velocity Dω > 0. The matter in the torus (indicated by the dotted area) is cooler. In the area around the torus a circulation system is created which mixes the matter of the torus with its surrounding.

Kippenhahn et al. (1980a) have estimated that this "self-destruction" by mixing is effective within the time during which the torus expands its big radius by a length which corresponds to its small diameter. In equ. (6) we therefore have to replace $\ell$ by d and one finds

$$|v_\omega| = \frac{2\chi H_p d}{(\nabla_{ad}-\nabla)\tau^* H_\Theta}, \quad H_\Theta = |ds/d\ln\Theta| \tag{7}$$

The redistribution is therefore given by mass elements which move outwards or inwards and mix with the surrounding after they have moved along a distance comparable with their own size*). One can, therefore, define a diffusion coefficient:

$$D = v_\omega \ell = v_\omega d = \frac{2\chi H_p d^2}{(\nabla_{ad}-\nabla)\tau^* H_\Theta} . \tag{8}$$

It should be mentioned that this diffusion coefficient does not depend on the size of the elements since the thermal adjustment time scale $\tau^*$ is proportional to $d^2$. The corresponding diffusion time scale over a distance W is given by

*) Here we have assumed that the characteristic mass element has the form of a torus with small diameter d. The same would hold if the mass element had the topology of a sphere with mean diameter d. If these little "drops" move off or towards the axis of rotation, the motion is no longer axisymmetric. But, as long as friction can be neglected, drops which have an excess velocity can move through the matter as easily as tori, since in a frictionless fluid by azimuthal pressure gradients there is no exchange of angular momentum for reasons which we have already discussed in the foregoing section.

$$\tau_{diff} = \frac{H_\Theta (\nabla_{ad} - \nabla)}{H_p} \frac{\tau^W_{KH}}{2\chi\zeta} \tag{9}$$

where we have introduced the thermal adjustment time scale of the mass of a shell of thickness W, which corresponds to the thermal adjustment time scale of a mass element of thickness d according to the formula

$$\tau^W_{KH} = \frac{W^2}{d^2} \tau^* \tag{10}$$

If W becomes the radius of the star we find for the diffusion time scale

$$\tau_{diff} = \frac{H_\Theta (\nabla_{ad} - \nabla)}{H_p} \frac{\tau_{KH}}{2\chi\zeta} \tag{11}$$

which indeed is of the order of the Eddington-Vogt time scale.

Similar arguments can be used to show that this is also the time scale if the condition $\partial\omega/\partial z = 0$ is violated.

If, therefore, the angular velocity distribution in a star violates the Goldreich-Schubert-Fricke condition, then mass elements of all sizes will start a random motion and redistribute angular momentum. One can describe this random motion as a diffusion process and its effect on mixing angular momentum as kind of turbulent friction. In appendix A we give an estimate of the turbulent viscosity, which is caused by GSF instability.

## How fast is circulation in surface-near regions?

Chemical anomalism in A-stars and correlation between pulsational variability and rotational velocity for A stars have provoked a series of papers in which the effect of a fractional sedimentation was used to explain these stars. For reference, see, for instance, the papers by Baglin (1972) and by Vauclair (1976). Although the outer convective regions of these stars are rather shallow and, therefore, convection does not contribute considerably to mixing, there is meridional circulation caused by the stellar rotation. Baglin (1972) takes this into account and uses shear instabilities caused by the circulation to explain why, in some stars, sedimentation seems to be effective but not in others.

In order to see the principles we assume an unevolved star of two solar masses with $R = 1.61\ R_\odot$, $X = 0.732$, $Y = 0.240$, $v_{equ} = 50$ km/sec. For this model we have $\chi = 0.011$ for surface-near equatorial regions. For our estimates we take the region with $\log P = 5.03$ as a representive layer, which is just below the stellar hydrogen convective zone. There we have

$H_p = 1.75\times10^8$, $1-\beta = 0.014$, $\nabla_{ad} - \nabla = 0.115$, $\ell_{opt} = (\kappa\rho)^{-1} = 1.23\times10^6$.

