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MAINTENANCE POLICY FOR A CONTINUOUSLY MONITORED DETERIORATING SYSTEM

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We consider a continuously monitored system that gradually and stochastically deteriorates. An alarm threshold is set on the system deterioration level for triggering a delayed preventive maintenance operation. A mathematical model is developed to find the value of the alarm threshold that minimizes the asymptotic unavailability. Approximations are derived to improve the numerical optimization.

1. INTRODUCTION

This article considers the maintenance of a technical device subject to a continuous gradual random deterioration. Assuming that the level of deterioration can be measured in practice, the degradation process of a system can be controlled by two different methods: continuous monitoring and inspections (whether periodic or not). In both cases, the maintenance actions are performed when the measured deterioration level exceeds an alarm threshold. Continuous monitoring is particularly justified for highly critical systems and the continuous information on the system condition can be exploited by the maintenance decision-maker in order to maximize the availability of the system or to minimize its long-run expected cost. The objec-

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tive of this article is precisely to propose a stochastic maintenance model that can be used to optimize the maintenance decision on a system subject to continuous monitoring and delayed maintenance actions. In our model, we assume that the deterioration condition of the device can be modeled by an aging stochastic process. We also suppose that, in the absence of repair or replacement actions, the aging variable evolves like a Gamma stochastic process, a special case of the Levy process. When the device is new, the aging variable is equal to zero. When the aging variable reaches a failure level L, a breakdown occurs. Moreover, the continuous monitoring is assumed to be "perfect"; that is, the condition of the system is known with no error or uncertainty at any time.

We define a preventive maintenance policy which depends on the state of the system. The preventive maintenance policy is fully determined by an alarm threshold A (A is lower than L). When the aging variable exceeds this threshold A, a preventive maintenance operation (i.e., replacement of the system) is planned. This operation occurs after a delay time τ (corresponding, e.g., to a maintenance setup time) and its duration ρ depends on the deterioration state of the system at the beginning of the maintenance action. The preventive maintenance operation replaces the system by a new identical one or repairs it to an as-good-as new state. Since the system still deteriorates during the delay time τ, a breakdown can occur between the alarm time and the onset of the maintenance action. Obviously, the choice of the maintenance threshold A will influence the performance of the maintenance policy. If the threshold A is close to L, the probability of breakdown is great. If the threshold A is low, the probability of breakdown is small, but unnecessary maintenance operations are done on a system with a long potential residual life. We compute the asymptotic unavailability of the maintained system and we find the preventive maintenance policy which minimizes this unavailability.

Several articles propose preventive policies for stochastically deteriorating systems modeled by Levy stochastic processes; see, for example, Abdel-Hameed [1], Grall, Dieulle, Bérenguer, and Roussignol [6], Newby and Dagg [7], van Noortwijk, Kok, and Cooke [10], Yang and Klutke [11], or Zuckerman [12]. Most of these articles consider inspection-based maintenance policies. The model presented in this article is similar to the model of Zuckerman [12] for the maintenance problem of a continuously monitored system. It is more general because it takes into account maintenance duration depending on the deterioration state of the system. Since we consider a Gamma deteriorating process, and not a general Levy process, an explicit expression (and several approximations) of the asymptotic unavailability is derived. Other similar models with particular interest in the availability have been developed by Bloch-Mercier [3] and Cocozza-Thivent [5] for systems which are not continuously known.

The model proposed in this article can also be connected with the so-called delay-time analysis framework for maintenance problems developed by Christer [4]. The delay time is defined as the elapsed time between the first detectable apparition of a defect and the failure of the system. In our analysis, the time between the exceeding of the alarm threshold A and the failure threshold L can thus be interpreted as a delay time in the sense of Christer. We compute probability characteristics of this delay time using the specific properties of Gamma processes.

This article is organized as follows. Section 2 is devoted to the description of the system characteristics. Then, a probabilistic analysis is performed in Section 3 and leads to an expression of the asymptotic unavailability. Two approximations of this expression are given in Section 4. Finally, Section 5 presents numerical experiments of the optimization of the maintenance policy.

2. DESCRIPTION OF THE SYSTEM

2.1. Assumptions and Maintenance Policy

We consider a stochastically deteriorating system with preventive or curative maintenance operations occurring at some random times.

We suppose that the following assumptions are verified:

1. The condition of the system at time t can be summarized by a scalar aging variable X_t. The aging variable of the system varies increasingly as the system deteriorates. The initial state X_0 is zero.
2. If the aging variable is greater than a given level L, the system is supposed to be in the failed state. In this situation, the system may still be in operation, but its high level of deterioration is unacceptable both for economic reasons (poor products quality, high consumption of raw material, etc.) and for safety reasons (high risk of hazardous breakdowns). Let us denote by σ_L the time at which the aging variable crosses the failure level L:

$$\sigma_L = \inf(t/X_t \geq L). \tag{2.1}$$

   We assume that the system is only subject to this wear-dependent failure mode; that is, we discard all failures not directly related to the deterioration level.

