EXTREME COVERINGS OF n-SPACE BY SPHERES

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It is well known that the problem of determining the most economical covering of n-dimensional Euclidean space, by equal spheres whose centres form a lattice, may be formulated in terms of positive definite quadratic forms, as follows:

Let $f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_n) = \boldsymbol{x}'A\boldsymbol{x}$ $(A' = A)$ be positive definite, and $d = d(f) = \det A$. For real $\boldsymbol{\alpha}$, set

$$(1.1) \qquad m(f; \alpha) = \min_{\boldsymbol{x}} f(\boldsymbol{x}+\boldsymbol{\alpha})$$

(the minimum being taken over integral $\boldsymbol{x}$),

$$(1.2) \qquad m(f) = \max_{\boldsymbol{\alpha}} m(f; \boldsymbol{\alpha}),$$

$$(1.3) \qquad \mu(f) = m(f)/d^{1/n}.$$

If now $A = P'P$, and Λ is the lattice spanned by the columns of P, then spheres of radius $(m(f))^{\frac{1}{2}}$ centred at the points of Λ cover space minimally; and, since

$$d(\Lambda) = |\det P| = d^{\frac{1}{2}},$$

the density $\theta(\Lambda)$ of the covering is given by

$$\theta(\Lambda) = J_n(\mu(f))^{\frac{1}{2}n}$$

(where J_n is the volume of the unit sphere).

Thus the problem of minimizing $\theta(\Lambda)$ is equivalent to that of determining

$$(1.4) \qquad \mu_n = \min_{f} \mu(f).$$

If $\mu(f)$ is a local minimum, i.e. if $\mu(g) \geqq \mu(f)$ for all forms g sufficiently close to f, we say that f is *extreme*; and if $\mu(f) = \mu_n$, we say that f is *absolutely extreme*. If f is extreme (absolutely extreme) so is any form equivalent under integral unimodular transformation to a positive multiple of f, and it is convenient to unite such forms into a single class.

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By a direct investigation of neighbouring forms, Bleicher [3] has shown that the form

$$n\sum_{i=1}^{n} x_i^2 - 2\sum_{i<j} x_i x_j \tag{1.5}$$

is extreme for all n; for $n = 2$ and $n = 3$, Barnes [2] showed that this is the only class of extreme forms, which Bambah [1] had previously shown to be absolutely extreme. Delone and Ryskov [4] have announced that the above form is also absolutely extreme when $n = 4$.

The first object of this paper is to establish a criterion for a form to be extreme. The criterion, which is stated in Theorem 1, bears a marked similarity to the condition for a form to be eutactic (which is part of the necessary and sufficient condition for a form to be extreme for the corresponding packing problem). However, there is here no analogue of a "perfect" form (see Voronoï [5]).

Our second main result (Theorems 2 and 3) is that a Voronoï domain Δ (see § 2) contains at most one interior extreme form f (other than the multiples of f), and the group of automorphisms of f is then the same as that of Δ. This result, together with the criterion for extremeness, provides a systematic method of finding all extreme forms in any given dimension when the Voronoï domains are known. One of us intends shortly to publish complete results for $n = 4$, based on this method.

The evidence we have obtained to date supports the conjecture that every Voronoï domain contains an interior extreme form; the truth of this conjecture would, with Theorem 2, imply that every extreme form is an interior form.

In § 2 we recall Voronoï's results, establish some necessary notation and state our theorems. In § 3, we analyze the neighbours of an interior form f, whence we deduce our theorems in §§ 4 and 5. Finally, in § 6, we use our results to show that the form (1.5) is extreme for all n, and further that it represents the only class of extreme forms in Voronoï's "principal domain".