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The classical Sweet formula (Sweet 1950) for this star would give a radial component of the circulation velocity in surface-near regions

$$v_r \approx 6.7\times 10^{-4}\chi = 7.4\times 10^{-6} \qquad (12)$$

this velocity would not prevent sedimentation, which according to Baglin's formula (1972) is $v_{sed} = 1.4\times 10^{-4}$. But as Baker and Kippenhahn (1959) have shown, for non-uniform rotation, an additional term becomes important in surface-near regions. If there is no solid body rotation, their estimate gives

$$v_r \approx 6.7\times 10^{-4}\,\frac{\bar{\rho}}{\rho}\,\chi = 1.65\times 10^{2} \qquad (13)$$

Here $\bar{\rho}$ is the mean density $3M/4\pi R^3$ of the star. But even if there is solid body rotation, another term appears if one takes into account second order effects in the small quantity $\chi$:

$$v_r \approx 4.5\times 10^{-4}\,\frac{\bar{\rho}}{\rho}\,\chi^2 = 1.21 \qquad (14)$$

This has been shown by Öpik (1951) and Mestel (1966).

As already mentioned, the Sweet term (12) does no harm to sedimentation, the other two would mix faster than separation by sedimentation. In the paper by Vauclair (1976) an argument by Osaki (1972) was used to ignore the dangerous $\bar{\rho}/\rho$-terms. In the following we show where the difficulties lie in getting the meridional circulation effects sufficiently small.

Osaki (1972) notes, correctly, that the estimates (12) - (14) are based on the assumption of a steady state. If time derivatives are taken into account in the energy equation, the situation is different. He considers the case where the terms containing time derivatives in the energy equation are large compared to the terms which describe the transport of energy by circulation. In this case one obtains an equation similar to the heat equation and one can assume that a steady state is reached after some time. But Osaki is aware that this steady state is GSF unstable and that, therefore, the angular velocity distribution will not remain in this state. We have a minor objection to the Osaki picture in the following sense: The equations from which Osaki derives his equations are identical with those from which the meridional motion caused by GSF instability is derived. Therefore, we do not think that first a steady state is reached and then instability sets in, but that Osaki's steady state is never reached. And, probably, this is what Osaki had in mind when he wrote that a "steady state may likely be established between the meridional circulation and the irregular motion due to this instability". Therefore if one wants to apply Osaki's reduction of the meridional circulation to the sedimentation problem, one must, at least, estimate how big the circulation is in the final state suggested by Osaki.

Recently we have rediscussed the problem and it seems that we arrive at a solution, which is probably equivalent to what Osaki has suggested. If one allows a small deviation $\omega^*$ from solid body rotation $\Omega$, one can determine this deviation in such a way that the Baker-Kippenhahn term and the Öpik-Mestel term cancel each other. As is shown in detail in appendix B, $\omega^*$ is given by

$$\frac{\omega^*}{\omega} = \frac{4}{15}\,\frac{\Omega^2 r^3}{GM}\,[4\,\cot^2\theta(1-\cos^3\theta)-1] \tag{15}$$

Its $\theta$-dependence is given in Fig. 4.

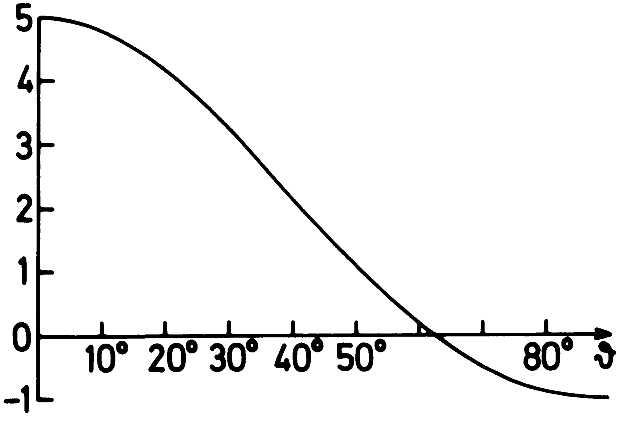


Figure 4 To demonstrate the $\theta$-dependence of the angular velocity $\omega$ the function $4\cot^2\theta(1-\cos^3\theta)-1$ is plotted as a function of $\theta$. It changes sign at $\theta = 62\overset{\circ}{.}2$.

If, in the surface-near regions of a star, the angular velocity distribution is given by $\omega = \Omega + \omega^*$ then the meridional velocity is given solely by the Sweet estimate (12) up to second order terms in $\chi$ and cannot overcome sedimentation.