The system is monitored continuously. In practice, this monitoring can be implemented, for example, using vibration analysis, temperature control, or conductivity measurements; see *Handbook of Condition Monitoring* [8]. According to the monitored aging variable, the maintenance policy states as follows (see Fig. 1):

1. When the aging variable becomes greater or equal to a critical threshold A ($A \leq L$), a maintenance operation is planned. Let σ_A denote the time at which the aging variable crosses the alarm level A ($\sigma_A \leq \sigma_L$):

$$\sigma_A = \inf(t/X_t \geq A). \tag{2.2}$$

2. When activated, a maintenance operation effectively begins after a delay time τ (i.e., at time $\sigma_A + \tau$). The time needed to begin the maintenance stands for a global maintenance setup time [e.g., diagnosis operations and maintenance resources mobilization (tools, spare parts, maintenance crew)].

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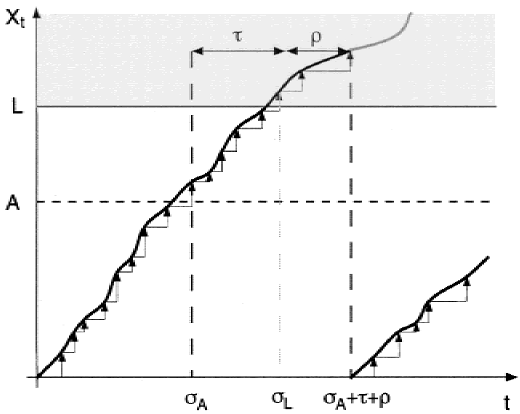

FIGURE 1. Description of the maintenance/monitoring policy.

3. The maintenance operation has a random duration ρ. The probability law of the random variable ρ can depend on the state of the aging variable at the beginning of the maintenance operation in such a way that

$$E(\rho) = \rho_1 + \rho_2 \mathbb{E}(X_{\sigma_A+\tau}), \tag{2.3}$$

where ρ_1 and ρ_2 are known parameters. A proper choice of the parameters ρ_1 and ρ_2 allows us to model a maintenance duration increasing with the level of deterioration of the system at the beginning of the maintenance operation. This assumption is plausible since the maintenance of a more deteriorated system is likely to be longer and more complicated.
4. Between σ_A and $\sigma_A + \tau$, the device deteriorates and a failure may occur before the maintenance operation begins. Depending on the occurrence of a failure, a preventive or a corrective maintenance action has to be performed.

   - If a failure occurs in this time interval (i.e., if $\sigma_L \leq \sigma_A + \tau$), the device is unavailable from the failure time until the time of the end of the maintenance operation $\sigma_A + \tau + \rho$.
   - If a failure does not occur (i.e., if $\sigma_L > \sigma_A + \tau$), the device is unavailable from the time $\sigma_A + \tau$ at which the maintenance operation begins until the time of the end of the maintenance operation $\sigma_A + \tau + \rho$.

   The unavailability duration U_1 of the system resulting from one maintenance operation is

$$U_1 = \rho \mathbb{I}_{\sigma_L > \sigma_A+\tau} + ((\sigma_A + \tau + \rho) - \sigma_L)\mathbb{I}_{\sigma_L \leq \sigma_A+\tau}, \tag{2.4}$$

   where $\mathbb{I}_{\mathrm{E}} = 1$ if E is true and 0 otherwise.
5. At the end of the maintenance operation, which can be either a true physical replacement or an overhaul or repair, the device is assumed to be as good as

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new and the aging variable is equal to zero. The system evolution after this time is independent of the past.

2.2. Stochastic Deteriorating Model

Between maintenance operations, the aging variable is supposed to evolve like a Gamma stochastic process $(\widetilde{X}_t)_{t\geq 0}$ such that the following hold:

1. $\widetilde{X}_0 = 0$.
2. For all $0 \leq s < t$, the random variable $\widetilde{X}_t - \widetilde{X}_s$ follows a Gamma probability density with shape parameter $\alpha(t-s)$ and scale parameter β:

$$f_{\alpha(t-s),\beta}(x) = \frac{1}{\Gamma(\alpha(t-s))} \beta^{\alpha(t-s)} x^{\alpha(t-s)-1} e^{-\beta x} \mathbb{I}_{\{x\geq 0\}}, \tag{2.5}$$

   where α and β are strictly positive parameters.
3. $(\widetilde{X}_t)_{t\geq 0}$ has independent increments.

We can note that before the first maintenance operation, the process $(X_t)_{t\geq 0}$ describing the evolution of the system subject to the previous maintenance policy is the same as the process $(\widetilde{X}_t)_{t\geq 0}$ describing the aging variable evolution between maintenance operations.

The relevance of the modeling of the aging variable as a Gamma stochastic process has been justified by several authors (e.g., van Noortwijk et al. [10] and Singpurwalla and Wilson [9]). A process that has independent increments, is not decreasing, and is homogeneous in time belongs to the class of the Levy processes; see Asmussen [2]. If our deteriorating process has these three properties, the choice of the Gamma process is quite natural, as advocated by van Noortwijk et al. [10]. The Gamma process is a Levy process which has explicit marginal probability density functions and this property permits computations. The two parameters α and β of the Gamma process can be estimated from degradation data by a statistical procedure.

3. PERFORMANCE OF THE MAINTENANCE POLICY AND STUDY OF THE SYSTEM EVOLUTION

The maintenance policy is driven by the choice of the critical threshold A. The value of this decision parameter has to be optimized according to a given indicator. The indicator used in this article to assess the performance of the proposed maintenance policy is the asymptotic unavailability. The optimal value of A aims at beginning maintenance operations in order to avoid both unexpected breakdowns (if A is close to L) and unnecessary maintenance operations done on a system with a long potential residual life (if A is low).

