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A HOMOTOPY TYPE OF A p-GROUP WITH CYCLIC CENTRE

S. B. CONLON

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Abstract

Let G be a p-group with cyclic centre $\mathscr{Z}(G) = Z$. Then $\mathscr{S}(G) = \{Z < H \leqslant G | H' \cap Z = (1)\}$, a poset ordered under inclusion. Then the associated simplicial complex $|\mathscr{S}(G)|$ is homotopic to a bouquet of spheres. A subgroup E of G is called a CES if $C_G(E) = Z = \mathscr{Z}(E)$ and if E/Z is elementary. Then $|\mathscr{S}(G)|$ is homotopic to the one-point union of the $|\mathscr{S}(E)|$ for all CES's E in G. If $|E/Z| = p^{2n}$, then $|\mathscr{S}(E)|$ is homotopic to a one-point union of p^{n^2} $(n-1)$-spheres.

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1. Introduction

A finite p-group G has a faithful irreducible representation ρ over the complex number field $\mathbf{C}$ if and only if its centre $\mathscr{Z}(G) = Z$ is cyclic. Indeed by Schur's lemma the restriction of ρ to Z consists of scalar matrices λI, where λ is a faithful linear representation of Z. Every representation of G is monomial and so one can ask if a "transitive" monomial representation with "stabiliser" H restricts to Z to give λI. The set of such H gives the poset

$$\mathscr{S}(G) = \{ Z < H \leqslant G | H' \cap Z = (1) \},$$

ordered under inclusion.

$\mathscr{S}(G)$ is unchanged by extension of Z to a larger cyclic group (by central amalgamation or equivalently by the addition of further scalar matrices to the irreducible p-group). Indeed $\mathscr{S}(G)$ is preserved under isoclinism.

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To a poset X is associated a simplicial complex $|X|$ whose vertices are the elements of X and simplices the nonempty chains in X. A morphism $f\colon X \to Y$ is an order preserving map and induces a simplicial map $|f|\colon |X| \to |Y|$ (more details are given in Quillen [2]). For $y \in Y$, we set $f/y = \{x \in X | f(x) \leqslant y\}$. We have the following theorem due to Quillen [1]:

(1.1) *If f/y is contractible $\forall y \in Y$, then $|f|$ is a homotopy equivalence.*

An elementary proof is given by Walker [3].

If instead of $\mathscr{S}(G)$, we consider those "transitive" monomial representations whose "stabilisers" are not contained in the centre Z, but which represent G faithfully, we are led to the poset

$$\mathscr{T}(g) = \{ H \leqslant G | H \nleqslant Z, H' \cap Z = (1)\}.$$

However $\mathscr{S}(G)$ and $\mathscr{T}(G)$ are homotopically equivalent. For consider the map $f\colon \mathscr{T}(G) \to \mathscr{S}(G)$, $H \mapsto HZ$. If $K \in \mathscr{S}(G)$, then f/K consists of all H in $\mathscr{T}(G)$ with $H \leqslant K$. But any simplex in $|f/K|$ is joined to the vertex K. Hence $|f/K|$ is a cone, contractible to K, and so by (1.1) |f | is a homotopy equivalence.

Thus we concentrate on $\mathscr{S}(G)$ and show that $|\mathscr{S}(G)|$ is a homotopic to a one-point union of spheres.

A p-group E is called an ES if $\mathscr{Z}(E)$ is cyclic and $E/\mathscr{Z}(E)$ is elementary. An ES is almost *e*xtra*s*pecial and *is* extraspecial if its cyclic centre has order p. For such an E, commutation defines a symplectic form on $E/\mathscr{Z}(E)$ into $\mathbf{F}_p$. If $|E/\mathscr{Z}(E)| = p^{2n}$, E is called an n-ES. E is a central amalgamated product of n 1-ES's.

A subgroup E of G is called a CES (*c*entralised ES) if $C_G(E) = \mathscr{Z}(E) = Z$ and E/Z is elementary. E is an n-CES of G if further E is an n-ES.

We will show that $|\mathscr{S}(E)|$ is homotopic to a one-point union of p^{n^2} $(n-1)$-spheres when E is an n-ES and that $|\mathscr{S}(G)|$ is homotopic to the one-point union of the $|\mathscr{S}(E)|$, and E runs through all CES's in G.