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The Voronoï polytope Π (Voronoï [6]) corresponding to a positive form f is the set of points $\boldsymbol{x}$ such that

$$f(\boldsymbol{x}) \leqq f(\boldsymbol{x}-\boldsymbol{l}) \text{ for all integral } \boldsymbol{l}. \tag{2.1}$$

A finite set $\pm\boldsymbol{l}_1, \pm\boldsymbol{l}_2, \cdots, \pm\boldsymbol{l}_\sigma$ of integral points suffices to define Π, which therefore has σ pairs of opposite parallel faces, with equations $f(\boldsymbol{x}) = f(\boldsymbol{x}\pm\boldsymbol{l}_i)$ $(i = 1, \cdots, \sigma)$. A given $\boldsymbol{l} \neq \boldsymbol{0}$ belongs to this set and so defines a face of Π if and only if

$$f(\boldsymbol{l}) = \min f(\boldsymbol{x})$$

taken over all integral $\boldsymbol{x} \equiv \boldsymbol{l} \pmod 2$ and this minimum is attained only at $\boldsymbol{x} = \pm \boldsymbol{l}$. In general, $\sigma = 2^n - 1$, and there is then one pair of faces for each congruence class of $\boldsymbol{l}$ modulo 2 other than $\boldsymbol{0}$; in this case we shall call f an *interior* form.

Voronoï [7] has shown that the $\frac{1}{2}n(n+1)$-dimensional space of positive quadratic forms may be partitioned into polyhedral cones (Δ) with the origin as vertex, possessing the following properties:

(i) no two cones have a common interior point;

(ii) an integral unimodular transformation of variables either leaves a cone invariant or transforms it into another cone of the system;

(iii) there exists a finite number of the cones, say $\Delta_0, \Delta_1, \cdots, \Delta_\tau$, such that any positive form is equivalent to a form lying in some Δ_i $(0 \leqq i \leqq \tau)$;

(iv) a cone Δ uniquely determines the set S of $2^n - 1$ pairs $\pm \boldsymbol{l}$ of integral points which define the polytope Π of a form f lying in the interior of Δ, and also determines the sets of n faces of Π which intersect in a vertex of Π.

Thus what we have called an interior form is simply a form lying in the interior of some Voronoï cone Δ. For an interior form, Π is primitive (i.e. each vertex of Π lies on just n faces). We shall denote generally by $\boldsymbol{v}$ a vertex of Π and by $\boldsymbol{l}_1, \cdots, \boldsymbol{l}_n$ the points of S specifying the n faces on which $\boldsymbol{v}$ lies. Then the matrix

$$L = [\boldsymbol{l}_1, \cdots, \boldsymbol{l}_n] \tag{2.2}$$

is non-singular, and $\boldsymbol{v}$ is uniquely determined by the n linear equations

$$f(\boldsymbol{v}) = f(\boldsymbol{v} - \boldsymbol{l}_i) \qquad (1 \leqq i \leqq n), \tag{2.3}$$

i.e.

$$2\boldsymbol{l}_i' A \boldsymbol{v} = f(\boldsymbol{l}_i) \qquad (1 \leqq i \leqq n). \tag{2.4}$$

For each vertex $\boldsymbol{v}$ of Π, we define $\boldsymbol{c}$ by

$$\boldsymbol{c} = L^{-1}\boldsymbol{v} \tag{2.5}$$

and c_0 by

$$\sum_{i=0}^{n} c_i = 1. \tag{2.6}$$

Then

$$\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{l}_i \tag{2.7}$$

i.e.

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$$v = \sum_{i=0}^{n} c_i \boldsymbol{l}_i, \text{ where } \boldsymbol{l}_0 = \boldsymbol{0}, \tag{2.8}$$

so that $c_0, c_1, \cdots, c_n$ are barycentric coordinates of v with respect to the simplex $\boldsymbol{l}_0, \boldsymbol{l}_1, \cdots, \boldsymbol{l}_n$.

Now, from the convexity of Π, it follows easily that

$$m(f) = \max_v m(f; v) = \max_v f(v), \tag{2.9}$$

the maximum being taken over all vertices of Π. We shall say that a vertex v is *maximal* if $f(v) = m(f)$.

THEOREM 1. *Let $f(x) = x'Ax$ be an interior form, and $F(x) = x'A^{-1}x$ its inverse. Then f is extreme if and only if F is expressible in the form*

$$F(x) = \sum_v \lambda_v \Big[\sum_{i=1}^{n} c_i (\boldsymbol{l}_i' x)^2 - (v'x)^2 \Big] \tag{2.10}$$

where v runs over all maximal vertices of Π,

$$\lambda_v \geqq 0 \text{ for all } v, \tag{2.11}$$

and c, $\boldsymbol{l}_i$ are defined in (2.5), (2.3).