Let U_∞ denote the asymptotic unavailability of the system. If $U(t)$ is the unavailability duration of the system before time t:

$$U_\infty = \lim_{t\to\infty} \frac{U(t)}{t}. \tag{3.1}$$

Its calculation as a function of A needs to know some probabilistic characteristics of the system evolution under the maintenance policy. We first give an expression of the asymptotic unavailability as a function of the data of the given problem $(A, L, \tau, \rho_1, \rho_2, \alpha, \beta)$. After proving several properties of the process $(X_t)_{t\geq 0}$, we propose a calculable expression of U_∞.

3.1. Asymptotic Unavailability

First, we note that the process $(X_t)_{t\geq 0}$ is a regenerative process with regeneration times being the dates of the end of maintenance. Indeed, at a time of end of maintenance, the process is equal to zero and the random evolution of the system after the end of the maintenance does not depend on the past. Thus, we can compute the asymptotic unavailability as the mean off time on a cycle (see Eq. (2.4)) divided by the mean duration of a cycle:

$$U_\infty = \frac{\mathbb{E}(\rho \mathbb{I}_{\{\sigma_L > \sigma_A + \tau\}} + (\sigma_A + \tau - \sigma_L + \rho)\mathbb{I}_{\{\sigma_L \leq \sigma_A + \tau\}})}{\mathbb{E}(\sigma_A + \tau + \rho)}. \tag{3.2}$$

As a consequence,

$$\begin{aligned} U_\infty &= \frac{\mathbb{E}(\rho) + \mathbb{E}((\sigma_A + \tau - \sigma_L)\mathbb{I}_{\{\sigma_L \leq \sigma_A + \tau\}})}{\mathbb{E}(\sigma_A) + \tau + \mathbb{E}(\rho)} \\ &= \frac{\mathbb{E}(\rho) + \tau - \mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A)))}{\mathbb{E}(\sigma_A) + \tau + \mathbb{E}(\rho)}. \end{aligned} \tag{3.3}$$

Considering the dependence of the random variable ρ on the state of the aging variable at the beginning of the maintenance operation (Eq. (2.3)) and the survival function $\bar{G}(s)$ of $\sigma_L - \sigma_A$, we obtain

$$U_\infty = \frac{\rho_1 + \rho_2 \mathbb{E}(X_{\sigma_A+\tau}) + \tau - \displaystyle\int_0^\tau \bar{G}(s)\, ds}{\mathbb{E}(\sigma_A) + \tau + \rho_1 + \rho_2 \mathbb{E}(X_{\sigma_A+\tau})}. \tag{3.4}$$

3.2. About Entrance Times

In order to develop the above expression of the asymptotic unavailability (Eq. (3.4)), we need to further investigate the expressions of the quantities $\mathbb{E}(\sigma_A)$, $\mathbb{E}(X_{\sigma_A+\tau})$ and the expression of the survival function $\bar{G}(s)$ of $\sigma_L - \sigma_A$.

3.2.1. Expression of $\mathbb{E}(\sigma_A)$. The survival function of the entrance time σ_A is easy to obtain from the Gamma stochastic process $(\widetilde{X}_t)_{t\geq 0}$ describing the aging variable evolution between maintenance operations:

$$\begin{aligned} \mathbb{P}(\sigma_A > t) &= \mathbb{P}(\widetilde{X}_t \leq A) \\ &= F_{\alpha t, \beta}(A), \end{aligned} \tag{3.5}$$

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where $F_{\alpha t,\beta}$ is the distribution function of the Gamma density probability function $f_{\alpha t,\beta}$.

By definition, the mean time of entrance $\mathbb{E}(\sigma_A)$ is

$$\begin{aligned}\mathbb{E}(\sigma_A) &= \int_0^\infty F_{\alpha t,\beta}(A)\,dt \\ &= \int_0^A \left(\int_0^\infty f_{\alpha t,\beta}(x)\,dt\right)dx.\end{aligned} \tag{3.6}$$

3.2.2. Expression of $\mathbb{E}(X_{\sigma_A+\tau})$. Let us now turn our attention to the quantity $\mathbb{E}(X_{\sigma_A+\tau})$, which is the mean aging variable value at the beginning of the maintenance operation.

PROPOSITION 3.1: *The mean deterioration level at the entrance time* $\sigma_A + \tau$ *is given by*

$$\mathbb{E}(X_{\sigma_A+\tau}) = \frac{\alpha}{\beta}\,(\mathbb{E}(\sigma_A) + \tau). \tag{3.7}$$

PROOF: Since the random process $\widetilde{X}_t$ has independent increments, we have

$$\mathbb{E}(\widetilde{X}_t - \widetilde{X}_s | \mathcal{F}_s) = \frac{\alpha}{\beta}\,(t-s), \tag{3.8}$$

where $\mathcal{F}_t$ denotes the natural filtration of the process $\widetilde{X}_t$. As a consequence,

$$\mathbb{E}\left(\widetilde{X}_t - \frac{\alpha t}{\beta}\,\middle|\,\mathcal{F}_s\right) = \widetilde{X}_s - \frac{\alpha s}{\beta} \tag{3.9}$$

and $\widetilde{X}_t - \alpha t/\beta$ is a 0-mean martingale. The use of the martingale stopping time theorem with the stopping time $\sigma_A + \tau$ proves the proposition. ■

3.2.3. Expression of $\int_0^\tau \bar{G}(s)$. The survival function $\bar{G}(s)$ of $\sigma_L - \sigma_A$ is defined by

$$\bar{G}(s) = \mathbb{P}(\sigma_L - \sigma_A > s). \tag{3.10}$$

An expression of $\bar{G}(s)$ as a function of the problem parameters is given in Proposition 3.2. A proof of the following proposition is given in Appendix A.