2. $|\mathscr{S}(G)|$ as a union of suspensions

For K such that $Z \leqslant K \leqslant G$, write

$$\mathscr{A}(K) = \{ H \in \mathscr{S}(G) | H \leqslant K, H/Z \text{ elementary}\}.$$

If $K = Z$, then $\mathscr{A}(K)$ is empty. In analogy to Proposition 2.1 of Quillen [2], we have:

PROPOSITION 2.1. *The inclusion $i\colon \mathscr{A}(G) \to \mathscr{S}(G)$ is a homotopy equivalence.*

PROOF. Take $K \in \mathscr{S}(G)$. Then

$$i/K = \mathscr{A}(K) = \{ H | Z < H \leqslant K, H/Z \text{ elementary}, H \text{ abelian} \}.$$

Take $L/Z \leqslant \mathscr{Z}(K/Z)$ with $|L/Z| = p$. If $H \in \mathscr{A}(K)$, then we show that $LH \in \mathscr{A}(K)$. For LH/Z is elementary and so $(LH)' \leqslant Z$. But $LH \leqslant K$ and $K' \cap Z = (1)$ and so $(LH)' = (1)$, i.e. $LH \in \mathscr{A}(K)$. Thus $|i/K|$ is a cone with vertex L and so is contractible. The result now follows from Quillen's Theorem (1.1).

We can now confine our attention to the homotopy type of $\mathscr{A}(G)$. Take $A/Z \leqslant \mathscr{Z}(G/Z)$, with A/Z of order p. Let

$$\mathscr{C} = \{ H \in \mathscr{A}(G) - \mathscr{A}(C_G(A)) | \, |H/Z| = p \}.$$

LEMMA 2.2. *$C_G(A)$ is maximal in G.*

PROOF. Take $a \in A - Z$. Then $g \mapsto [g, a]$ is a group homomorphism from G onto the subgroup of Z of order p. Its kernel is $C_G(A)$ which is thus maximal.

PROPOSITION 2.3. *$|\mathscr{A}(G)| \simeq \bigvee_{H \in \mathscr{C}} S|\mathscr{A}(C_G(A, H))|$ where $\vee$ is the one-point union, S is the (two-point) suspension and $C_G(A, H) = C_G(\langle A, H \rangle)$.*

PROOF. For $H \in \mathscr{C}$, set

$$\mathscr{D}(H) = \{H\} \cup \bigcup_{K \in \mathscr{A}(C_G(A, H))} \{\langle K, H \rangle\},$$

a subposet of $\mathscr{A}(G)$. Then

$$\mathscr{A}(G) = \mathscr{A}(C_G(A)) \cup \bigcup_{H \in \mathscr{C}} \mathscr{D}(H).$$

This follows as every L in $\mathscr{A}(G)$ satisfies either

(i) $[L, A] = (1)$, whence $L \in \mathscr{A}(C_G(A))$ or

(ii) $[L, A] > (1)$, whence $K = C_G(A) \cap L$ is maximal in L and so $\exists H \in \mathscr{C}$ with $L = \langle K, H \rangle$ (we allow $L = H$ and $K = Z$).

Set $\mathscr{B} = \mathscr{A}(C_G(A)) \dot{\cup} \dot{\bigcup}_{H \in \mathscr{C}} \mathscr{D}(H)$, $\dot{\cup}$ denoting abstract disjoint union. We consider $\mathscr{B}$ as a poset where the order relation within each $\dot{\cup}$-summand is that of inclusion; if $K \in \mathscr{A}(C_G(A, H)) \subseteq \mathscr{A}(C_G(A))$, then K in $\dot{\cup}$-summand $\mathscr{A}(C_G(A))$ is $\leqslant \langle K, H \rangle$ in $\dot{\cup}$-summand $\mathscr{D}(H)$.

It is claimed that the map f: $\mathscr{B} \to \mathscr{A}(G)$ obtained by removing dots, is a homotopy equivalence. For take $L \in \mathscr{A}(G)$ and look at f/L. If $[L, A] = (1)$, then $L \in \mathscr{A}(C_G(A))$ and f/L is a cone with vertex L lying in the subspace $|\mathscr{A}(C_G(A))|$ of $|\mathscr{B}|$. If $[L, A] > (1)$, then either $|L/Z| = p$ and so $L \in \mathscr{C}$ and $f/L = \{L\}$, which is a point and so contractible, or $N = L \cap C_G(A) > Z$. In this latter case, if $H \in \mathscr{C}$ and $H < L$, then $C_G(A, H) \cap L = N$ so that

$$f/L = \mathscr{A}(N) \cup \bigcup_{\substack{H \in \mathscr{C} \\ H < L}} \Big(\{H\} \cup \bigcup_{K \in \mathscr{A}(N)} \{\langle K, H \rangle\} \Big).$$

In $|\mathscr{B}|$ this is a cone with vertex N and so is contractible. By (1.1) f is then a homotopy equivalence.