THEOREM 2. *If f is an extreme form in the interior of a Voronoï cone Δ, then every extreme form in Δ is a multiple of f.*

THEOREM 3. *If f is an extreme form in the interior of a Voronoï cone Δ, then f and Δ have the same group of automorphisms.*

Before proceeding to the proof of these results, we note some alternative formulations of the criterion of Theorem 1.

First, defining two vertices of Π to be congruent if their difference is integral, it is easy to verify that each vertex v has $n+1$ congruent vertices; specifically, if v is determined by the simplex $(\boldsymbol{l}_0, \boldsymbol{l}_1, \cdots, \boldsymbol{l}_n)$ $(\boldsymbol{l}_0 = \boldsymbol{0})$, then v is congruent to

$$v_j = v - \boldsymbol{l}_j \qquad (0 \leqq j \leqq n) \qquad (v_0 = v) \tag{2.12}$$

and v_j is determined by the simplex $(\boldsymbol{l}_0 - \boldsymbol{l}_j, \cdots, \boldsymbol{l}_n - \boldsymbol{l}_j)$. Further

$$f(v) = f(v_j) \qquad (0 \leqq j \leqq n) \tag{2.13}$$

and, from (2.8), (2.6),

$$v_j = \sum_{i=0}^{n} c_i (\boldsymbol{l}_i - \boldsymbol{l}_j). \tag{2.14}$$

Thus congruent vertices have the same barycentric coordinates $c_0, c_1, \cdots, c_n$ (with the above ordering of the simplexes), and all are maximal if one is.

If now we set

$$\psi_v(x) = \sum_{i=1}^{n} c_i(l_i'x)^2 - (v'x)^2 = \sum_{i=0}^{n} c_i(l_i'x)^2 - (v'x)^2, \tag{2.15}$$

it is easy to verify that

$$\psi_{v_j}(x) = \psi_v(x) \qquad (0 \leqq j \leqq n) \tag{2.16}$$

$$\psi_v(x) = \tfrac{1}{2}\sum_{i,j=0}^{n} c_i c_j(l_i'x - l_j'x)^2 = \sum_{0\leqq i<j\leqq n} c_i c_j(l_i'x - l_j'x)^2. \tag{2.17}$$

Since trivially $\psi_v(x) = \psi_{-v}(x)$, we therefore have:

COROLLARY 1. *It suffices in the sum* (2.10), *to consider only one vertex* v *from the set of* $2(n+1)$ *vertices congruent to a given maximal vertex or its negative.*

COROLLARY 2. *The summand in* (2.10) *may be replaced by*

$$\sum_{\substack{i,j=0\\ i<j}}^{n} c_i c_j(l_i'x - l_j'x)^2.$$

3. Analysis of neighbouring forms

Let $f(x) = x'Ax$ be an interior form and $g(x) = x'Bx$ a neighbouring form. Then

$$B = A + \varepsilon T \tag{3.1}$$

for some symmetric T with, say, $\max |t_{ij}| = 1$. We shall suppose throughout that $\varepsilon \neq 0$, so that $g \neq f$, and that ε is so small that g is also an interior form of the cone Δ in which f lies.

Then, to each vertex v of $\Pi = \Pi_f$ with defining points $l_1, \cdots, l_n$, there corresponds uniquely a vertex w of Π_g with the same defining points, so that

$$2l_i'Bw = g(b_i) \qquad (1 \leqq i \leqq n). \tag{3.2}$$

LEMMA 3.1 (Minkowski). *For all small* ε,

$$d(g) = d(f)\left(1 + k_1\varepsilon + k_2\varepsilon^2 + O(\varepsilon^3)\right),$$

where

$$k_1 = \operatorname{tr}(A^{-1}T)$$

and

$$k_2 < 0 \quad \textit{if} \quad k_1 = 0 \quad \textit{and} \quad T \neq 0.$$

PROOF. Since A is positive definite, we may choose P so that

$$A = P'P,\ T = P'DP,$$

where

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$$D = \operatorname{diag}(d_1, d_2, \cdots, d_n).$$