PROPOSITION 3.2: *The survival function* $\bar{G}(s)$ *of* $\sigma_L - \sigma_A$ *is*

$$\bar{G}(s) = -\iint_{\{A<x<L,0<x+y<L,0<y\}} \left(\int_0^\infty f_{\alpha u,\beta}(x)\,du\right) \frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\,dx\,dy. \tag{3.11}$$

By integration of Eq. (3.11), we obtain the following proposition.

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Proposition 3.3:

$$\int_0^{\tau} \bar{G}(s)\,ds = \int_0^{L-A} \bar{F}_{\alpha\tau,\beta}(z)\left(\int_0^{\infty} f_{\alpha t,\beta}(L-z)\,dt\right)dz. \tag{3.12}$$

The proof is given in Appendix B.

4. DEVELOPMENT AND APPROXIMATIONS OF THE ASYMPTOTIC UNAVAILABILITY

The results of Section 3 lead to an expression of U_∞, which is numerically achievable.

However, some computation and especially the integration of the survival function $\bar{G}(s)$ can be numerically burdensome. In this section, a computable expression of the asymptotic unavailability is developed. Two approximations of the quantity U_∞ are proposed in order to make its computation faster and easier.

U_∞ will be denoted hereafter as the "exact" expression of the asymptotic unavailability, as opposed to the approximations.

4.1. Computable Expression of U_∞

From Section 3, the exact expression of the asymptotic unavailability can be written as

$$\begin{aligned} U_\infty = {} & \frac{1}{\rho_1 + \left(1 + \rho_2 \dfrac{\alpha}{\beta}\right)\left(\tau + \displaystyle\int_0^{A}\left(\int_0^{\infty} f_{\alpha t,\beta}(x)\,dt\right)dx\right)} \\ & \times \left(\rho_1 + \left(1 + \rho_2 \frac{\alpha}{\beta}\right)\tau + \rho_2 \frac{\alpha}{\beta}\int_0^{A}\left(\int_0^{\infty} f_{\alpha t,\beta}(x)\,dt\right)dx \right. \\ & \left. - \int_0^{L-A}\left(\int_0^{\infty} f_{\alpha t,\beta}(L-x)\,dt\right)\bar{F}_{\alpha\tau,\beta}(x)\,dx\right). \end{aligned} \tag{4.1}$$

If $f_{\alpha t,\beta}$ is the Gamma probability density with shape parameter αt and scale parameter β, we have

$$f_{\alpha t,\beta}(x) = \beta f_{\alpha t,1}(\beta x), \tag{4.2}$$

and, obviously,

$$\int_0^{\infty} f_{\alpha t,\beta}(x)\,dt = \frac{\beta}{\alpha}\int_0^{\infty} f_{u,1}(\beta x)\,du. \tag{4.3}$$

Let us define the function φ by

$$\varphi(x) = \int_0^{\infty} f_{t,1}(x)\,dt. \tag{4.4}$$

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We have respectively

$$\mathbb{E}(\sigma_A) = \frac{1}{\alpha}\int_0^{\beta A} \varphi(y)\,dy, \tag{4.5}$$

$$\begin{aligned}\mathbb{E}(\inf(\tau,\sigma_L-\sigma_A)) &= \int_0^{\tau} \bar{G}(s)\,ds\\ &= \frac{1}{\alpha}\int_0^{\beta(L-A)} \bar{F}_{\alpha\tau,1}(y)\varphi(\beta L-y)\,dy.\end{aligned} \tag{4.6}$$

Equation (4.1) can be written with respect to φ:

$$U_\infty = \frac{\rho_1 + \left(1+\rho_2\dfrac{\alpha}{\beta}\right)\tau + \rho_2\dfrac{1}{\beta}\displaystyle\int_0^{\beta A}\varphi(y)\,dy - \frac{1}{\alpha}\int_0^{\beta(L-A)}\varphi(\beta L-y)\bar{F}_{\alpha\tau,1}(y)\,dy}{\rho_1 + \left(1+\rho_2\dfrac{\alpha}{\beta}\right)\left(\tau+\dfrac{1}{\alpha}\displaystyle\int_0^{\beta A}\varphi(y)\,dy\right)}, \tag{4.7}$$

where $\bar{F}_{\alpha\tau,1}$ is the survival function of the Gamma probability density function $f_{\alpha\tau,1}$.

The function φ is a decreasing function and we can show the following:

1. $\lim_{x\to\infty}\varphi(x) = 1$.
2. $\int_0^\infty(\varphi(x)-1)\,dx = \frac{1}{2}$.