Unfortunately, the angular velocity distribution as given by equ. (15) is GSF unstable since $\omega$ varies on cylinders coaxial with the axis of rotation. As a consequence of this, we expect that turbulent friction will occur as given by equ. (A5). This friction would immediately cause deviations from the angular velocity distribution given by eq. (15), and, therefore, the compensation given by eq. (B23) would be distorted. If one demands a steady state the circulation must compensate the flux of angular momentum caused by friction:

$$s\rho(\underline{v}\,\underline{\nabla})(\omega s^2) = \underline{\nabla}\,(\eta_t s^3 \underline{\nabla}\omega). \tag{16}$$

The $\omega$ on the left-hand side of eq. (16) is roughly equal to $\Omega$ = const. while $\underline{\nabla}\omega$ on the right can be replaced by $\underline{\nabla}\omega^*$. Then an estimate for the right-hand side is given by

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$$|\underline{\nabla}\ (\eta_t s^3 \underline{\nabla}\omega)| \approx \chi\Omega\ s^2\eta_t/H_p \tag{17}$$

where we have made use of the following estimates:

$$|\nabla\eta_t| \approx \eta_t/H_p\ , \quad s \approx r \approx R, \quad |\underline{\nabla}\omega| \approx \chi/R \tag{18}$$

and where we have assumed $H_p << r$. This gives with eq. (A5)

$$v_s \approx \frac{\chi\eta_t}{2\rho H_p} \approx \frac{16}{5}\frac{(1-\beta)\chi^3 c}{(\nabla_{ad}-\nabla)\zeta R\kappa\rho}\ . \tag{19}$$

Since $v_s \approx v_\Theta$ and $v_r \approx H_p v_\Theta/R$ we find

$$v_r \approx \frac{16}{5}\frac{(1-\beta)}{(\nabla_{ad}-\nabla)\zeta}\frac{H_p c}{\kappa\rho R^2}\chi^3 . \tag{20}$$

For the stellar model used here we find with $\zeta = 1/6$ (spherical turbulent elements)

$$v_r \approx 1.4\times10^{-3}. \tag{21}$$

The mixing due to circulation is about ten times more effective than separation by sedimentation. However our estimates are only approximate, and we think, therefore, that sedimentation as a mechanism for the separation of chemical elements in surface-near regions of stars with ineffective outer convective zones cannot be excluded completely. In our picture there are two effects which reduce the circulation velocity: the special angular velocity distribution which kills the $\bar{\rho}/\rho$-effects and the ineffectivness of the unstables modes in GSF unstable region due to self-destruction.

In the foregoing considerations we have shown only that there exists an angular velocity distribution which strongly reduces the circulation speed in surface near regions but we have not been able to show how out of a given initial angular velocity distribution such a special state of rotation could evolve.

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APPENDIX A: The turbulent viscosity induced by GSF instabilities

We define two characters with length scales:

$$H_\omega = |\partial z / \partial \ln \omega|$$

$$H_\theta = \begin{cases} -\partial s/\partial \ln\theta \ , & \text{if } \partial s/\partial \ln\theta < 0 \\ \infty \ , & \text{if } \partial s/\partial \ln\theta \geq 0 \ , \ \theta = s^2\omega . \end{cases} \tag{A1}$$

They are measures for the degree of violation of the GSF instabilities. In case of stability they are infinite. The smaller they are the more unstable is the angular velocity distribution. In principle we should derive the turbulent viscosity separately for the two cases in which each of the two GSF instability conditions is violated. Instead of this we use the minimum of the two quantities

$$H = \mathrm{Min}(H_\omega, H_\theta). \tag{A2}$$

Indeed as one can see from the detailed derivation in the paper by Kippenhahn et al. (1980a) the velocity of a turbulent element of size d which has moved along a distance of its size in both cases can be described by

$$v_t = \frac{H_p}{H} \frac{2\chi}{\nabla_{ad} - \nabla} \frac{d}{\tau^*} \tag{A3}$$

If we then define an eddy viscosity by

$$\eta_t = \rho v_t d \tag{A4}$$

we find

$$\eta_t = \frac{16}{3} \frac{\chi}{\nabla_{ad} - \nabla} \frac{H_p}{H} \frac{1}{\zeta} \frac{acT^3}{c_p \kappa \rho} \tag{A5}$$

where again $\zeta$ is a dimensionless quantity of the order 1 which depends on the geometry of the elements. With this eddy viscosity one can define a timescale over which this viscosity can change a GSF unstable angular velocity distribution over a certain distance. This timescale then is of the order of the diffusion timescale already derived in eq. (9).