Then

$$d(g) = d(f) \det (I+\varepsilon D)$$

and

$$\det (I+\varepsilon D) = 1+\varepsilon \sum d_i+\varepsilon^2 \sum_{i<j} d_i d_j+O(\varepsilon^3).$$

Hence

$$k_1 = \sum d_i = \operatorname{tr} D = \operatorname{tr} (P^{-1}DP) = \operatorname{tr} (A^{-1}T).$$

Finally, if $\sum d_i = 0$, then

$$2k_2 = 2\sum_{i<j} d_i d_j = (\sum d_i)^2-\sum d_i^2 = -\sum d_i^2 < 0$$

if $D \neq 0$, i.e. if $T \neq 0$.

Write for convenience

$$\phi(\mathbf{x}) = \mathbf{x}'T\mathbf{x},$$

so that

$$g(\mathbf{x}) = f(\mathbf{x})+\varepsilon\phi(\mathbf{x}).$$

LEMMA 3.2. *If* $\mathbf{v}$, $\mathbf{w}$ *are corresponding vertices of* Π_f, Π_g *respectively, defined by the integral points* $\mathbf{l}_1, \cdots, \mathbf{l}_n$, *then*

$$\mathbf{w} = \mathbf{v}+\varepsilon\boldsymbol{\alpha}+\varepsilon^2\boldsymbol{\beta}+O(\varepsilon^3) \tag{3.3}$$

where

$$\boldsymbol{\beta} = -A^{-1}T\boldsymbol{\alpha}, \tag{3.4}$$

$$\boldsymbol{\alpha} = \boldsymbol{\gamma}-A^{-1}T\mathbf{v}, \tag{3.5}$$

and $\boldsymbol{\gamma}$ *is defined by*

$$2\mathbf{l}_i' A\boldsymbol{\gamma} = \phi(\mathbf{l}_i) \qquad (1 \leqq i \leqq n). \tag{3.6}$$

PROOF. We may write $\mathbf{w}$ in the form (3.3) and determine $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ from (3.2), i.e.

$$2\mathbf{l}_i'(A+\varepsilon T)(\mathbf{v}+\varepsilon\boldsymbol{\alpha}+\varepsilon^2\boldsymbol{\beta}+O(\varepsilon^3)) = \mathbf{l}_i'(A+\varepsilon T)\mathbf{l}_i \qquad (1 \leqq i \leqq n).$$

Equating coefficients of ε and of ε^2 gives

$$2\mathbf{l}_i' A\boldsymbol{\alpha}+2\mathbf{l}_i' T\mathbf{v} = \mathbf{l}_i' T\mathbf{l}_i = \phi(\mathbf{l}_i) \qquad (1 \leqq i \leqq n), \tag{3.7}$$

$$2\mathbf{l}_i' A\boldsymbol{\beta}+2\mathbf{l}_i' T\boldsymbol{\alpha} = 0 \qquad (1 \leqq i \leqq n). \tag{3.8}$$

Now (3.4) follows from (3.8), since $L = [\mathbf{l}_1, \cdots, \mathbf{l}_n]$ is non-singular, and, defining $\boldsymbol{\gamma}$ by (3.6), we obtain (3.5) from (3.7).

LEMMA 3.3. *With the notation of Lemma* 3.2, *we have*

$$g(\mathbf{w}) = f(\mathbf{v})+\varepsilon(2\mathbf{v}'A\boldsymbol{\gamma}-\phi(\mathbf{v}))+\varepsilon^2 f(\boldsymbol{\alpha})+O(\varepsilon^3). \tag{3.9}$$

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PROOF. From (3.3) we obtain

$$\begin{aligned} g(\mathbf{w}) &= \mathbf{w}'(A+\varepsilon T)\mathbf{w} \\ &= f(\mathbf{v})+\varepsilon(2\mathbf{v}'A\alpha+\mathbf{v}'T\mathbf{v})+\varepsilon^2\alpha'A\alpha+2\varepsilon^2(\mathbf{v}'A\beta+\mathbf{v}'T\alpha)+O(\varepsilon^3). \end{aligned}$$

Inserting the expressions (3.4), (3.5) for β and α gives (3.9).