These two properties of the function φ lead to first-order approximations of the mean entrance time $\mathbb{E}(\sigma_A)$ and $\mathbb{E}(X_{\sigma_A+\tau})$. Equations (3.7) and (4.5) and property (2) of φ successively lead to the two following propositions.

PROPOSITION 4.1: *The mean entrance time of $\widetilde{X}_t$ in $[A,+\infty[$ is*

$$\mathbb{E}(\sigma_A) = \frac{1}{\alpha}\left(\beta A + \frac{1}{2} - \int_{\beta A}^{\infty}(\varphi(y)-1)\,dy\right). \tag{4.8}$$

PROPOSITION 4.2: *The mean level of the aging variable X_t at time $\sigma_A+\tau$ is given by*

$$\mathbb{E}(X_{\sigma_A+\tau}) = A + \frac{1}{2\beta} - \frac{1}{\beta}\int_{\beta A}^{\infty}(\varphi(y)-1)\,dy + \frac{\alpha\tau}{\beta}. \tag{4.9}$$

Because of its fast decrease, the function $\varphi(x)$ is close to 1 for $x \geq 1$. If βA is large enough (at least greater than 1), the following approximations of $\mathbb{E}(\sigma_A)$ and $\mathbb{E}(X_{\sigma_A+\tau})$ can be derived :

$$\mathbb{E}(\sigma_A) \simeq \frac{1}{\alpha}\left(\beta A + \frac{1}{2}\right), \tag{4.10}$$

$$\mathbb{E}(X_{\sigma_A+\tau}) \simeq A + \frac{1}{2\beta} + \frac{\alpha\tau}{\beta}. \tag{4.11}$$

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As a first-order simplification of Eq. (4.7), we have

$$U_\infty \simeq \frac{\rho_1 + \left(1+\rho_2\dfrac{\alpha}{\beta}\right)\tau + \dfrac{\rho_2}{\beta}\left(\beta A + \dfrac{1}{2}\right) - \dfrac{1}{\alpha}\displaystyle\int_0^{\beta(L-A)} \varphi(\beta L - y)\bar{F}_{\alpha\tau,1}(y)\,dy}{\rho_1 + \left(1+\rho_2\dfrac{\alpha}{\beta}\right)\left(\tau + \dfrac{1}{\alpha}\left(\beta A + \dfrac{1}{2}\right)\right)}. \quad \textbf{(4.12)}$$

The two approximations given by Eqs. (4.10) and (4.11) will be used in the sequel. We now focus on the remaining integral in the previous expression of the asymptotic unavailability, which is connected to the survival function of $\sigma_L - \sigma_A$. Two different approximations of $\mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A)))$ are proposed in the next two subsections.

4.2. First Approximation of the Asymptotic Unavailability

The first proposed approximation of the asymptotic unavailability is based on a heuristic approximation of the difference between the entrance times, $\sigma_L - \sigma_A$.

If the trajectories of the stochastic process $(X_t)_{t\geq 0}$ were continuous, then $\sigma_L - \sigma_A$ would have the same probability law as σ_{L-A}. Hence, $\mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A)))$ can be replaced at first sight by $\mathbb{E}(\inf(\tau, \sigma_{L-A}))$. Since $X_{\sigma_A} \geq A$, the mean deterioration level at time σ_A is greater than A (see Fig. 2). A correction is needed and leads us to use $\mathbb{E}(X_{\sigma_A})$ rather than A. As a consequence, the proposed approximation is

$$\mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A))) \approx \mathbb{E}(\inf(\tau, (\sigma_{L-\mathbb{E}(X_{\sigma_A})}))). \quad \textbf{(4.13)}$$

From Eq. (4.11) taking $\tau = 0$, we have $\mathbb{E}(X_{\sigma_A}) \simeq A + 1/2\beta$ and

$$\mathbb{E}(\inf(\tau, (\sigma_{L-\mathbb{E}(X_{\sigma_A})}))) \simeq \int_0^\tau F_{\alpha u,\beta}\left(L - A - \frac{1}{2\beta}\right) du. \quad \textbf{(4.14)}$$

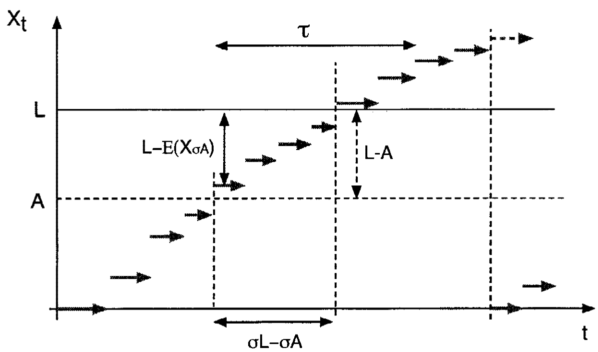

FIGURE 2. Approximation 1 of $\mathbb{E}(\inf \tau, (\sigma_L - \sigma_A))$.

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The first approximated value of the asymptotic unavailability as a function of the alarm threshold A is

$$U_\infty^1(A) = \frac{\rho_1 + \left(1 + \rho_2 \dfrac{\alpha}{\beta}\right)\tau + \rho_2\left(A + \dfrac{1}{2\beta}\right) - \displaystyle\int_0^\tau F_{\alpha u,\beta}\left(L - A - \dfrac{1}{2\beta}\right) du}{\rho_1 + \left(1 + \dfrac{\alpha\rho_2}{\beta}\right)\left(\tau + \dfrac{A\beta}{\alpha} + \dfrac{1}{2\alpha}\right)}. \tag{4.15}$$

In this expression, only one integral remains to be evaluated.

