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MODULAR REPRESENTATIONS OF FINITE GROUPS WITH UNSATURATED SPLIT (B, N)-PAIRS

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1. Introduction. Let p be a prime number. A finite group $G = (G, B, N, R, U)$ is called a *split* (B, N)*-pair of characteristic* p *and rank* n if

(i) G has a (B, N)-pair (see [**3**, Definition 2.1, p. B-8]) where $H = B \cap N$ and the Weyl group $W = N/H$ is generated by the set $R = \{w_1, \ldots, w_n\}$ of "special generators."

(ii) $H = \bigcap_{n \in N} n^{-1}Bn$.

(iii) There exists a p-subgroup U of G such that $B = UH$ is a semi-direct product, $U \trianglelefteq B$ and H is abelian with order prime to p.

A (B, N)-pair satisfying (ii) is called a *saturated* (B, N)-pair. We call a finite group G which satisfies (i) and (iii) an *unsaturated* split (B, N)-pair. (Unsaturated means "not necessarily saturated".) If $C = \bigcap_{n \in N} n^{-1}Un$ then G has a (saturated) split (B, N)-pair if and only if $C = 1$.

We are interested in studying the irreducible modular representations of unsaturated split (B, N)-pairs over an algebraically closed field k of characteristic p. Curtis ([**3**]) and Richen ([**7**]) have dealt with the saturated case. Unsaturated split (B, N)-pairs arise, for example, as the (B, N)-pairs of parabolic subgroups of groups with split (B, N)-pairs. It is true that any unsaturated split (B, N)-pair G can be replaced by a saturated pair $(B, \tilde{N})$ where $\tilde{N} = N\tilde{H}$ and $\tilde{H} = CH$ (see [**3**, Proposition 2.6, p. B-10]) and the two pairs have isomorphic Weyl groups. However, $B = U\tilde{H}$ may no longer satisfy (iii) above and we cannot apply the Curtis-Richen results on irreducible kG-modules directly.

However, if $C \trianglelefteq G$ then $(G/C, B/C, N, R, U/C)$ is a saturated split (B, N)-pair and since C is a p-group there exists a bijection between the set of isomorphism classes of irreducible kG-modules and the set of isomorphism classes of irreducible $k(G/C)$-modules. So in the case when $C \trianglelefteq G$ we could apply the Curtis-Richen theory to the group G/C.

There are three major advantages in a theory where we only assume G has an unsaturated split (B, N)-pair:

(1) Unsaturated groups appear even in the course of the Curtis-Richen treatment of the saturated case (see [**3**, Corollary 5.1, p. B-24]

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and [7, Lemma 3.23, p. 453]) and so we get some simplification of proofs by dealing with unsaturated groups from the start.

(2) The unsaturated theory applies at once to parabolic subgroups G_J $(J \subseteq R)$ of G (see [9]).

(3) In case $p = 2$ there exist unsaturated G for which the group C is not normal in G (see below). The irreducible kG-modules for such a group G cannot be reached by the "saturated" theory.

Let $G = (G, B, N, R, U)$ be an unsaturated split (B, N)-pair. All k-spaces are assumed to be finite dimensional. We study the modular representations of G over k in the following way: Let $Y = \mathrm{Ind}_U{}^G(k_U)$ and $E = \mathrm{End}_{kG}(Y)$ where k_U is the trivial U-module k. Sawada ([8]) was the first to examine Y and E for groups with split (B, N)-pairs and he established a bijective correspondence between the set of isomorphism classes of irreducible left kG-modules and the set of isomorphism classes of irreducible right E-modules. In doing so, Sawada relied upon work done by Curtis and Richen. We begin by discussing the right E-modules directly, and we are able to recover most of the results of Curtis, Richen and Sawada by using a recent theorem of Green ([5]). By this method we are able to discard the saturation condition from the general theory.

The main results of this paper are the following theorems concerning the algebra E:

a) E is Frobenius (3.7).

b) Every simple right E-module is one-dimensional and is thus given by a multiplicative character $\psi : E \rightarrow k$ (3.13).

c) Each such ψ is determined by a vector $(\chi, \mu_1, \ldots, \mu_n)$ where χ is a linear character of B and $\mu_i \in k$ $(1 \leqq i \leqq n)$ (3.22).

These vectors $(\chi, \mu_1, \ldots, \mu_n)$ correspond exactly to Curtis' "weights" (see [3, Definition 4.2, p. B-17, B-18]) and many, but not all, of the arguments used in the proofs of b) and c) are based on Curtis' work. The application of our results concerning the irreducible representations of E, to the irreducible representations of G, is given at the end of Section 3.

Using a result of Kantor and Seitz on 2-transitive permutation groups we show that if p is odd the group C is normal in G for all unsaturated split (B, N)-pairs (4.5). Lastly we give an example of a rank one unsaturated split (B, N)-pair when $p = 2$ but C is not normal in G.

Notations. Since H is abelian, $B = UH$ and U is a p-group all modular irreducible characters of B are linear. Let $k^* = k \backslash \{1\}$ and $\hat{B} = \mathrm{Hom}(B, k^*)$. If $x, g \in G$ then $x^g = g^{-1}xg$. For any subset T of G, $[T] = \sum_{t \in T} t \in kG$ and $T^g = g^{-1}Tg$. Let $w \in W$ and $(w) \in N$ be such that $(w)H = w$. For X any subgroup of G containing H, we write Xw for $X(w)$ (similarly for wX, XwX). If A is any subgroup of G normalized by H, then $A^{(w)} = A^{h(w)}$ for any $h \in H$ so we write A^w. The Weyl group W acts on the elements of H by $h^w = h^{(w)}$ since H is abelian. If H is a subgroup of G and M is any kG-module, $M|H$ denotes the restriction of M to H and we sometimes write $\rho|H$ if ρ is the character afforded by M.

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2. Unsaturated split (B, N)**-pairs.** Let p be a prime number and let $G = (G, B, N, R, U)$ be an unsaturated split (B, N)-pair of characteristic p and rank n. Therefore, we allow that $\bigcap_{n \in N} B^n \neq H$. We assume, unless otherwise stated, that k is an algebraically closed field of characteristic p. Let $v: N \rightarrow W$ be the natural epimorphism where $W = N/H$ is the Weyl group of the (B, N)-pair. The set $R = \{w_1, \ldots, w_n\}$ generates W and the length of $w \in W$ as a product of such generators is denoted $l(w)$. The unique element of maximal length in W will be denoted w_0.

Let $Y \cong \mathrm{Ind}_U{}^G(k_U)$ and let y correspond to the element $1_{kG} \otimes_{kU} 1_k$. If $\{g_i | \ i \in I\}$ is a left transversal for the cosets of U in G, then $Y = kG \cdot y$ has k-basis $\{g_i y | \ i \in I\}$. Let $E = \mathrm{End}_{kG}(Y)$.

We assume that $\{(w) | \ w \in W\}$ is a fixed, but arbitrary, set of coset representatives of H in N.