It is of interest to compare the eddy viscosity given in eq. (A5) with the radiative viscosity given by

$$\eta_R = \frac{2}{15} \frac{aT^4}{c\kappa\rho} . \tag{A6}$$

The ratio of the two viscosities then is given by simple dimensionless factors

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$$\frac{\eta_t}{\eta_R} = 40 \frac{\chi}{\nabla_{ad}-\nabla} \frac{H_p}{H} \frac{1}{\zeta} \frac{c^2}{v_c^2}$$

where $v_c$ is the velocity of sound.

## APPENDIX B: The angular velocity distribution with highly reduced circulation

We start from Eq. (7) of Baker & Kippenhahn (1959)

$$(\text{div}\ \underline{F})^{(2)} = \frac{d}{d\psi}\left(-\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\right)(\text{grad}^2\psi)^{(2)} + + \frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\left(\frac{1}{s}\frac{\partial(\omega^2 s^2)}{\partial s}\right)^{(2)} \tag{B1}$$

This formula holds for angular velocity distributions which have conservative centrifugal acceleration. $\psi$ is the sum of gravitational and centrifugal potential. Any axisymmetric function A can be considered as the sum of a function $A^{(o)}$ which is constant on $\psi$-surfaces and a function $A^{(2)}$ which has vanishing mean values on all $\psi$-surfaces. Eq. (B1) is a relation between these latter types of functions. We apply eq. (B1) not only to the angular velocity distribution $\omega = \Omega$ = const. (which has a conservative centrifugal field) but also to distributions in the neighbourhood of that: $\omega = \Omega+\omega^*(s,z)$. As long as $|\omega^*/\Omega|<\chi$ the formula is still valid in second order in $\chi$.

We want to show that the second order term of the first summand produced by solid body rotation (Öpik, 1951; Mestel, 1966), which does not depend on the density, can be compensated by a small correction $\omega$ to solid body rotation with $\Omega$ = const. via the second summand. For this we have to evaluate the first summand to second order in the rotation parameter $\chi = \Omega^2 r^3/GM$.

The luminosity is defined as

$$L = \int_\psi \underline{F}\, d\underline{\sigma} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\int_\psi |\nabla\psi|\, d\sigma. \tag{B2}$$

where $d\sigma$ is the surface element of an equipotential surface. With $M_\psi$ defined as the total mass interior to the surface $\psi$ = const. one has

$$\frac{\partial\psi}{\partial r} = \frac{GM_\psi}{r^2} - \Omega^2 r \sin^2\theta \tag{B3}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\Omega^2 r \sin\theta\cos\theta \tag{B4}$$

and therefore

$$d\sigma = 2\pi r^2 \sin\theta\, d\theta \sqrt{1 + \left(\frac{dr}{r d\theta}\right)^2}\ . \tag{B5}$$

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Since $dr/rd\theta$ is of the order $\chi$ it can be neglected. And combining Eqs. (B3) and (B4) we obtain

$$|\nabla\psi| = \frac{GM_\psi}{r^2}(1 - \chi\sin^2\theta) + O(\chi^2). \tag{B6}$$

The integration then gives

$$\int_\psi |\nabla\psi| d\sigma = 4\pi GM_\psi(1 - \frac{2}{3}\chi_e) \tag{B7}$$

with $\chi_e = \Omega^2 r_e{}^3/GM_\psi$, $r_e$ being the equatorial radius of the surface $\psi$ = const. Therefore, we can write

$$-\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi} = \frac{L}{4\pi GM_\psi}(1 + \frac{2}{3}\chi_e) \tag{B8}$$

and, to evaluate its derivative with respect to $\psi$, we need to compute $dM_\psi/d\psi$ and $d\chi_e/d\psi$. One has

$$dM_\psi = \rho dV_\psi = \rho d\psi \int_\psi \frac{d\sigma}{|\nabla\psi|} \tag{B9}$$

and

$$\int_\psi \frac{d\sigma}{|\nabla\psi|} = \frac{4\pi}{GM_\psi}\int_0^{\frac{\pi}{2}} d\theta \sin\theta\, r^4 (1 + \chi\sin^2\theta). \tag{B10}$$