As a remark, note that for $L - A < 1/2\beta$,

$$\int_0^\tau F_{\alpha u,\beta}\left(L - A - \frac{1}{2\beta}\right) du = 0. \tag{4.16}$$

As a consequence, the approximated value U_∞^1 overestimates the asymptotic unavailability when the alarm threshold A is close to the failure threshold L.

4.3. Second Approximation of the Asymptotic Unavailability

In order to avoid an integration of the Gamma density with respect to the shape parameter in Eq. (4.15), the following approximation is proposed, which is derived from the properties of the function φ.

Since the function φ decreases quickly and tends to 1, it comes for $\beta L - y \geq \beta A$, and, hence, for $0 \leq y \leq \beta(L - A)$,

$$\varphi(\beta L - y) \simeq 1.$$

From Eq. (4.6), we have

$$\begin{aligned}\int_0^\tau \bar{G}(s)\, ds &= \frac{1}{\alpha}\int_0^{\beta(L-A)} \bar{F}_{\alpha\tau,1}(y)\, \varphi(\beta L - y)\, dy \\ &\simeq \frac{1}{\alpha}\int_0^{\beta(L-A)} \bar{F}_{\alpha\tau,1}(y)\, dy.\end{aligned} \tag{4.17}$$

We know that (see Eq. (3.7))

$$\mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A))) = \frac{\beta}{\alpha}\, \mathbb{E}(X_{\inf(\tau,(\sigma_L - \sigma_A))}).$$

Hence, this approximation aims at replacing the mean deterioration level at time $t = \inf(\tau, \sigma_L - \sigma_A)$ by the minimum between the difference $L - A$ and the mean deterioration at time τ (see Fig. 3):

$$\mathbb{E}(X_{\inf(\tau,(\sigma_L - \sigma_A))}) \approx \mathbb{E}(\inf(X_\tau, (L - A))).$$

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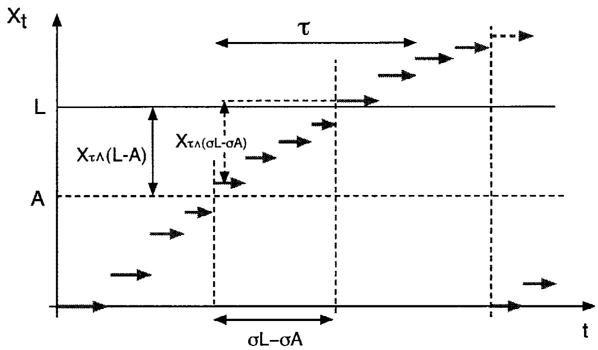

FIGURE 3. Approximation 2 of $\mathbb{E}(\inf \tau, (\sigma_L - \sigma_A))$.

The second approximated value of the asymptotic unavailability as a function of the alarm threshold A is given by

$$U_\infty^2(A) = \frac{\rho_1 + \left(1 + \rho_2 \dfrac{\alpha}{\beta}\right)\tau + \rho_2\left(A + \dfrac{1}{2\beta}\right) - \dfrac{1}{\alpha}\displaystyle\int_0^{\beta(L-A)} \bar{F}_{\alpha\tau,1}(y)\,dy}{\rho_1 + \left(1 + \dfrac{\alpha\rho_2}{\beta}\right)\left(\tau + \dfrac{A\beta}{\alpha} + \dfrac{1}{2\alpha}\right)}. \quad \textbf{(4.18)}$$

5. NUMERICAL EXPERIMENTS

The alarm threshold is set on the system deterioration level for triggering a delayed preventive maintenance operation. The mathematical model developed in the previous sections aims at finding the value of this alarm threshold that minimizes the asymptotic unavailability of the system:

$$A^* = \arg\min_A U_\infty(A). \quad \textbf{(5.1)}$$

Several numerical optimization have been performed successfully for different values of the problem parameters L, τ, ρ_1, ρ_2, α, and β using a quasi-Newton method. As an example, Table 1 shows the evolution of the optimal alarm threshold A^* for degradation characteristics with a fixed given mean per time unit ($\alpha/\beta = 2$) and a decreasing variance per time unit (α/β^2). As the variance decreases, the optimal value A^* increases and the associated asymptotic unavailability $U_\infty^* = U_\infty(A^*)$ decreases. For a small variance, the value of the degradation mean per time unit is a significant parameter. An increase of the variance lead to choosing a more conservative alarm threshold in order to avoid frequent unexpected failures.

The given optimization results (see Table 1) have been obtained with the "exact" expression of the asymptotic unavailability. This leads to burdensome and slow numerical computation. The two proposed approximations give the same numerical

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TABLE 1. Evolution of the Optimal Asymptotic Unavailability as a Function of the Variance of the Deterioration per Unit of Time [$\mathbb{E}(\widetilde{X}_1) = \alpha/\beta = 2$, $L = 20$, $\tau = 2$, $\rho_1 = 2$, $\rho_2 = 0.1$].