In this preliminary section we state results which, though proved in [**3**] and [**7**] under the assumption of saturation, do not actually depend on that condition. For example, statements in [**7**, Chapter II] which do not involve $H = B \cap N$ will be true in the unsaturated case. We also make adjustments to other results when necessary to suit our unsaturated hypothesis.

The reader will notice that the proofs of certain technical lemmas in Sections 1 and 2 have been deferred to Section 5 where the specific rank 1 case is discussed.

Notation. Let $w \in W$. Then ${}_wB^+ = B \cap B^w$; ${}_wU^+ = U \cap U^w$; ${}_wB^- = B \cap B^{w_0 w}$; and ${}_wU^- = U \cap U^{w_0 w}$. Write U_w^+, U_w^- for ${}_{w^{-1}}U^+$, ${}_{w^{-1}}U^-$ respectively.

Remark 1. Notice that ${}_wB^+ = {}_wU^+H$, ${}_wB^- = {}_wU^-H$ (see [**7**, proof of Theorem 3.3(b), p. 444]) and that H normalizes ${}_wU^+$, ${}_wU^-$ for any $w \in W$.

2.1 Lemma. *The intersection of the N-conjugates of B is $B \cap B^{w_0}$. Also*

$$\bigcap_{n \in N} U^n = \bigcap_{w \in W} U^w = U \cap U^{w_0}.$$

Proof. We need only show that $B \cap B^{w_0} \subseteq {}_wB^+$ for all $w \in W$. A proof of this fact can be found in [**7**, proof of Lemma 2.4, p. 441]. The second statement follows from the remark above.

Remark 2. Let $C = {}_{w_0}U^+$. Then $C^w = C$ for all $w \in W$ by 2.1.

2.2 LEMMA. *Let* $wv \in W$ *satisfy* $l(vw) = l(v) + l(w)$. *Then*

$$_{vw}U^- = {}_wU^-({}_vU^-)^w \text{ and } {}_wU^- \cap ({}_vU^-)^w = C.$$

Proof. The first part follows by an easy induction on $l(w)$ from [**7**, proof of Theorem 3.3(a), p. 444]. By 2.1, $C \subseteq {}_wU^- \cap ({}_vU^-)^w$ and

$$\begin{aligned} {}_wU^- \cap ({}_vU^-)^w &= U \cap U^{w_0w} \cap U^{w_0vw} \cap U^w \\ &\subseteq U^{w_0w} \cap U^w \\ &= (U^{w_0} \cap U)^w \\ &= C \text{ by remark 2.} \end{aligned}$$

2.3 COROLLARY. *Let* $w \in W$. *Then* $U = {}_wU^+{}_wU^-$ *and* ${}_wU^+ \cap {}_wU^- = C$. *Hence* $|U|c = |{}_wU^+||{}_wU^-|$ *where* $c = |C|$.

Proof. Let $v = w_0w^{-1}$ and apply 2.2.

2.4 LEMMA. *Let* $w \in W$. *Let* Ω_w *be a left transversal* (*containing* 1) *of* U_w^- *by* C. *Then* Ω_w *is automatically a transversal of* U *by* U_w^+ *by* 2.3 *and* $|\Omega_w| = |U_w^-|/C$. *Also* $BwB = UwB = \Omega_wwB$.

Notation. For $w_i \in R$, write Ω_i for Ω_{w_i}, B_i for ${}_{w_i}B^-$ and U_i for ${}_{w_i}U^-$.

The following short lemmas are consequences of results proven in the rank one case (see 5.1–5.4) and the Bruhat Decomposition Theorem (see [**1**, Theorem 1, p. 25]).

2.5 LEMMA. *Let* $w \in W$. *Then* $\Omega_w^{(w)} \cap B = 1$.

2.6 LEMMA. *Let* $w_1, w_2 \in W$, $u_1, u_2 \in U$, $h_1, h_2 \in H$. *Then*

$$u_1h_1(w_1)U = u_2h_2(w_2)U \Leftrightarrow w_1 = w_2,\ u_2^{-1}u_1 \in U_{w_1}^+,\ h_1 = h_2.$$

The set $\Gamma = \{u_wh(w) \mid h \in H, u_w \in \Omega_w, w \in W\}$ *is a transversal for the left cosets of* U *in* G.

2.7 LEMMA. *Every element of* G *can be uniquely expressed as* $g = u(w)hu'$ *where* $w \in W$, $u \in \Omega_w$, $h \in H$ *and* $u' \in U$.

The next lemma is a consequence of 2.6 and 2.7.

2.8 LEMMA. *The elements of* N *form a transversal for the* U-U *double cosets of* G.

3. The endomorphism algebra E. In this section we characterize the simple right E-modules.

By 2.8 E has k-basis $\{A_n | n \in N\}$ where $A_n(y) = p_ny$ and p_n is the sum of those $\gamma \in \Gamma$ which lie in UnU (see, for example, [**8**, p. 32]). The elements A_n ($n \in N$) are clearly independent of the choice of transversal of the cosets of U in G. Therefore, using 2.6,

3.1 $$A_n(y) = [\Omega_w]ny$$
$$p_n = [\Omega_w]n \text{ where } v(n) = w.$$

Clearly $p_h = h$ for all $h \in H$. Multiplication in E is given by the formulae

3.2 $$A_m A_n = \sum_{t \in N} c_{mnt} A_t \quad (m, n \in N)$$

where $c_{mnt} = z_{mnt} 1_k$ and $z_{mnt} \in \mathbf{Z}$ is the number of pairs $(\gamma, \xi) \in \Gamma \times \Gamma$ such that $\gamma \in UnU$, $\xi \in UmU$ and $\gamma\xi \in tU$ since $A_t(y)$ is the sum of all the distinct U-translates of ty and $gy = g'y \Leftrightarrow gU = g'U$ for any $g, g' \in G$. The following lemma is immediate.

3.3 LEMMA. *If $t, m, n \in N$ are such that $UtU \not\subseteq UnUmU$, then the coefficient of A_t in $A_m A_n$ is zero.*

3.4 LEMMA. *Let $n, m \in N$ with $v(n) = v$, $v(m) = w$ be such that $l(vw) = l(v) + l(w)$. Then $A_m A_n = A_{nm}$.*

Proof. We know

$$A_m A_n(y) = [\Omega_v]n[\Omega_w]my$$
$$= [\Omega_v]n[\Omega_w]n^{-1}nmy.$$

By 2.2 $U_{vw}^- = U_v^- \cdot nU_w^- n^{-1}$ and $|U_{vw}^-|c = |U_v^-| \cdot |U_w^-|$. We see that $A_m A_n(y)$ is the sum of $|\Omega_v||\Omega_w|$ U-translates of nmy by our choice of transversals (2.4). Therefore $A_m A_n = \lambda A_{nm}$ where λ is the integer $|\Omega_v||\Omega_w|/|\Omega_{vw}|$. By 2.4

$$\lambda = \frac{|U_v^-||U_w^-|}{c^2} \cdot \frac{c}{|U_{vw}^-|}$$

so that $\lambda = 1$ as required.