Since this term depends on $\rho$ we take only the lowest order, i.e. $r^4 = r_e{}^4$ = const. and obtain

$$\int_\psi \frac{d}{|\nabla\psi|} = \frac{4\pi r_e{}^4}{GM_\psi}. \tag{B11}$$

From the definition of $\chi_e$ one gets

$$\frac{d\chi_e}{d\psi} = 3\frac{\chi_e}{r_e}\frac{dr_e}{d\psi} \tag{B12}$$

with

$$\frac{dr_e}{d\psi} \approx \frac{r_e{}^2}{GM_\psi}. \tag{B13}$$

Differentiating Eq. (B8) leads to

$$\frac{d}{d\psi}\left(-\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\right) = \frac{L}{4\pi GM_\psi}\left(-\frac{1}{M_\psi}\frac{dM_\psi}{d\psi} + \frac{2}{3}\frac{d\chi_e}{d\psi}\right) \tag{B14}$$

and inserting Eqs. (B9), (B11), (B12), and (B13) one has

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$$\frac{d}{d\psi}\left(-\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\right) = \frac{L}{4\pi GM_\psi}\left(-\frac{4\pi r_e{}^4\rho}{GM_\psi{}^2} + 2\,\chi_e\,\frac{r_e}{GM_\psi}\right). \qquad \text{(B15)}$$

If we define a mean density by $M_\psi = \frac{4}{3}\pi\, r_e{}^3\,\bar{\rho}$, we obtain finally

$$\frac{d}{d\psi}\left(-\frac{4acT^3}{3}\frac{dT}{d\psi}\right) = -\frac{L\rho}{M_\psi}\frac{r_e{}^4}{G^2M_\psi{}^2}\left(1-\frac{2}{3}\chi_e\frac{\bar{\rho}}{\rho}\right) \quad . \qquad \text{(B16)}$$

To obtain $(\mathrm{grad}^2\psi)^{(2)}$ we have to split $(\nabla\psi)^2$ into one part which is constant on $\psi$-surfaces and another for which the mean value over a $\psi$-surface is zero. We do this at our level of approximation by substracting from $\mathrm{grad}^2\psi$ a term proportional to the Legendre polynomial $P_2$ such that a term is left which depends on $\psi$ only. For this we need the function $r = r(\theta)$ for $\psi$ = const. From Eqs. (B3) and (B4) one gets

$$\frac{dr}{rd\theta} = \frac{\Omega^2 r^3}{GM_\psi}\ \sin\theta\ \cos\theta + O(\chi^2) \qquad \text{(B17)}$$

and after integration

$$\frac{r_e{}^3}{r^3} = 1 + \frac{3}{2}\chi_e\ \cos^2\theta. \qquad \text{(B18)}$$

For the gradient of $\psi$ we have

$$(\nabla\psi)^2 = \frac{G^2M_\psi{}^2}{r^4}(1-2\,\chi_e\ \sin^2\theta) + O(\chi_e{}^2) \qquad \text{(B19)}$$

and by inserting Eq. (B18) we obtain

$$(\nabla\psi)^2 = \frac{G^2M_\psi{}^2}{r_e{}^4}(1+2\chi_e - 4\chi_e\ \sin^2\theta) \qquad \text{(B20)}$$

which we split into two terms according to the rule given above:

$$(\nabla\psi)^2 = \frac{G^2M_\psi{}^2}{r_e{}^4}\left(1-\frac{2}{3}\chi_e\right) + \frac{8}{3}\chi_e\frac{G^2M_\psi{}^2}{r_e{}^4}\left(1-\frac{3}{2}\sin^2\theta\right). \qquad \text{(B21)}$$

So we now have

$$(\mathrm{grad}^2\psi)^{(2)} = \frac{8}{3}\chi_e\frac{G^2M_\psi{}^2}{r_e{}^4}\left(1-\frac{3}{2}\sin^2\theta\right) \qquad \text{(B22)}$$

and therefore can equate the two terms of Eq. (B1) which do not depend on the density:

$$\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\psi}\left(\frac{1}{s}\frac{\partial(\omega^2 s^2)}{\partial s}\right)^{(2)} = -\frac{16}{9}\frac{L\bar{\rho}}{M_\psi}\chi_e^{\,2}\left(1-\frac{3}{2}\sin^2\theta\right). \tag{B23}$$

With the help of Eq. (B8) this transforms to

$$\frac{1}{\Omega^2}\left(\frac{1}{s}\frac{\partial(\omega^2 s^2)}{\partial s}\right)^{(2)} = \frac{16}{3}\frac{\Omega^2 r^3}{GM_\psi}\left(1-\frac{3}{2}\sin^2\theta\right). \tag{B24}$$