4. Proof of Theorem 1

With the notation of § 3, we first prove

LEMMA 4.1. *The interior form f is extreme if and only if there exists no symmetric T such that*

$$\text{tr}\,(A^{-1}T) \geqq 0 \tag{4.1}$$

and, for every maximal vertex $\mathbf{v}$ of Π_f,

$$2\mathbf{v}'A\gamma-\phi(\mathbf{v}) < 0 \tag{4.2}$$

(*where γ is defined by* (3.6)).

PROOF. (i) Suppose first that there exists a T satisfying the stated conditions.

Defining $g(\mathbf{x}) = \mathbf{x}'(A+\varepsilon T)\mathbf{x} = f(\mathbf{x})+\varepsilon\phi(\mathbf{x})$ as in § 3, we take ε sufficiently small and *positive*. Then, by (4.1) and Lemma 3.1, *either*

$$\begin{aligned} &\text{tr}\,(A^{-1}T) > 0 \quad \text{and} \quad d(g) > d(f), \quad \textit{or} \\ &\text{tr}\,(A^{-1}T) = 0 \quad \text{and} \quad d(g) = d(f)+O(\varepsilon^2). \end{aligned} \tag{4.3}$$

Next, since the values of $g(\mathbf{w})$ (at vertices $\mathbf{w}$ of Π_g) are arbitrarily close to the corresponding $f(\mathbf{v})$, we have

$$m(g) = \max_{\mathbf{w}} g(\mathbf{w}),$$

where the maximum is taken over those $\mathbf{w}$ which correspond to maximal vertices $\mathbf{v}$ of Π_f. From (3.9) and the inequalities (4.2), it follows that

$$m(g) = m(f)-k\varepsilon+O(\varepsilon^2) \tag{4.4}$$

for some $k > 0$.

From (4.3) and (4.4), it follows at once that for all sufficiently small $\varepsilon > 0$,

$$\mu(g) < \mu(f),$$

whence f is not extreme.

(ii) Suppose next that there exists no T satisfying (4.1), (4.2). With the previous notation, any sufficiently close neighbour g of f can be written as

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$$g(x) = x'(A+\varepsilon T)x \text{ with } \varepsilon > 0.$$

We may suppose that g is not a multiple of f.

Replacing g by a suitable multiple of g, we can ensure that[1]

$$\text{tr } (A^{-1}T) = 0,$$

and $T \neq 0$ since g is not a multiple of f. Then, by Lemma 3.1,

$$d(g) = d(f)(1-|k_2|\varepsilon^2+O(\varepsilon^3)), \; |k_2| > 0. \tag{4.5}$$

Since T satisfies (4.1), it follows from our hypothesis that the inequality

$$2v'A\gamma-\phi(v) \geqq 0$$

holds for some maximal vertex v of Π_f. Let w be the corresponding vertex of Π_g; w may not be maximal, but in any case

$$m(g) \geqq g(w).$$

It now follows from (3.9), since $f(\alpha) \geqq 0$, that for all sufficiently small $\varepsilon > 0$

$$m(g) > m(f) \quad or \quad m(g) = m(f)+O(\varepsilon^3). \tag{4.6}$$

From (4.5) and (4.6) we obtain at once

$$\mu(g) > \mu(f),$$

showing that f is extreme.