3.5. COROLLARY. *Let $h \in H$, $n \in N$. Then*

$$A_n A_h = A_{hn} = A_{n^{-1}hn} A_n.$$

3.6 COROLLARY. *The set $\{A_h, A_{(w_i)} \mid h \in H, w_i \in R\}$ k-algebra generates E.*

We can now state and prove one of the main results of this paper. The proof is due to Green who proved it for the saturated case. Notice that the proof relies only on 3.4 and is therefore true for any field.

3.7 PROPOSITION. *Let G be a finite group with an unsaturated split (B, N)-pair of characteristic p and rank n. Let k be any field. Then E is a Frobenius algebra.*

Proof. Let $q \in N$ satisfy $v(q) = w_0$, the unique element of maximal length in W. Let $f: E \times E \to k$ be given as follows: For $\alpha, \beta \in E$, $f(\alpha, \beta)$

is to be the coefficient of A_q in the expression of $\alpha\beta$ as a linear combination of the basis elements $\{A_n | n \in N\}$. Certainly f is bilinear and associative and we need only show that f is non-degenerate. Let $\{Z_n | n \in N\}$ be the basis of E given by $Z_n = A_{n^{-1}q}$. We require the following lemma:

3.8 LEMMA. *Let $n, n' \in N$, $v(n) = w$, $v(n') = w'$. Then $f(Z_n, A_{n'})$ is zero if either* (i) $l(w) > l(w')$ *or* (ii) $l(w) = l(w')$ *but $w \neq w'$. In the case $w = w'$, $f(Z_n, A_{n'}) = \delta_{n,n'}$ (that is, 1 for $n = n'$ and 0 otherwise).*

Proof. By 3.3 the coefficient of A_q in $Z_nA_{n'} = A_{n^{-1}q}A_{n'}$ is 0 if $UqU \not\subseteq Un'Un^{-1}qU$. So $f(Z_n, A_{n'})$ is certainly 0 if

$$Bw_0B \not\subseteq Bw'Bw^{-1}w_0B. \tag{*}$$

Since

$$l(w'w^{-1}w_0) \leqq l(w') + l(w^{-1}w_0) = l(w') + l(w_0) - l(w)$$

(*) holds in (i) or (ii) (see [**1**, Lemme 1, p. 23]). If $w = w'$, we see that

$$A_{n^{-1}q}A_{n'} = A_{n'n^{-1}q}$$

by 3.4 since

$$l(w'w^{-1}w_0) = l(w_0) = l(w') + l(w^{-1}w_0).$$

Hence $f(Z_n, A_{n'})$ is 0 or 1 depending upon whether $n \neq n'$ or $n = n'$ and 3.8 is proved.

Now the elements of N can be totally ordered so that

$$l(v(n)) < l(v(n')) \Rightarrow n < n'.$$

So if for $n, n' \in N$ we have $n \geqq n'$ then we must have $l(v(n)) \geqq l(v(n'))$. By 3.8 $f(A_n, A_n') = \delta_{n,n'}$ and we see that the matrix

$$(f(Z_n, A_{n'}))_{n,n' \in N}$$

is unitriangular and hence nonsingular. We have shown that f is non-degenerate and the proof of Proposition 3.7 is completed.

Definition. Let $w_i \in R$. Define $G_i = \langle U, U_i^{w_i}\rangle$, $H_i = G_i \cap H$.

3.9 LEMMA. (see [**3**, Proposition 3.7, p. B-15]) *Let $w_i \in R$. We can arrange that $(w_i) \in G_i$. In this case $G_i = UH_i \cup \Omega_iH_i(w_i)U$.*

Proof. Consider $P_i = B \cup Bw_iB$ and any representative $(w_i)'$ of w_i. Let $1 \neq u \in \Omega_i$. Then $u^{(w_i)'} \in P_i$ and if $u^{(w_i)'} \in B$ then $u = 1$ by 2.5. Therefore

$$u^{(w_i)'} \in Bw_iB = \Omega_iw_iB.$$

Hence there exists a representative $(w_i) \in G_i$ because the subgroup

$$\langle U, \Omega_i^{(w_i)'}\rangle$$

does not depend on $(w_i)'$ since

$$U_i^{(w_i)'} = U_i^{w_i} = \Omega_i^{(w_i)'}C^{(w_i)'} = \Omega_i^{(w_i)'}C \text{ and}$$
$$\langle U, \Omega_i^{(w_i)'}\rangle = \langle U, \Omega_i^{(w_i)'}C\rangle = \langle U, U_i^{w_i}\rangle.$$

The subgroup G_i has the required form since $G_i \subset P_i$.

We assume from now on that $(w_i) \in G_i$, for every $w_i \in R$.
For proofs of the following two lemmas see 5.6 and 5.10.

3.10 STRUCTURAL EQUATIONS IN G. *Let* $w_i \in R$, $\Omega_i^* = \Omega_i\backslash\{1\}$. *There exist functions* $f_i : \Omega_i^* \to \Omega_i^*$, $g_i : \Omega_i^* \to U$, $h_i : \Omega_i^* \to H$ *where* f_i *is a bijection, such that for every* $u \in \Omega_i^*$

$$(w_i)u(w_i) = f_i(u)h_i(u)(w_i)g_i(u).$$

Since $(w_i) \in G_i$, $h_i(u) \in H_i$ *for all* $u \in \Omega_i^*$.

3.11 LEMMA. *Let* $w_i \in R$. *Then*

$$A_{(w_i)}{}^2 = A_{(w_i)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \textit{where} \quad b(i) = |\Omega_i^*|$$

and $u_{i_1}, \ldots, u_{i_{b(i)}}$ *are certain elements of* Ω_i^* *(not necessarily distinct).*

The following formulae were first determined by Sawada ([**8**, Proposition 2.6, p. 34]) for the saturated case.

3.12 FORMULAE. *Let* $n \in N$, $v(n) = w$.
(i) *If* $l(w_iw) = l(w) + 1$, *then* $A_nA_{(w_i)} = A_{(w_i)n}$.

(ii) *If* $l(w_iw) = l(w) - 1$, *then* $A_nA_{(w_i)} = A_n \sum_{s=1}^{b(i)} A_{h_i}(u_{i_s})$.

(iii) *If* $l(ww_i) = l(w) + 1$, *then* $A_{(w_i)}A_n = A_{n(w_i)}$.

(iv) *If* $l(ww_i) = l(w) - 1$, *then* $A_{(w_i)}A_n = \sum_{s=1}^{b(i)} A_{(w_i)^{-1}h_i(u_{i_s})(w_i)}A_n$

Proof. Parts (i) and (iii) follow from 3.4. For (ii) let $w = w_iv$ with $l(v) = l(w) - 1$. Then $(w_i)^{-1}n = m \in N$, $v(m) = v$ and

$$A_n = A_{(w_i)m} = A_mA_{(w_i)}$$

by 3.4. Therefore

$$\begin{aligned} A_nA_{(w_i)} &= A_mA_{(w_i)}{}^2 \\ &= A_mA_{(w_i)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by 3.11} \\ &= A_n \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by 3.4.} \end{aligned}$$

Part (iv) is proved similarly using Lemma 3.5.