It suffices to look at $\mathscr{B}$. The picture of $|\mathscr{B}|$ is first of all a cone $|\mathscr{A}(C_G(A))|$ with vertex A, together with a separate cone cap with vertex H and section $|\mathscr{A}(C_G(A, H))| (\subseteq |\mathscr{A}(C_G(A))|)$ for each $H \in \mathscr{C}$.

We now contract the cone $|\mathscr{A}(C_G(A))|$ to its vertex A. For each $H \in \mathscr{C}$, the corresponding cone cap of section $|\mathscr{A}(C_G(A, H))|$ becomes the suspension $S(|\mathscr{A}(C_G(A, H))|)$ of this section from the two vertices A and H. Thus we obtain the one-point union of suspensions

$$|\mathscr{B}| \simeq \bigvee_{H \in \mathscr{C}} S\big(|\mathscr{A}(C_G(A, H))|\big),$$

with common point A, as required.

3. Case when G is an ES

Suppose that G is an n-ES and so G is the central amalgamated product of n 1-ES's. In applying (2.3), we choose $A/Z \leqslant \mathscr{Z}(G/Z)$ of order p. $C_G(A)/Z$ has order p^{2n-1} and $|\mathscr{C}| = p^{2n-1}$ = number of points in a projective space of dimension $2n - 1$ over $\mathbf{F}_p$ lying outside a hypersurface. Hence there are p^{2n-1} $\vee$-summands in (2.1). For $H \in \mathscr{C}$, $C_G(A, H)/Z$ has order $p^{2(n-1)}$ and $C_G(A, H)$ is an $(n-1)$-ES. By induction on n we can suppose that $|\mathscr{A}(C_G(A, H))|$ is homotopic to a one-point union of p^{2n-1} $(n-2)$-spheres. Each $(n-2)$-sphere suspends to give a $(n-1)$-sphere. Thus the total number of $(n-1)$-spheres in $|\mathscr{A}(G)|$ is $p^{2n-1} \times p^{(n-1)^2} = p^{n^2}$ and the induction proceeds. The induction starts when $n = 0$, $G = Z$ and $\mathscr{A}(G)$ is void. $S(\varnothing)$ is the pair of suspending points and so is a 0-sphere. Summarizing, we have

PROPOSITION 3.1. *If G is an n-ES, then $|\mathscr{S}(G)|$ is homotopically equivalent to a one-point union of p^{n^2} $(n-1)$-spheres.*

This structure of p^{n^2} $(n-1)$-spheres is the Tits' building for the symplectic group $Sp(2n, \mathbf{F}_p)$ acting on G/Z with the symplectic form being given by commutation. The $(n-1)$-dimensional homology group has rank p^{n^2} and the induced action of $Sp(2n, \mathbf{F}_p)$ on this gives a realisation of the Steinberg character.

4. $\mathscr{S}(G)$ as a one-point union of spheres

THEOREM 4.1. *Let G be a p-group with cyclic centre Z and set $\mathscr{S}(G) = \{Z < H \leqslant G \mid H' \cap Z = (1)\}$, ordered under inclusion. Then $|\mathscr{S}(G)|$ is homotopically equivalent to a one-point union of spheres.*

PROOF. By (2.1), $|\mathscr{S}(G)| \simeq |\mathscr{A}(G)|$ and by (2.3) we have

$$|\mathscr{A}(G)| \simeq \bigvee_{H \in \mathscr{C}} S(|\mathscr{A}(C_G(A, H))|), \tag{4.2}$$

where $\mathscr{C} = \{H \in \mathscr{A}(G) - \mathscr{A}(C_G(A)) \mid |H/Z| = p\}$. We apply this same result (4.2) to each $\vee$-summand $\mathscr{A}(C_G(A, H))$ in turn and so on. We thus obtain $\vee$-sums over sequences $(A, H; A_1, H_1; \dots)$ and we look at how these sequences terminate. For a particular choice of A, H we look at $S(|\mathscr{A}(C_G(A, H))|)$.

(i) *Case* $\mathscr{Z}(C_G(A, H)) > Z$. Take $B \leqslant \mathscr{Z}(C_G(A, H))$ with $|B/Z| = p$. Then $|\mathscr{A}(C_G(A, H))|$ is contractible to the vertex B. As $S(\text{point}) \simeq$ point, such an ending gives no contribution to final one-point union.