In order to deduce Theorem 1 from Lemma 4.1, we first note that, using the definitions (3.6) of γ and (2.5) of c, we may write the expression on the left of (4.2) as

$$\begin{aligned} 2v'A\gamma-\phi(v) &= 2\sum_1^n c_i l_i' A\gamma-\phi(v) \\ &= \sum_1^n c_i\phi(l_i)-\phi(v) \\ &= L_v(T), \quad \text{say.} \end{aligned}$$

Then

$$L_v(T) = \sum_1^n c_i l_i' T l_i - v'Tv \tag{4.7}$$

is a linear form in the elements t_{ij} ($i \leqq j$) of T. Thus (4.1), (4.2) form, in the variables t_{ij}, a system of linear inequalities

[1] Geometrically, this amounts to projecting from the origin onto the tangent plane, at f, to the determinantal surface $d = d(f)$. Algebraically, we replace T by $U = T-1/n \text{ tr } (A^{-1}T)A$, and then $A+\varepsilon T$ is a multiple of $A+\eta U$, where η is small with ε.

$$\text{(4.8)}\qquad \begin{cases} \operatorname{tr}(A^{-1}T) \geqq 0 \\ L_v(T) < 0 \qquad (v \in \mathscr{M}) \end{cases}$$

(where $\mathscr{M}$ denotes the set of maximal vertices of Π_f).

It follows from the classical theory of linear inequalities that the system (4.8) has no solution T if and only if there exist numbers,

$$\text{(4.9)}\qquad \lambda \geqq 0, \qquad \lambda_v \geqq 0 \qquad (v \in \mathscr{M}), \qquad \lambda_v \text{ not all zero}$$

satisfying

$$\text{(4.10)}\qquad \lambda \operatorname{tr}(A^{-1}T) \equiv \sum_{v \in \mathscr{M}} \lambda_v L_v(T).$$

Further, the relations (4.9), (4.10) imply that $\lambda > 0$. For we have

$$\begin{aligned} L_v(A) &= \sum_1^n c_i f(\mathbf{l}_i) - f(v) \\ &= 2\sum_1^n c_i \mathbf{l}_i' A v - f(v) \\ &= 2v'Av - f(v) = f(v); \end{aligned}$$

hence, taking $T = A$ in (4.10) gives

$$n\lambda = \lambda \operatorname{tr}(A^{-1}A) = \sum_{v \in \mathscr{M}} \lambda_v f(v) = m(f) \sum \lambda_v > 0.$$

Hence, dividing through by λ, we may replace (4.9), (4.10) by

$$\text{(4.11)}\qquad \operatorname{tr}(A^{-1}T) = \sum_{v \in \mathscr{M}} \lambda_v L_v(T)$$

where

$$\text{(4.12)}\qquad \lambda_v \geqq 0 \text{ for all } v \in \mathscr{M}.$$

Finally, (4.11) holds for all symmetric T if and only if it holds for all T of the form $\mathbf{x}\mathbf{x}'$; inserting $T = \mathbf{x}\mathbf{x}'$ in (4.11) gives the required condition (2.10) of Theorem 1.

5. Proof of Theorems 2 and 3

For the proof of Theorem 2, it suffices to show that a Voronoï cone Δ cannot contain two extreme forms f_0, f_1, of which f_0 is an interior form and f_1 is either an interior or a boundary form (not a multiple of f_0). Suppose to the contrary that two such forms exist, and consider the line segment joining them:

$$f_t = (1-t)f_0 + tf_1 \qquad (0 \leqq t \leqq 1).$$

Then $\mu(f_t)$ is a continuous function of t on the interval [0, 1] and so attains its maximum value at some point t_0 of the interval. Since f_0 and f_1 are extreme, $\mu(f_t) \geqq \mu(f_0)$ for all sufficiently small t, and $\mu(f_t) \geqq \mu(f_1)$ for all t sufficiently close to 1. Hence the maximum is attained at some t_0 with $0 < t_0 < 1$.

Writing $f = f_{t_0}$, $\phi = f_0 - f_1$, it now follows that f is an interior form of Δ and

$$\mu(f+\varepsilon\phi) \leqq \mu(f) \tag{5.1}$$

for all sufficiently small ε (of either sign). As in the proof of Theorem 1, write

$$f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x}, \quad \phi(\boldsymbol{x}) = \boldsymbol{x}'T\boldsymbol{x}, \quad g(\boldsymbol{x}) = f(\boldsymbol{x})+\varepsilon\phi(\boldsymbol{x})$$

and suppose without loss of generality that

$$\text{tr}\,(A^{-1}T) = 0;$$

then $T \neq 0$, since ϕ is not a multiple of f.