Definition. Let $\chi \in \hat{B}$, $w \in W$. Then ${}^w\chi \in \hat{B}$ where ${}^w\chi(hu) = \chi(h^w u)$ for $h \in H$, $u \in U$.

The proof of the following lemma is based on [**3**, proof of Theorem 4.3a, p. B-20].

3.13 LEMMA. *Every irreducible right E-module X is one-dimensional and if $X = kx$ there exists a character $\chi \in \hat{B}$ uniquely defined by $xA_h = \chi(h)x$ for all $h \in H$.*

Proof. Every one-dimensional right E-module will uniquely determine a character of B since by 3.4

$$A_h A_{h'} = A_{h'h} = A_{h'} A_h \ (h, h' \in H).$$

Let

$$\chi \in \hat{B}, \quad E_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h^{-1}) A_h.$$

Then

$$E_\chi A_h = \chi(h) E_\chi \text{ for all } h \in H \text{ and}$$

$$1_E = \sum_{\chi \in \hat{B}} \oplus E_\chi.$$

Since

$$X = \sum_{\chi \in \hat{B}} \oplus X E_\chi$$

there exists $\chi \in \hat{B}$ with $XE_\chi \neq 0$. Take any $z \in X$ for which $zE_\chi \neq 0$ and let $t = zE_\chi$. Then $tA_h = \chi(h)t$ for all $h \in H$.

Choose $w \in W$ of maximal length so that $x = tA_{(w)} \neq 0$. Then x affords the character ${}^w\chi$: that is

$$\begin{aligned} xA_h &= {}^w\chi(h)x \text{ since } xA_h = tA_{(w)}A_h \\ &= tA_{(w)^{-1}h(w)}A_{(w)} \text{ by 3.5} \\ &= {}^w\chi(h)tA_{(w)}. \end{aligned}$$

We now consider $xA_{(w_i)}$ for $w_i \in R$.

Case 1. $l(w_i w) > l(w)$. Then

$$\begin{aligned} xA_{(w_i)} &= tA_{(w)}A_{(w_i)} \\ &= tA_{(w_i)(w)} \text{ by 3.12 (i)} \\ &= tA_{(w_i w)h} \text{ for some } h \in H \text{ since } v((w_i)(w)) = v((w_i w)) \\ &= tA_h A_{(w_i w)} \text{ by 3.4} \\ &= \chi(h) tA_{(w_i w)} \\ &= 0 \text{ by choice of } w. \end{aligned}$$

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Case 2. $l(w_i w) < l(w)$. Then

$$\begin{aligned} xA_{(w_i)} &= tA_{(w)}A_{(w_i)} \\ &= tA_{(w)} \sum_{s=1}^{b(i)} A_{h_i(u_i)} \quad \text{by 3.12 (ii)} \\ &= x \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \\ &= \sum_{s=1}^{b(i)} {}^{w}\chi(h_i(u_{i_s}))x. \end{aligned}$$

Therefore x generates a one-dimensional right E-submodule of X by 3.6. But X irreducible implies $X = kx$.

We are able to formulate more results based on the rank one case, the first being the following crucial lemma.

3.14 LEMMA. *Fix* $\chi \in \hat{B}$, $w_i \in R$. *Let* $d_i = \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s}))$. *If* $d_i \neq 0$ *then* $\chi|H_i = 1$. *Hence* $d_i = -1$.

Proof. By Theorem 5.12 there exists a one-dimensional $P_i = B \cup Bw_iB$-module M such that if M affords $\xi: P_i \to k^*$ then $\xi|H = \chi|H$. Now G_i is generated by p-groups so that $\xi|G_i = 1$ and $\xi|H_i = 1$. Therefore $\chi|H_i = 1$ and since $h_i(u_{i_s}) \in H_i$ ($s = 1, \ldots, b(i)$) (by 3.10) and $b(i) = |\Omega_i| - 1$, the result follows since $1 < |\Omega_i|$ is a power of p.

3.15 LEMMA. *Let* ψ *be any multiplicative character* $\psi: E \to k$. *Then there exist* $\chi \in \hat{B}$, $\mu_1, \ldots, \mu_n \in k$ *such that*

$$\left.\begin{array}{l} \text{(i) } \psi(A_h) = \chi(h) \text{ for all } h \in H \\ \text{(ii) } \psi(A_{(w_i)}) = \mu_i \ (1 \leqq i \leqq n) \end{array}\right\} (*)$$

Moreover, $\mu_i = 0$ *or* -1 *and* $\mu_i \neq 0$ *implies* $\chi|H_i = 1$.

Proof. Part (i) follows from 3.13 and (ii) follows from 3.11 and 3.14.

We might call the sequence $(\chi, \mu_i, \ldots, \mu_n)$ the "weight of ψ" to correspond with Curtis' terminology.

Definition. Let $J \subseteq R$. Then $W_J = \langle w_i | \, w_i \in J \rangle$.

3.16 LEMMA. *Let* $\chi \in \hat{B}$, $J \subseteq R$. *Suppose* $\chi|H_i = 1$ *for every* $w_i \in J$. *Then* ${}^{w}\chi = \chi$ *for all* $w \in W_J$.

Proof. It is sufficient to show ${}^{w_i}\chi = \chi$ for all $w_i \in J$. Since $\chi|H_i = 1$,

$$d_i = \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s})) \neq 0$$

for every $w_i \in J$ and the result follows by Lemma 5.11.

The above lemma is also proved in [**3**, Lemma 5.4, p. B-26] and [**7**, Corollary 3.22, p. 453] under the saturation condition.

We wish to prove the converse of 3.15; that is, given any sequence $(\chi, \mu_i, \ldots, \mu_n)$ where $\chi \in \hat{B}$, $\mu_i \in k$ $(1 \leqq i \leqq n)$ and where $\mu_i = 0$ or -1 with $\mu_i \neq 0$ implying $\chi|H_i = 1$, then there exists a multiplicative character $\psi: E \to k$ with properties (*). In order to do this we place additional restrictions on the choice of coset representations $\{(w_i)|w_i \in R\}$.

The following lemma is due to Tits. A proof can be found in [**4**, (1G), p. 5].

3.17 LEMMA. *Let $w_i \in R$. Then $B_i \cup B_i w_i B_i$ is a subgroup of G.*

Remark. Notice that the above lemma does not depend on a saturated condition since $B_i = U_i H$, $U \cap U^{w_0}$ is normalized by H and $U \cap U^{w_0} \subset U_i$ $(w_i \in R)$.