Here we have replaced $r_e$ by r because the $\theta$-dependance of r is of higher order. We now write $\omega = \Omega+\omega^*(s,z)$ and obtain with the approximation $\omega^* \ll \Omega$

$$\frac{1}{s}\frac{\partial(\omega^2 s^2)}{\partial s} = 2\Omega^2 + 2\Omega\,\frac{1}{s}\frac{\partial(\omega^* s^2)}{\partial s} \tag{B25}$$

which, together with Eq. (B24) leads to

$$\frac{\partial(\omega^* s^2)}{\partial s} = \frac{8}{3}\frac{\Omega^3(s^2+z^2)^{3/2}s}{GM_\psi}\left(1-\frac{3}{2}\frac{s^2}{s^2+z^2}\right). \tag{B26}$$

Integration over s results in

$$\omega^* = \frac{4}{3}\frac{\Omega^3}{GM}\frac{1}{s^2}\left[\frac{1}{5}(4z^2-s^2)(s^2+z^2)^{3/2} + f(z)\right], \tag{B27}$$

where we have replaced $M_\psi$ by the total mass M, which corresponds to neglecting a term of the order $\rho/\bar{\rho}$. To determine the function f(z) we require $\omega^*$ to remain finite for $s \to 0$, so

$$f(z) = -\frac{4}{5}z^5 \tag{B28}$$

and we finally have

$$\omega^* = \frac{4}{15}\frac{\Omega^3}{GM}\frac{1}{s^2}\left[(4z^2-s^2)(s^2+z^2)^{3/2} - 4z^5\right], \tag{B29}$$

or, in spherical polar coordinates

$$\omega^* = \frac{4}{15}\frac{\Omega^3 r^3}{GM}\left[4\cot^2\theta(1-\cos^3\theta) - 1\right]. \tag{B30}$$

The angular velocity distribution $\Omega+\omega^*$ has to be distinguished from the so-called circulation-free angular velocity distributions as they are discussed by Schwarzschild (1942), Kippenhahn (1963), Roxburgh (1964). The latter describe states of rotation which up to first order in $\chi$ do not cause circulation. The velocity distribution determined in this appendix demands a circulation which up to second order in $\chi$ has no $\rho/\bar{\rho}$-term. Nevertheless it gives the normal Sweet type of circulation velocity in the first order. But this circulation is unimportant for the problem of sedimentation.

## REFERENCES

Baglin, A.: 1972, Astron. Astrophys. 19, p. 45.
Baker, N., Kippenhahn, R.: 1959, Z. Astrophys. 48, p. 140.
Chandrasekhar, S.: 1961, in: "Hydrodynamic and Hydrogmagnetic Stability", Oxford Clarendon Press
Colgate, S.A.: 1968, Astrophys. J. (Letters) 153, p. L81.
Cowling, T.G.: 1951, Astrophys. J. 114, p. 272.
Cottrell, p.L., Norris, J.: 1978, Astrophys. J. 221, p. 896.
Endal. A.S., Sofia, S.: 1976, Astrophys. J. 210, p. 184.
Endal, A.S., Sofia, S.: 1978, Astrophys. J. 220, p. 279.
Fricke, K.: 1968, Z. Astrophys. 68, p. 317
Goldreich, P., Schubert, G.: 1967, Astrophys. J. 150, p. 571.
Hardorp, J.: 1974, Astron. Astrophys. 32, p. 133.
James, R.A., Kahn, F.D.: 1970, Astron. Astrophys. 5, p. 232.
James, R.A., Kahn, F.D.: 1971, Astron. Astrophys. 12, p. 332.
Joseph, D.D.: 1976, "Stability of Fluid Motions, Springer, Heidelberg.
Kippenhahn, R., Thomas, H.-C., Weigert, A.: 1965, Z. Astrophys. 61, p. 241.
Kippenhahn, R., Meyer-Hofmeister, E., Thomas, H.-C.: 1969, Astron. Astrophys. 5, p. 155.
Kippenhahn, R., Thomas, H.-C.: 1978, Astron. Astrophys. 63, p. 265.
Kippenhahn, R., Ruschenplatt, G., Thomas, H.-C.: 1980a, Astron. Astrophys. (in press).
Kippenhahn, R., Ruschenplatt, G., Thomas, H.-C.: 1980b, Astron. Astrophys. (in press).
Kippenhahn, R.: 1963, Proc. of the Varenna Conference "Star Evolution", ed. L. Gratton, p. 330.
Kippenhahn, R.: 1969, Astron. Astrophys. 2, p. 309.
Law, W.Y.: 1980, Yale Thesis.
Mestel, L.: 1966, Z. Astrophys. 63, p. 196.
Osaki, Y.: 1972, Publ. Astr. Soc. Japan 24, p. 509.
Paczynski, B.: 1973, Acta Astronomica 23, p. 191.
Papaloizou, J., Pringle, J.E.: 1979, Discussion remarks on the paper by Sparks and Kutter, IAU Coll. No. 53: "White Dwarfs and Variable Degenerate Stars", Ed. Van Horn and Weidemann, Rochester, p. 294.
Öpik, E.J.: 1951, M.N.R.A.S. 111, p. 278.
Roxburgh, I.W.: 1964, M.N.R.A.S. 128, p. 157.
Schwarzschild, M.: 1942, Astrophys. J. 95, p. 44.
Sweet, P.A.: 1950, M.N.R.A.S. 110, p. 548.
Sweigart, A.V., Mengel, J.G.: 1979, Astrophys. J. 229, p. 624.
Vauclair, G.: 1976, Astron. Astrophys. 50, p. 435.
Zahn, J.P.: 1975, Mém. Soc. Roy. des Sc. de Liège, 6e ser., tome VIII, p. 31.