By Lemma 2.1,

$$d(g) = d(f)\bigl(1-|k_2|\varepsilon^2+O(\varepsilon^3)\bigr), \qquad |k_2| > 0; \tag{5.2}$$

and, in particular

$$d(g) < d(f) \tag{5.3}$$

for all small $\varepsilon \neq 0$.

Next, let $\boldsymbol{v}$ be any maximal vertex of Π_f, and adopt the notation of § 3. Then, for the corresponding vertex $\boldsymbol{w}$ of Π_g we have, from Lemmas 3.2 and 3.3,

$$\boldsymbol{w} = \boldsymbol{v}+\varepsilon\boldsymbol{\alpha}+\varepsilon^2\boldsymbol{\beta}+O(\varepsilon^3) \tag{5.4}$$

$$g(\boldsymbol{w}) = f(\boldsymbol{v})+p_1\varepsilon+p_2\varepsilon^2+O(\varepsilon^3), \qquad p_2 = f(\boldsymbol{\alpha}). \tag{5.5}$$

Also

$$m(f) = f(\boldsymbol{v}), \quad m(g) \geqq g(\boldsymbol{w}). \tag{5.6}$$

We now show, from (5.2)—(5.6), that (5.1) cannot in fact hold for all small ε, whence Theorem 2 follows at once.

Suppose first that, in (5.5), $p_1 \neq 0$. Then, for all small ε of the same sign as p_1, we have

$$g(\boldsymbol{w}) > f(\boldsymbol{v}); \tag{5.7}$$

hence, by (5.6), (5.3),

$$\mu(g) > \mu(f), \tag{5.8}$$

contradicting (5.1).

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Next suppose that $p_1 = 0$, $p_2 \neq 0$. Then $p_2 = f(\alpha) > 0$, and again we obtain (5.7) and (5.8). Hence (5.1) is false for all small $\varepsilon \neq 0$.

Finally suppose that $p_1 = p_2 = 0$. Then

$$m(g) \geqq g(w) = m(f) + O(\varepsilon^3)$$

and this, with (5.2), again shows that

$$\mu(g) > \mu(f)$$

for all small $\varepsilon \neq 0$.

Theorem 3 is now easily deduced from Theorem 2. Let f be an extreme form interior to a Voronoï cone Δ, and let $G(f)$, $G(\Delta)$ be the groups of (integral unimodular) automorphisms of f, Δ respectively.

Suppose that $U \in G(f)$. Then U transforms Δ into a Voronoï cone Δ'; since $f = Uf \in \Delta'$, Δ and Δ' have a common interior form f and so are identical. Hence $U \in G(\Delta)$.

Conversely, suppose that $U \in G(\Delta)$ and let $Uf = f'$. Then f' is an interior form of Δ; and since f' is equivalent to f, f' is an extreme form. It now follows from Theorem 2 that f' is a multiple of f, whence $f' = f$ and so $U \in G(f)$.

6. The principal domain

For all $n \geqq 2$, the set of forms ϕ expressible in the form

$$\phi(x) = \sum_{i=1}^{n} \rho_i x_i^2 + \sum_{i<j} \rho_{ij}(x_i - x_j)^2, \quad \rho_i \geqq 0, \quad \rho_{ij} \geqq 0 \quad (i, j = 1, \cdots, n) \tag{6.1}$$

is a Voronoï cone Δ, called by Voronoï the "principal domain" and discussed fully in [7] We may write it more symmetrically as

$$\phi(x) = \sum_{\substack{i,j=0 \\ i<j}}^{n} \rho_{ij}(x_i - x_j)^2, \quad x_0 = 0, \quad \rho_{ij} \geqq 0 \quad (0 \leqq i < j \leqq n). \tag{6.2}$$

The set S of integral points defining Π for any interior form consists of the $2^n - 1$ points with all coordinates 0 or 1 (other than $\mathbf{0}$), and their negatives.