3.18 LEMMA. *Let $w_i \in R$. Then coset representative (w_i) can be chosen in $\langle U_i, U_i^{w_i}\rangle$.*

Proof. Clearly

$$\langle U_i, U_i^{w_i}\rangle \subset B_i \cup B_i w_i B_i = U_i H \cup U_i H w_i U.$$

If $U_i^{w_i} \subset U_i H$ then $U_i^{w_i} = U_i$ so that

$$\begin{aligned} B^{w_i} &= U_i^{w_i}({}_{w_i}U^+)^{w_i} H \\ &= U_{i\,w_i} U^+ H \\ &= B, \end{aligned}$$

contrary to the (B, N)-pair axioms. Hence $U_i^{w_i} \cap U_i H w_i U_i$ is non-empty and there exists a coset representative n_i and $u_1, u_2, u_3 \in U_i$ such that

$$u_1^{w_i} = u_2 n_i u_3. \text{ Hence}$$

3.19. The coset representative (w_i) can be chosen in $U_i U_i^{w_i} U_i$ and the proof of 3.18 is completed.

Remark. Statement 3.19 is important since we are able to choose the coset representatives $\{(w_i)|\, w_i \in R\}$ in the same way whether the (B, N)-pair is saturated or not (see [**2**, Lemma 2.2, p. 351] or [**3**, Definition 3.9, p. B-16]).

We assume from now on that coset representatives $\{(w_i)|\, w_i \in R\}$ are chosen according to 3.19.

The next lemma (see [**7**, Lemma 3.28, p. 456]) holds in the unsaturated case:

3.20 LEMMA. *Let $J \subseteq R$. Coset representatives $\{(w)|w \in W_J\}$ can be* *chosen so that if $w, w' \in W_J$ then*

$$(w)(w')(ww')^{-1} \in H_J = \langle H_i^w | w \in W_J, w_i \in J\rangle.$$

Definition. For any $\chi \in \hat{B}$, let $e(\chi) = \sum_{h\in H} \chi(h^{-1})A_h$.

Sawada proved the following theorem (see [**8**, Proposition 3.1, p. 36]) for the saturated case. The theorem remains true for the unsaturated case using results above and we omit the proof.

3.21 THEOREM. *Let $J \subseteq R$ and let coset representatives $\{(w) | w \in W_J\}$ be chosen according to* 3.20. *Let $\chi \in \hat{B}$ and suppose $\chi|H_i = 1$ for all $w_i \in J$. Let*

$$z(J, \chi) = e({}^{w_0}\chi) \sum_{w\in W_J} A_{(w)(w_0)}.$$

Then $z = z(J, \chi)$ generates a one-dimensional right E-module (right ideal of E) with the following properties:

(i) $\quad zA_h = \chi(h)z \ (h \in H)$

(ii) $\quad zA_{(w_i)} = \begin{cases} 0 & w_i \in J \text{ or } \chi|H_i \neq 1 \\ -z & w_i \notin J \text{ and } \chi|H_i = 1. \end{cases}$

We can now prove the converse of 3.15, one of the main results of this paper. We might call the sequence $(\chi, \mu_1, \ldots, \mu_n)$ an *admissible vector* if $\chi \in \hat{B}$, all $\mu_i \in \{0, -1\}$ and $\mu_i \neq 0$ implies $\chi|H_i = 1$.

3.22 THEOREM. *Let G be a finite group with an unsaturated split (B, N)-pair of characteristic p and rank n, and let k be an algebraically closed field of the same characteristic. Given any sequence $(\chi, \mu_1, \ldots, \mu_n)$ where $\chi: B \to k^*$ is a homomorphism, $\mu_i \in k$ $(1 \leqq i \leqq n)$ such that $\mu_i = 0$ or -1, there exists a multiplicative character $\psi: E \to k$ given by $\psi(A_h) = \chi(h)$ $(h \in H)$ and $\psi(A_{(w_i)}) = \mu_i$ $(1 \leqq i \leqq n)$ if and only if for any $i \in \{1, \ldots, n\}$ with $u_i \neq 0$ we have $\chi|H_i = 1$.*

Proof. ($\Rightarrow$) This follows by 3.15.

($\Leftarrow$) Let $J = \{w_i \in R | \mu_i = 0 \text{ and } \chi|H_i = 1\}$.

Let $z(J, \chi)$ be as in Theorem 3.21 and the result follows.

Remark. We have shown that $(\chi, \mu_1, \ldots, \mu_n)$ is the weight of some multiplicative character $\psi: E \to k$ if and only if it is an admissible vector.

Definition. Let $\chi \in \hat{B}$, $J \subseteq M(\chi) = \{w_i \in R| \chi|H_i = 1\}$. Then (J, χ) is called an *admissible pair.*

By 3.21 each admissible pair (J, χ) determines an admissible vector $(\chi, \mu_1, \ldots, \mu_n)$ where $\mu_i = 0$ (for $w_i \in J$ or $\chi|H_i \neq 1$) or $\mu_i = -1$ (for

$w_i \notin J$ and $\chi|H_i = 1$). If for each admissible vector $(\chi, \mu_1, \ldots, \mu_n)$ we let

$$J = \{w_i \in R | \mu_i = 0 \text{ and } \chi|H_i = 1\}$$

we see by 3.22 that the correspondence

$$(J, \chi) \leftrightarrow (\chi, \mu_1, \ldots, \mu_n)$$

described above is a bijective one between the set of all admissible pairs and the set of all admissible vectors. We now show how such weights and vectors correspond to Curtis' weights (see [**3**, Definition 4.2, p. B-17, B-18]) and find a full set of irreducible left kG-modules in Y.

Definition. Let M be any finite dimensional left kG-module. Let

$$F(M) = \{m \in M | um = m, \text{all } u \in U\}.$$

Green ([**5**, 1.3]) describes how $F(M)$ may be regarded as a right E-module. In fact, if $m \in F(M)$ and $\alpha \in E$,

$$m\alpha = p_\alpha m \text{ where } \alpha(y) = p_\alpha y \ (p_\alpha \in kG).$$

In particular (by 3.1)

$$3.23 \quad \begin{aligned} mA_{(w_i)} &= [\Omega_i](w_i)m & (w_i \in R) \\ mA_h &= hm & (h \in H) \end{aligned}$$

for all $m \in F(M)$.

Green proves ([**5**, Theorem 2]) that the correspondence $M \rightarrow F(M)$ induces a bijection between the set of isomorphism classes of irreducible left kG-modules and the set of isomorphism classes of simple right E-modules. Since we have shown that all simple right E-modules are one-dimensional (3.13), $F(M)$ is one dimensional if M is an irreducible kG-module and $F(M)$ is associated with an admissible vector $(\chi, \mu_1, \ldots, \mu_n)$ by 3.22. By 3.23 this vector coincides with the Curtis-Richen weight of M and any non-zero $m \in F(M)$ is called a *weight element* of weight $(\chi, \mu_1, \ldots, \mu_n)$. In other words $F(M)$ is precisely the set of all weight elements in M and if M is irreducible then M has a unique U-stable (B-stable) line.

The following theorem was first proved by ([**8**]). His proof uses [**3**], [**7**] and therefore applies only to saturated split (B, N)-pairs.