	Variance α/β^2		
	4	2	1
Optimal alarm threshold $A^\star$	13.6012	14.1137	14.5656
Optimal asymptotic unavailability $U_\infty^\star$	0.3094	0.3027	0.2976

results with a significant time savings. As an example, Figure 4 shows the evolution of the asymptotic unavailability as a function of the alarm threshold for an average amount of deterioration per time unit $\alpha/\beta = 1$. In this configuration, the approximations of $\mathbb{E}(\sigma_A)$ and $\mathbb{E}(X_{\sigma_A+\tau})$ can be used ($A > 1$ and $L - A > \frac{1}{2}$). Such a set of parameters makes the two approximations of $\mathbb{E}(\inf(\tau, (\sigma_L - \sigma_A)))$ accurate and the results point out that the three computed values of the asymptotic unavailability are very close to each other.

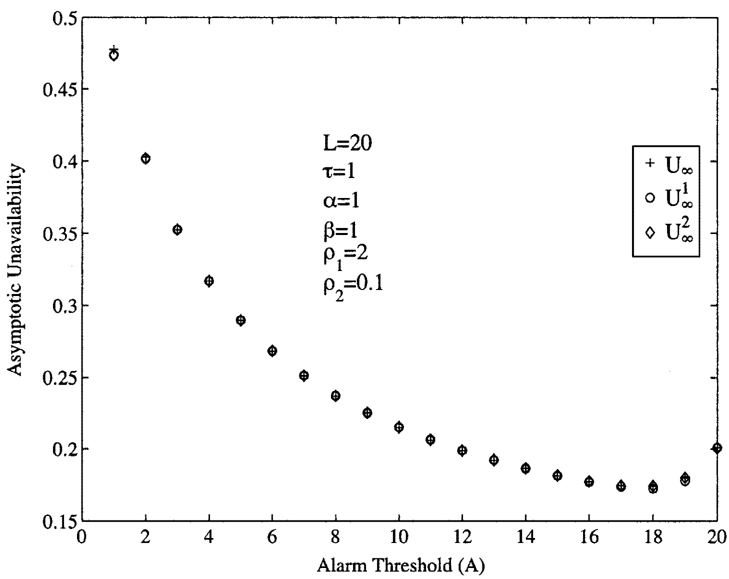

FIGURE 4. Comparison of the numerical computations of the asymptotic unavailability.

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6. CONCLUSION

We have studied a condition-based maintenance policy based on a control limit structure for a continuously deteriorating/continuously monitored system. A mathematical model has been developed in order to evaluate the asymptotic unavailability of the maintained system and to allow the optimization of the maintenance parameter (i.e., of the threshold value of the control limit rule). The mathematical model is based on the regenerative properties of the maintained system state. It has been elaborated in the special case of a deterioration modeled by Gamma stochastic processes. Numerical experiments have shown that minimizing the asymptotic unavailability, an optimal setting of the alarm threshold value can adapt the maintenance policy to the deterioration characteristics of the system.

References

1. Abdel-Hameed, M. (1987). Inspection and maintenance policies of devices subject to deterioration. *Advances in Applied Probability* 19: 917–931.
2. Asmussen, S. (1987). *Applied probability and queues*. Wiley Series in Probability and Mathematical Statistics. New York: Wiley.
3. Bloch-Mercier, S. (2002). A preventive maintenance policy with sequential checking procedure for a Markov deteriorating system. *European Journal of Operational Research* 142: 548–576.
4. Christer, A. (1999). Developments in delay time analysis for modelling plant maintenance. *Journal of the Operational Research Society* 50: 1120–1137.
5. Cocozza-Thivent, C. (2000). A model for a dynamic preventive maintenance policy. *Journal of Applied Mathematics and Stochastics Analysis* 13(4): 321–346.
6. Grall, A., Dieulle, L., Bérenguer, C., & Roussignol, M. (2002). Continuous-time predictive-maintenance scheduling for a deteriorating system. *IEEE Transactions on Reliability* 51: 141–150.
7. Newby, M. & Dagg, R. (2001). Optimal inspection policies in the presence of covariates. In *Proceedings of the European Safety and Reliability Conference—ESREL'2002*, pp. 131–138.
8. Rao, B. (ed.). (1996). *Handbook of condition monitoring*. Amsterdam: Elsevier.
9. Singpurwalla, N. & Wilson, S. (1998). Failure models indexed by two scales. *Advances in Applied Probability* 30: 1058–1072.
10. van Noortwijk, J., Kok, M., & Cooke, R. (1997). Optimal maintenance decisions for the sea-bed protection of the Eastern-Scheldt Barrier. In R. Cooke, M. Mendel, & H. Vrijling (eds.), *Engineering probabilistic design and maintenance for flood protection*. Boston: Kluwer Academic, pp. 25–51.
11. Yang, Y. & Klutke, G.-A. (2000). Lifetime-characteristics and inspections-schemes for Lévy degradation processes. *IEEE Transactions on Reliability* 49(4): 377–382.
12. Zuckerman, D. (1978). Optimal replacement policy for the case where the damage process is one-sided Levy process. *Stochastic Processes and Their Applications* 7: 141–151.