The group $G(\Delta)$ of automorphisms of Δ has order $2(n+1)!$ and is transitive on the edge-forms $(x_i - x_j)^2$ $(0 \leqq i < j \leqq n)$ of Δ. $G(\Delta)$ is in fact generated by (i) all permutations of $x_1, x_2, \cdots, x_n$; (ii) $x_i \to -x_i$ $(1 \leqq i \leqq n)$; (iii) $x_1 \to x_1$, $x_i \to x_1 - x_i$ $(2 \leqq i \leqq n)$.

It follows from Theorem 3 that any interior extreme form of Δ has all ρ_{ij} equal, and so is a multiple of [2]

[2] The principal domain appears to be exceptional in having a sufficiently large group of automorphisms to determine the extreme form f completely.

$$(6.3)\qquad f(x) = \sum_{\substack{i,j=0\\ i<j}}^{n} (x_i - x_j)^2 = n\sum_{i=1}^{n} x_i^2 - 2\sum_{1\leq i<j\leq n} x_i x_j .$$

As we remarked in § 1, this form is indeed extreme; we now give an alternative proof of this, using Theorem 1.

The integral points

$$(6.4)\qquad \boldsymbol{l}_i = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_i \qquad (1 \leqq i \leqq n)$$

(where $\boldsymbol{e}_i$ is the ith unit vector) determine the vertex

$$(6.5)\qquad \boldsymbol{v} = \frac{1}{n+1}(n, n-1, \cdots, 2, 1)'$$

of Π_f; and, from (2.5), we obtain

$$\boldsymbol{c} = \frac{1}{n+1}(1, 1, \cdots, 1)',$$

whence

$$(6.6)\qquad c_i = \frac{1}{n+1} \quad \text{for} \quad 0 \leqq i \leqq n.$$

By permuting coordinates in (6.5), we obtain a set of $n!$ distinct vertices, which contains just 1 vertex from each set of congruent vertices. It follows that every vertex of Π is maximal, and, in applying the criterion of Theorem 1, we may take $\mathscr{M}$ to be this set of $n!$ vertices.

It is here a little more convenient in applying Theorem 1, to use the expression (2.17). With the $\boldsymbol{l}_i$, c_i given by (6.4), (6.6) (and $\boldsymbol{l}_0 = \boldsymbol{0}$) we have for the vertex (6.5)

$$\psi_v(x) = \sum_{0\leq i<j\leq n} c_i c_j (\boldsymbol{l}_i' \boldsymbol{x} - \boldsymbol{l}_j' \boldsymbol{x})^2,$$

$$(n+1)^2 \psi_v(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} (x_i + x_{i+1})^2 + \cdots + \sum_{i=1}^{n-r} (x_i + x_{i+1} + \cdots + x_{i+r})^2 + \cdots$$
$$\cdots + (x_1 + x_2 + \cdots + x_n)^2.$$

Summing this over all permutations of the x_i, we obtain

$$(6.7)\qquad (n+1)^2 \sum_{v\in\mathscr{M}} \psi_v(x) = \sum_{r=1}^{n} r!(n+1-r)!\Big\{\sum_{1\leq i_1<\cdots<i_r\leq n} (x_{i_1} + x_{i_2} + \cdots + x_{i_r})^2\Big\}$$

$$(6.8)\qquad = \tfrac{1}{12}(n+2)!\Big\{2\sum_{i=1}^{n} x_i^2 + 2\sum_{i<j} x_i x_j\Big\},$$

since, on the right of (6.7), the coefficient of each x_i^2 is

$$\sum_{r=1}^{n} r!(n+1-r)!\binom{n-1}{r-1} = \tfrac{1}{6}(n+2)!,$$

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and the coefficient of each $2x_i x_j$ $(i < j)$ is

$$\sum_{r=2}^{n} r!\,(n+1-r)!\binom{n-2}{r-2} = \tfrac{1}{12}(n+2)!$$

Since the form in braces in (6.8) is just $F(\boldsymbol{x})$, the inverse of $f(\boldsymbol{x})$, we have therefore expressed $F(\boldsymbol{x})$ in the form (2.10) (with the λ_v all equal), whence f is extreme.

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