3.24 THEOREM. *Let G be a finite group with an unsaturated split (B, N)-pair of characteristic p and rank n. Let k be an algebraically closed field of the same characteristic. There exist bijective correspondences between the following:*

(i) *the set of admissible vectors,*
(ii) *the set of admissible pairs,*
(iii) *the set of isomorphism classes of simple right E-modules, and*
(iv) *the set of isomorphism classes of irreducible left kG-modules.*

These correspondences are given by:

$$(\chi, \mu_1, \ldots, \mu_n) \leftrightarrow (J, \chi) \leftrightarrow kz(J, \chi) \leftrightarrow kGz(J, \chi)(y).$$

Proof. We need only verify the correspondence between (iii) and (iv). Green ([**5**, 1.3c]) proves that the map $E \to F(Y)$ given by $\beta \to \beta(y)$ $(\beta \in E)$ is a right E-isomorphism. Let (J, χ) be an admissible pair. Since $z(J, \chi)$ generates a one-dimensional right ideal of E (3.21), $kz(J, \chi)(y)$ is a one-dimensional right E-submodule of $F(Y)$. Therefore by [**5**, 2.6a], $kGz(J, \chi)(y)$ is an irreducible left kG-module and

$$F(kGz(J, \chi)(y)) = kz(J, \chi)(y).$$

If M is any irreducible left kG-module, there exists an admissible pair (J, χ) with

$$F(M) \cong kz(J, \chi) \cong kz(J, \chi)(y)$$

as right E-modules. But M irreducible implies

$$M \cong kGz(J, \chi)(y).$$

Therefore $\{kGz(J, \chi)(y) | (J, \chi) \text{ admissible}\}$ is a full set of irreducible left kG-modules. (Curtis also determines such a set in [**3**, Corollary 6.12, p. B-37].)

4. Normality of C: A counter-example. In this short section we examine the subgroup $C = U \cap U^{w_0} = \bigcap_{n \in N} U^n$. We know that $C = 1$ if and only if G has a saturated split (B, N)-pair. In cases where $C \trianglelefteq G$, the Curtis-Richen theory can be applied to the saturated split (B, N)-pair $(G/C, B/C, N, R, U/C)$. Since C is normalized by H and N (see Remark 2 of Section 2.1), $C \trianglelefteq G$ if and only if $C \trianglelefteq U$. We show that if $C_i = U \cap U^{w_i} \trianglelefteq U$ for all $w_i \in R$ then $C \trianglelefteq U$; that is, $C \trianglelefteq G$ if this condition is satisfied for all rank 1 parabolic subgroups of G (Lemma 4.4). Using a theorem of Kantor and Seitz we show that $C \trianglelefteq G$ if p is odd (Lemma 4.5) and we give an example of a rank 1 (B, N)-pair when $p = 2$ and C is not normal in G.

4.1 LEMMA. $U = \langle (U_i)^{w^{-1}} | w \in W, w_i \in R, l(ww_i) = l(w) + 1 \rangle.$

Proof. Let $w = w_{i_1} \ldots w_{i_t}$ be a reduced expression for $w \in W$. It follows from 2.2 that

$${}_wU^- = (U_{i_t})(U_{i_{t-1}})^{w_{i_t}} \ldots (U_{i_1})^{w_{i_2} \cdots w_{i_t}}$$

since $l(w_{i_t} \ldots w_{i_s} w_{i_{s-1}}) = l(w_{i_t} \ldots w_{i_s}) + 1$ for any $2 \leqq s \leqq t - 1$. Since $U = {}_{w_0}U^-$ we have

$$U \subseteq \langle (U_i)^{w^{-1}} | w \in W, l(ww_i) = l(w) + 1 \rangle.$$

Also if $l(ww_i) = l(w) + 1$ then $(U_i)^{w^{-1}} \subseteq U$ by [**7**, Lemma 2.8, p. 441] which doesn't depend on saturation.

Since $C^w = C$ for all $w \in W$ we have

4.2 LEMMA. *$C \trianglelefteq U$ if and only if $C \trianglelefteq U_i$ for all $w_i \in R$.*

4.3 LEMMA. *Let $w_i \in R$. Assume $C_i \trianglelefteq U$. Then $C \trianglelefteq U_i$.*

Proof. We have

$$\begin{aligned} C &= U \cap U^{w_0} \cap U^{w_i w_0} \\ &= U \cap (U \cap U^{w_i})^{w_i w_0}. \end{aligned}$$

By assumption $(U \cap U^{w_i})^{w_i w_0} \trianglelefteq U^{w_i w_0}$ so that

$$C = C^{w_0 w_i} \trianglelefteq U \cap U^{w_0 w_i} = U_i.$$

The next lemma is immediate by 4.2 and 4.3.

4.4 LEMMA. *Suppose say $C_i \trianglelefteq P_i = B \cup Bw_iB$ for all $w_i \in R$. Then $C \trianglelefteq G$.*

The following discussion will prove one of the main results of this paper:

4.5 THEOREM. *If p is odd, $C \trianglelefteq G$ for all unsaturated split (B, N)-pairs.*

In order to prove 4.5 we may restrict our attention to the rank 1 case by Lemma 4.4. Suppose then that $G = B \cup BwB$ where $(G, B, N, \{w\}, U)$ is an unsaturated split (B, N)-pair. Then

a) G acts 2-transitively on $\Omega = G/B$, the space of cosets gB $(g \in G)$ and

b) $G^* = G/Z$ acts faithfully and 2-transitively on Ω where $Z = \bigcap_{g \in G} B^g$.

Let $\alpha = B, \beta = wB$. Notice that $|\Omega| = 1 + p^t$ where $2 \leqq |U/C| = p^t$ and that $B/Z = (G^*)_\alpha$, the stabilizer in G^* of α. Since U is a normal p-subgroup of B, the group B/Z contains a normal nilpotent subgroup $Q = UZ/Z$. Since $BwB = UwB$, the group Q acts transitively on $\Omega \backslash \{\alpha\}$. By [**6**, Theorem C', p. 131] of Kantor and Seitz we consider the following two cases.

(1) Q is regular on $\Omega\backslash\{\alpha\}$. Therefore $Q_\beta = 1$. But

$$\begin{aligned} Q_\beta &= \{uZ |\ u \in U, u(wB) = wB\} \\ &= \{uZ |\ u \in U, u^w \in B\} \\ &= CZ/Z. \end{aligned}$$

Hence $C \subseteq Z \cap U \subseteq B^w \cap U \subseteq U^w \cap U = C$. Therefore $C = U \cap Z$ and in this case $C \trianglelefteq G$ since $Z \trianglelefteq G$.

(2) G^* contains a regular normal subgroup of order q^2 where q is a

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Mersenne prime. Let $q = 2^r - 1$ where r is prime. Therefore $|\Omega| = q^2$ is an odd integer and p^t is even. Hence $p = 2$.

Thus, we have proved Theorem 4.5.

The argument in [**6**, proof of Corollary 1, p. 139] leads to the following example of a rank 1 unsaturated split (B, N)-pair where $p = 2$ and C is not normal in G.