APPENDIX

A. PROOF OF PROPOSITION 3.2

It is easy to obtain the survival function $\bar{H}(u,v)$ of (σ_A, σ_L) for $v > u > 0$:

$$\begin{aligned}\bar{H}(u,v) &= \mathbb{P}(\sigma_A > u, \sigma_L > v)\\ &= \mathbb{P}(\widetilde{X}_u \le A, \widetilde{X}_v \le L)\\ &= \iint_{\{0<x<A, 0<x+y<L, 0<y\}} f_{\alpha u, \beta}(x)\, f_{\alpha(v-u), \beta}(y)\, dx\, dy.\end{aligned}$$

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Then,

$$\bar{G}(s) = \iint_{\{v-u>s\}} \frac{\partial^2 H}{\partial u\, \partial v}(u,v)\, du\, dv$$
$$= \int_s^\infty \left(\frac{\partial H}{\partial v}(v-s,v) - \lim_{u\to 0} \frac{\partial H}{\partial v}(u,v) \right) dv.$$

Since the Gamma probability law $f_{\alpha u,\beta}(x)\,dx$ converges toward the Dirac probability law at zero if u tends to zero:

$$\lim_{u\to 0} \frac{\partial H}{\partial v}(u,v) = \frac{\partial F_{\alpha v,\beta}(L)}{\partial v}(v).$$

Then,

$$\bar{G}(s) = \int_s^\infty \left(\iint_{\{0<x<A,0<x+y<L,0<y\}} f_{\alpha(v-s),\beta}(x) \frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\, dx\, dy \right) dv + F_{\alpha s,\beta}(L)$$
$$= \int_0^\infty \left(\iint_{\{0<x<A,0<x+y<L,0<y\}} f_{\alpha v,\beta}(x) \frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\, dx\, dy \right) dv + F_{\alpha s,\beta}(L).$$

We know that the convolution of the functions $f_{\alpha v,\beta}(\cdot)$ and $f_{\alpha u,\beta}(\cdot)$ is equal to $f_{\alpha(v+u),\beta}(\cdot)$. Then, we can write

$$\int_0^\infty \left(\iint_{\{0<x,0<x+y<L,0<y\}} f_{\alpha v,\beta}(x)\, f_{\alpha s,\beta}(y)\, dx\, dy \right) dv$$
$$= \int_0^\infty \left(\int_0^L f_{\alpha(s+v),\beta}(z)\, dz \right) dv$$
$$= \int_0^L \left(\int_s^\infty f_{\alpha w,\beta}(z)\, dw \right) dz$$

and by differentiating,

$$\int_0^\infty \left(\iint_{\{0<x,0<x+y<L,0<y\}} f_{\alpha v,\beta}(x) \frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\, dx\, dy \right) dv$$
$$= -\int_0^L f_{\alpha s,\beta}(z)\, dz$$
$$= -F_{\alpha s,\beta}(L).$$

We obtain

$$\bar{G}(s) = -\int_0^\infty \left(\iint_{\{A<x<L,0<x+y<L,0<y\}} f_{\alpha v,\beta}(x) \frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\, dx\, dy \right) dv.$$

This ends the proof. ∎

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B. PROOF OF PROPOSITION 3.3

By integration of Eq. (3.11) with respect to s between any $\epsilon > 0$ and τ,

$$\begin{aligned}\int_{\epsilon}^{\tau} \bar{G}(s)\,ds &= -\int_{\epsilon}^{\tau}\left(\iint_{\{A<x<L,0<x+y<L,0<y\}}\left(\int_0^{\infty} f_{\alpha u,\beta}(x)\,du\right)\frac{\partial f_{\alpha s,\beta}(y)}{\partial s}\,dx\,dy\right)ds\\ &= -\iint_{\{A<x<L,0<x+y<L,0<y\}}\left(\int_0^{\infty} f_{\alpha u,\beta}(x)\,du\right)(f_{\alpha\tau,\beta}(y)-f_{\alpha\epsilon,\beta}(y))\,dx\,dy\\ &= -\int_A^L\left(\int_0^{\infty} f_{\alpha u,\beta}(x)\,du\right)\left(\int_0^{L-x}(f_{\alpha\tau,\beta}(y)-f_{\alpha\epsilon,\beta}(y))\,dy\right)dx.\end{aligned}$$

With $z = L - x$, we obtain, after integration with respect to y,

$$\int_{\epsilon}^{\tau} \bar{G}(s)\,ds = -\int_0^{L-A}\left(\int_0^{\infty} f_{\alpha u,\beta}(L-z)\,du\right)(F_{\alpha\tau,\beta}(z)-F_{\alpha\epsilon,\beta}(z))\,dz,$$

where $F_{\alpha t,\beta} = 1 - \bar{F}_{\alpha t,\beta}$ is the distribution function of the Gamma probability function $f_{\alpha t,\beta}$. Let ϵ tend to zero. The Gamma probability law $f_{\alpha\epsilon,\beta}$ converges toward the Dirac probability law at zero. Hence, $F_{\alpha\epsilon,\beta}(z)$ tends to one if $z > 0$. As a conclusion,

$$\int_0^{\tau} \bar{G}(s)\,ds = \int_0^{L-A}\left(\int_0^{\infty} f_{\alpha u,\beta}(L-z)\,du\right)\bar{F}_{\alpha\tau,\beta}(z)\,dz. \qquad \blacksquare$$

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