DISCUSSION

Osaki: I would like to comment on the three points which Prof. Kippenhahn has raised. Firstly, I am happy to hear your argument that differential rotation on the equipotential surface is not sufficient to induce instability. I think that we must be more careful in discussing the viscosity and the stability of the accretion disk. Secondly, do you agree with me that the second term, $\partial s/\partial t$, in the energy equation which governs the meridional circulation velocity becomes important near the surface zone? How did you estimate the order of magnitude of the circulation velocity when $v \sim 10^{-2}$ cm/sec?

Kippenhahn: Concerning your second point, the term $\partial s/\partial t$ can become important, but I have difficulty understanding your conclusion that this term will lead to a circulation-free $\omega$-distribution for the reasons I have given in our paper. We have estimated the circulation velocity by introducing an eddy viscosity caused by the Goldreich-Schubert-Fricke instability (using our value for the mean free path determined by self destruction). This viscosity gives a deviation $\tilde{\omega}$ from $\omega = \Omega + \omega^*$ which in a steady state demands an additional circulation velocity of the order I had mentioned.

Osaki: Thirdly, how sensitive is your result to your assumed form of perturbations? I ask this question because the perturbation you assumed does not conform to an eigenfunction of the linear mode. Did you consider the salt-finger type perturbation as well as the blob-type perturbation?

Kippenhahn: Concerning your third point, we have started our work with the problem of thermohaline mixing (the paper will appear in Astronomy and Astrophysics). There, as well as here, we have not investigated perturbations which correspond to the eigenfunctions of the linear theory. What we did is to investigate the fully developed nonlinear motion. We have made our estimate with a torus-like perturbation, but the mechanism of self destruction does not depend on the detailed geometry. All kinds of volumes that are in hydrostatic equilibrium with the surroundings, but have different angular velocities, will be destroyed by the circulation which they create.

Roxburgh: I understand that the most unstable modes in the Goldreich Schubert-Fricke unstable star are long thin modes, so I doubt the validity of an axially symmetric perturbation analysis. Since the stability analysis gives the growth rates, an estimate (or lower limit) on the diffusion rate can be obtained by determining the amplitude of the linear growing mode at which the nonlinear terms become important. Does this not give a reasonable estimate and has it been done?

Kippenhahn: The axisymmetry is not essential to our analysis. Self destruction appears in all kinds of perturbations you can make up. In the kind of analysis you suggest, you can learn how long it takes for the nonlinear terms to become important but you do not learn what will happen afterwards. You would not even find that a mechanism like self destruction exists.

Schatzman: Is the core of the sun rotating fast? Can you give an upper limit to the angular velocity of the solar core?

Kippenhahn: I do not know the upper limit by heart. However, it arises from the lack of an observed solar oblateness. An estimate has been given by D. Bortenverfer in a paper published in A & A about 6 years ago. You can probably find the latest value in one of the solar neutrino review papers.