Let $a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $a, b \in GL(2, 3)$ and $U = \langle a, b \rangle$ has defining relations $b^8 = a^2 = 1$, $a^{-1}ba = b^3$. Moreover, U is a Sylow 2-subgroup of $GL(2, 3)$. Let $M = V(2, 3)$, the space of 2-dimensional column vectors over $GF(3)$. We have a map $\tau : U \rightarrow \text{Aut}\ (M)$ given by $x \rightarrow \tau_x$ where $\tau_x(m) = xm$ $(x \in U, m \in M)$. Let G be the semi-direct product of M and U and let $U_1 = \{(0, x) | x \in U\}$,

$$w = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right),$$

and $N = \{1, w\}$. It can be easily verified that $G = (G, U_1, N, \{w\}, U_1\}$ is an unsaturated split (B, N)-pair. Furthermore, $C = U_1 \cap U_1^w$ is not normalized by $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, b \right)$.

5. The rank one case. If G is a finite group with an unsaturated split (B, N)-pair (G, B, N, R, U) then for each $w_i \in R$ the parabolic subgroup $P_i = B \cup B_{w_i}B$ has an unsaturated split (B, N)-pair $(P_i, B, N_i, \{w_i\}, U)$ of rank 1 where $N_i = H \cup w_iH$. Let $(w_i) \in N$ satisfy $(w_i)H = w_i$. As in the general case the set $\{A_h', A'_{(w_i)} | h \in H\}$ k-algebra generates

$$E_i = \text{End}_{kP_i}\ (Y_i)$$

where

$$Y_i \cong \text{Ind}_U{}^{P_i}(k_U)$$

(see 5.7). There exists an injective k-linear algebra homomorphism

$$\gamma : E_i \rightarrow E$$

given by

$$A_h' \rightarrow A_h\ (h \in H)$$

$$A_{(w_i)}' \rightarrow A_{(w_i)}$$

since the set $\{h, h(w_i) | h \in H\}$ forms part of the transversal for the U-U double cosets in G (see 3.2). Therefore, results proved for the rank one case can be extended to G.

It becomes necessary (5.11, 5.12) to examine $d = \sum_{s=1}^{b} \chi(h(u_s))$ where $\chi \in \hat{B}$ is fixed and the $h(u_s)$ $(s = 1, \ldots, b)$ are certain elements of H

determined by (w_i) and Richen's "structural equations." Since these equations exist for every $w_i \in R$, we refer in Section 3 to

$$d_i = \sum_{s=1}^{b(i)} \chi(h(u_{i_s})).$$

Therefore we now assume G has an unsaturated split (B, N)-pair of rank 1. Let $W = \{1, w\}$ and let Y, y, E and (w) be as in previous sections. We must give some technical lemmas concerning the structure of $G = B \cup BwB$.

5.1 Lemma. *Let Ω be any left transversal (containing 1) of U by ${}_wU^+$. Then*

$$\Omega^{(w)} \cap B = 1.$$

Proof. Since $\Omega \cap B^w \subset B \cap B^w = (U \cap U^w)H$, the result follows.

Remark. Note that $|\Omega| > 1$, for otherwise $U = {}_wU^+$, $wBw = B$, contrary to the (B, N)-pair axioms.

5.2 Lemma. *Cosets of the form gU $(g \in G)$ contained in $BwB = BwU$ are of the form $uh(w)U$ for some $u \in U$, $h \in H$. Moreover, if $u_1, u_2 \in U$ and $h_1, h_2 \in H$ then*

$$u_1h_1(w)U = u_2h_2(w)U \Leftrightarrow u_2^{-1}u_1 \in {}_wU^+ \text{ and } h_1 = h_2.$$

Proof. Clearly $u_1h_1(w)U = u_2h_1(w)U$ if $u_1 = u_2u$ for some $u \in {}_wU^+$ since H normalizes U and ${}_wU^+$.

Say $u_1h_1(w) = u_2h_2(w)u$ $(u \in U)$. Then

$$\begin{aligned} u_2^{-1}u_1 &= h_2(w)u(w)^{-1}h_1^{-1} \\ &= (w)h_2^{\,w}u(h_1^{-1})^w(w)^{-1} \end{aligned}$$

so that

$$u_2^{-1}u_1 \in B^w \cap B = {}_wU^+H.$$

Therefore $u_2^{-1}u_1 \in {}_wU^+$ since it is an element whose order is a power of p. Therefore

$$h_2^{\,w}u(h_1^{-1})^w \in {}_wU^+ \subset U$$

so that

$$(h_2^{\,w}u(h_2^{-1})^w)(h_2^{\,w}(h_1^{-1})^w) \in U.$$

Therefore $h_2^{\,w}(h_1^{-1})^w \in U$ and $h_2 = h_1$.

The following facts are easily verified:

5.3. Let $\Gamma = \{h, u(w)h | h \in H, u \in \Omega\}$. Then Γ is a set of representatives of left cosets of U in G.

5.4. Every element g of G can be uniquely expressed as $g = u_1h$ or $g = u(w)hu_2$ with $u_1, u_2 \in U$, $u \in \Omega$ and $h \in H$.

5.5. The elements of N form a transversal for the U-U double cosets in G.

Richen determines "structural equations" in the saturated case and these equations can be adapted to the unsaturated case, but we omit the detailed proof (see [**7**, p. 445]).

5.6. *Structural Equations in* G. Let $\Omega^* = \Omega \backslash \{1\}$. There exist functions

$$f : \Omega^* \to \Omega^*, g : \Omega^* \to U, h : \Omega^* \to H$$

where f is a bijection and

$$(w)u(w) = f(u)h(u)(w)g(u)$$

for any $u \in \Omega^*$.

We now examine the endomorphism algebra E.

As in Section 3 the set $\{A_n' | \, n \in N\}$ is a k-basis for E where for $h \in H$

5.7
$$A_h'(y) = hy$$
$$A'_{h(w)}(y) = [\Omega]h(w)y.$$

It is easy to see that

5.8 $\quad A_h'A_{(w)}' = A'_{(w)h}$ and $A_{(w)}'A_h' = A'_{h(w)}$ for any $h \in H$.

Therefore

5.9. The set $\{A_h', A_{(w)}' | \, h \in H\}$ k-algebra generates E.

5.10 Lemma. *There exist elements $u_1, \ldots, u_b$ (not necessarily distinct) belonging to Ω^* such that*

$$A_{(w)}'^2 = A_{(w)}' \sum_{s=1}^{b} A'_{h(u_s)}$$

where $b = |\Omega| - 1$.

Proof. We can write $A_{(w)}'^2 = \sum_{h \in H} \lambda_h A_h' + \sum_{h \in H} \lambda_{h(w)} A'_{h(w)}$ where $\lambda_h, \lambda_{h(w)} \in k$ for all $h \in H$. Fix $h \in H$. We show

(i) if $\lambda_h \neq 0$ then $h = (w)^2$ and $\lambda_{(w)^2} = |\Omega| \cdot 1_k$

(ii) if $\lambda_{h(w)} \neq 0$ then $h = h(u)$ for some $u \in \Omega^*$.