(ii) *Case* $G_1 = C_G(A, H) > Z$ *and* $\mathscr{Z}(G_1) = Z$. Choose $A_1/Z \leqslant \mathscr{Z}(G_1/Z)$ with $|A_1/Z| = p$. If $\mathscr{A}(G_1) \nsubseteq \mathscr{A}(C_{G_1}(A_1))$, then a choice of H_1 is possible and sequence proceeds. If however we have $\mathscr{A}(G_1) \subseteq \mathscr{A}(C_{G_1}(A_1))$, then every element of $\mathscr{A}(G_1)$ commutes with A_1 and $|\mathscr{A}(G_1)|$ is homotopic to a cone with vertex A_1 and so is contractible to a point. As in (i), this gives no contribution to the final one-point union.

(iii) *Case* $C_G(A, H) = Z$. Thus $\mathscr{A}(C_G(A, H))$ is void and $S(\varnothing)$ is the two-point 0-sphere. Continuing suspensions give higher dimensional spheres (as in Section 3).

Hence nontrivial contributions to $|\mathscr{A}(G)|$ come from sequences $A_1, H_1; \dots; A_n, H_n$, where, if we set $E = \langle A_1, H_1, \dots, A_n, H_n, Z\rangle$, then $C_G(E) = Z$ and so E is an n-CES for some n. Each such $\vee$-summand is homotopic to the n-fold suspension $S^n(\varnothing)$ which is a $(n-1)$-sphere and so $|\mathscr{A}(G)|$ is a one-point union of spheres, as required.

5. Critical roles of the CES's

(5.1) An $(n-1)$-spherical $\vee$-summand in (4.1) corresponds to a sequence $A_1, H_1; \dots; A_n, H_n$. These subgroups together with Z generate an n-CES E of G.

We now collect together summands in (4.1) according to the CES that they generate.

LEMMA 5.2. *If A/Z lying in $\mathscr{Z}(G/Z)$ has exponent p and if E is CES in G, then $A \leqslant E$.*

PROOF. As an elementary group is generated by its subgroups of order p, it is sufficient to show the result when A/Z has order p.

Suppose E is an n-CES and so E/Z has order p^{2n}. If $A \nleqslant E$, then AE/Z is elementary of order p^{2n+1}. Commutation defines a symplectic form on AE/Z into $\mathbf{F}_p$ and as this has odd dimension over $\mathbf{F}_p$ it has a singular subspace Y/Z. Then $Z < Y \leqslant C_G(E)$, contrary to the fact that E is a CES. Hence $A \leqslant E$, as required.

THEOREM 5.3. *Let G be a p-group with cyclic centre Z. Then $|\mathscr{S}(G)|$ is homotopic to a one-point union*

$$|\mathscr{S}(G)| \simeq \vee |\mathscr{S}(E)|,$$

where E runs through the CES's of G. If E is an n-CES, then $|\mathscr{S}(E)|$ is homotopic to a one-point union of p^{n^2} $(n-1)$-spheres.

PROOF. Let E be an n-CES of G. As $|\mathscr{S}(G)|$ and $|\mathscr{S}(E)|$ are one-point unions of spheres, it is sufficient to see that there are sufficient (i.e. p^{n^2}) $\vee$-summands in (4.1), indexed by sequences $(A_1, H_1; \dots; A_n, H_n)$, such that $E = \langle A_1, \dots, H_n, Z\rangle$.

Take r with $0 < r < n$ and write $M_r = \langle A_1, \dots, H_r, Z\rangle$ and $N_r = \langle A_{r+1}, \dots, H_n, Z\rangle$. Then E is the central amalgamated product of the ES's M_r and N_r. At the $(r+1)$st stage of analysing and forming the summands in (4.1) we have to look at $C_G(A_1, H_1, \dots, A_r, H_r) = C_G(M_r)$ and choose A_{r+1}/Z in $\mathscr{Z}(C_G(M_r))$ of order p. Now N_r is a CES in $C_G(M_r)$ and so by (5.2) $A_{r+1} \leqslant N_r$. A union is then taken over all H_{r+1}/Z of order p in $C_G(M_r)$ with $[A_{r+1}, H_{r+1}] > (1)$. Considering only those H_{r+1}/Z which lie in a given n-CES E, we see that their number is independent of the rest of G as is the same as if E were considered in isolation. Hence that part of the one-point union $|\mathscr{S}(G)|$ in 4.1 coming from all sequences $A_1, \dots, H_n, Z$ which generate E is homotopic to $|\mathscr{S}(E)|$. This completes the proof.

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Department of Pure Mathematics
University of Sydney
Sydney, N.S.W. 2006
Australia

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