Proof of (i). By 3.2 there exist $u_1, u_2 \in \Omega$ such that $u_1(w)u_2(w) \in hU$. We must have $u_2 = 1$ for otherwise

$$(w)^{-1}u_2(w) \in (w)^{-2}hU \subset B$$

contradicting 5.1. Now $u_1(w)^2 \in hU$ if and only if $(w)^2 = h$. It follows that

$$\lambda_{(w)^2} = |\Omega|.1_k.$$

Proof of (ii). If $\lambda_{h(w)} \neq 0$ there exist $u_1, u_2 \in \Omega$ such that

$$u_1(w)u_2(w) \in h(w)U.$$

Therefore by 5.6 $Uh(u_2)(w)U = Uh(w)U$ so that $h = h(u_2)$ by 5.3.

We know that $A_{(w)}'^2(y)$ is a sum of $|\Omega|^2$ G-translates of y; that is $|\Omega|^2$ terms of the form $gy = \gamma y$ ($\gamma \in \Gamma$, $g \in \gamma U$). If the term γy appears, ($\gamma \notin H$), so will each of its distinct G-translates of which there are $|\Omega|$ in number. If we call γy and its set of distinct U-translates an "orbit" then by (i) and because $|\Omega|^2 - |\Omega| = |\Omega|\,(|\Omega| - 1)$, we see that there are $|\Omega| - 1$ such orbits in $\sum_{h \in H} \lambda_{h(w)} A'_{h(w)}$. By (ii) $A_{(w)}'^2$ has the required form since $1 < |\Omega|$ is a power of p.

Definition. For $\chi \in \hat{B}$, let $e(\chi) = \sum_{h \in H} \chi(h^{-1}) A_h'$. Notice that

$$A_h' e(\chi) = e(\chi) A_h' = \chi(h) e(\chi) \text{ for any } h \in H.$$

We now fix $\chi \in \hat{B}$ and examine $d = \sum_{s=1}^{b} \chi(h(u_s))$. Remember that ${}^w\chi \in \hat{B}$ where ${}^w\chi(hu) = \chi(h^w u)$ for any $h \in H$, $u \in U$.

5.11 LEMMA. *Assume $d \neq 0$. Then ${}^w\chi = \chi$.*

Proof. Let $v = e({}^w\chi) A_{(w)}'$. By 5.9 v generates a one-dimensional right E-module since

$$\begin{aligned}
\text{(i) } vA_h' &= e({}^w\chi) A_{(w)}' A_h' \\
&= e({}^w\chi) A'_{h(w)} \text{ by } 5.8 \\
&= e({}^w\chi) A'_{(w)(w)^{-1}h(w)} \\
&= e({}^w\chi) A'_{(w)^{-1}h(w)} A_{(w)}' \text{ by } 5.8 \\
&= \chi(h) v \text{ for all } h \in H, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } vA_{(w)}' &= e({}^w\chi) A_{(w)}'^2 \\
&= e({}^w\chi) A_{(w)}' \sum_{s=1}^{b} A'_{h(u_s)} \quad \text{by } 5.10 \\
&= \sum_{s=1}^{b} \chi(h(u_s)) v \quad \text{by part (i)} \\
&= dv.
\end{aligned}$$

Therefore there exists a multiplicative character $\phi : E \to k$ such that

$$\phi(A_{(w)}') = d \text{ and } \phi(A_h') = \chi(h) \text{ for all } h \in H.$$

But

$$\phi(A_{(w)}' A_h') = \phi(A'_{(w)^{-1}h(w)} A_{(w)}') \qquad \text{for any } h \in H$$

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by 5.8 so that

$$\phi(A_{(w)}')\phi(A_h') = \phi(A'_{(w)^{-1}(w)})\phi(A_{(w)}') \text{ for any } h \in H$$

and so

$$d\chi(h) = {}^w\chi(h)d \text{ for all } h \in H$$

and the result follows.

5.12 THEOREM. *Assume $d \neq 0$. Then there exists a one-dimensional kG-module M_0 affording the character $\xi: G \to k^*$ with $\xi|H = \chi|H$.*

Proof. By 5.11 $A_{(w)}'$ commutes with $e(\chi)$. Hence $e(\chi)$ is in the centre of E and

$$e(\chi)E = e(\chi)Ee(\chi) = ke(\chi) \oplus ke(\chi)A_{(w)}'$$

is an algebra which has basis $e = e(\chi)$ and $t = e(\chi)A_{(w)}'$. Now $e^2 = e$, $et = te = t$, $t^2 = dt$ and $e = e_0 + e_1$ is a decomposition of e into primitive idempotents in $e(\chi)E$ where $e_0 = (1/d)(de - t)$ and $e_1 = (1/d)t$. Let $Y_\chi = e(\chi)Y$. Then Y_χ is a kG-module of dimension $|G:B| = |\Omega| + 1$ since $Y_\chi \cong \mathrm{Ind}_B{}^G(L_\chi)$ where L_χ is a kB-module affording the character χ. Let $m_0 = e_0(Y)$ and $M_1 = e_1(Y)$. Then $Y_\chi = M_0 \oplus M_1$ where M_0 and M_1 are indecomposable left kG-modules. We show that the dimension of M_0 is one by showing the dimension of M_1 is $|\Omega|$. Let $x_1 = e_1(y)$. Then x_1 is U-invariant and

$$\begin{aligned}
[\Omega](w)x_1 &= [\Omega](w)e_1(y) \\
&= e_1([\Omega](w)y) \\
&= e_1A_{(w)}'(y) \\
&= (1/d)e(\chi)A_{(w)}'^2(y) \\
&= (1/d)e(\chi)A_{(w)}' \sum_{s=1}^{b} A'_{h(u_s)}(y) \quad \text{by 5.10} \\
&= (1/d)e(\chi)A_{(w)}' \sum_{s=1}^{b} \chi(h(u_s))y \quad \text{since } e(\chi) \text{ and } A_{(w)}' \text{commute} \\
&= de_1(y) \\
&= dx_1 \neq 0 \text{ as } d \neq 0.
\end{aligned}$$

Therefore M_1 contains an element $x = (w)x_1$ such that $[\Omega]x \neq 0$ and x is stabilized by ${}_wU^+$. Let $L = \mathrm{Ind}_T{}^U\,(k_T)$ where $T = {}_wU^+$. Then there exists a surjective kU-map $\theta: L \to kUx$ given by $\theta(z) = x$ where $z = 1 \otimes 1$. Hence

$$\theta\left(\sum_{\omega\in\Omega} \omega z\right) = \sum_{\omega\in\Omega} \omega x \neq 0.$$

The socle of L is its space of U-invariants $[\Omega]z$. Therefore θ is a bijection

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and the k-space kUx has dimension $|\Omega|$. But $kUx \subset M_1$ and

$$\text{dimension } M_1 = \text{dimension } Y_\chi - \text{dimension } M_0 \leqq |\Omega|$$

so that the dimension of M_1 is $|\Omega|$.

Assume M_0 affords the character $\xi: G \rightarrow k^*$ and let $v = e_0(y)$. Then $M_0 = kv$ and if $h \in H$ it is easily verified that $hv = \chi(h)v$. Hence

$$\xi|H = \chi|H.$$

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