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A VARIATIONAL PROBLEM FOR CURVES ON FINSLER SURFACES

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Abstract

We study the variational problem for N-parallel curves on a Finsler surface by means of exterior differential systems using Griffiths' method. We obtain the conditions when these curves are extremals of a length functional and write the explicit form of Euler–Lagrange equations for this type of variational problem.

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1. Introduction

The theory of variations is a central topic in optimization theory, mechanics, differential geometry and other fields of mathematics [1]. In particular, the search for the extremals of arc length functionals for curves is the archetypal problem in Riemannian and Finsler geometry.

It is well known that the extremals of the energy functional or a Riemannian or Finsler manifold are *geodesics* and that in fact these coincide with the extremals of the arc length functional for unit speed curves. An equivalent characterization is that a unit speed curve γ on a Riemannian or Finsler manifold is a geodesic if and only if the tangent vector is parallel along γ with respect to a certain connection.

In the Riemannian case, this is equivalent to the fact that the normal vector along the unit speed curve γ is also parallel with respect to the Levi-Civita connection, but this property does not extend to Finsler manifolds due to the dependency on direction of the Finslerian inner product.

We recall that, in a previous paper [10], we have encountered a family of curves γ on a Finsler surface characterized by the property that the Finslerian normal vector N is parallel along the curve γ with respect to the Chern connection with reference vector N. We have called these curves N-*parallels* and have shown that these curves are deeply related to the Gauss–Bonnet-type theorems in Finsler geometry, proving in this way the importance of them. However, due to the difficulties of the problem setting and the complexity of computations, the study of such curves on Finsler manifolds is not an easy task.

Motivated by these facts, we consider the following problem.

Are the N-parallel curves extremals of some length functional?

Let M be a two-dimensional differentiable manifold. We give a first answer to this question by studying the functional

$$\mathcal{L}_N(\gamma) := \int_a^b \sqrt{g_N(T,T)}\,dt, \tag{1.1}$$

where $T(t) := \dot{\gamma}(t)$ and $N(t)$ are the tangent and normal vectors along the curve $\gamma : [a,b] \to M$, respectively. Here $g_N := g_{ij}(\gamma(t), N(t))\,dx^i \otimes dx^j$ is the fundamental tensor of the Finsler surface (M,F) evaluated in the normal direction (see Section 2 for details). We determine the extremals, called *N-extremals*, of this functional for variations $\gamma : (-\varepsilon,\varepsilon) \times [a,b] \to M$, $(u,t) \mapsto \gamma(u,t)$, $\gamma(0,t) = \gamma(t)$ subject to some end-point condition.

One might be tempted to consider this variational problem on the surface M as in the classical variational problem for the usual arc length variation (see [3, 12, 15]). However, one can easily see that along the variation curve $\gamma(u,t)$, the variation vector field and the normal vector field both belong to the normal bundle

$$\{w \in T_\gamma M : T(t) \text{ and } w(t) \text{ are linearly independent}\}$$

and hence the treatment from the classical variational problem does not apply. Instead, we consider this variational problem 'upstairs' on the indicatrix bundle Σ by Griffiths' formalism and obtain in this way the Euler–Lagrange equations for this variational problem.

The novelty of the present research lies in the following results.

(1) We have formulated the variational problem for the functional (1.1) and we have computed the corresponding Euler–Lagrange equations.

(2) We have shown that the extremals of the functional (1.1) form a family of curves on the base manifold M that are different from the usual geodesics and have determined the conditions when they coincide with the N-parallels. This is one of the main differences between the arc length variational problem in Riemannian and Finsler settings.

(3) We have successfully used Griffiths' formalism for variational problems based on exterior differential systems in order to solve an intractable variational problem in the classical setting.

We point out that this method can be applied in the study of other variational problems in Riemannian geometry or in the general theory of calculus of variations. In future we intend to use this approach in the study of other variational problems.

Here is the structure of our paper. We review the basics of Finsler surfaces in Section 2 and present Griffiths' formalism in Section 3. In Section 4, we describe the geometry of N-parallel curves on a Finsler surface and express these curves as integral manifolds of an exterior differential system as well as second-order differential equations.

Section 5 contains the study of variational problems for the functional (1.1). Here we obtain the Euler–Lagrange equations for this functional (Theorem 5.4) and we prove that these equations are necessary and sufficient conditions for the extremals of our functional (Theorem 5.3). The relation of the extremal curves with the geodesic curvature is given and this leads us to a new class of Finsler surfaces for which the N-parallels and extremals of the functional (1.1) coincide.

Finally we discuss the N-extremals for some special Finsler surfaces in Section 6.

2. Finsler surfaces

A *Finsler surface* is a pair (M, F), where M is a two-dimensional differentiable manifold and $F : TM \to [0, \infty)$ is a function, called *the fundamental function*, that is positive and smooth away from the zero section, has the *homogeneity property* $F(x, \lambda v) = \lambda F(x, v)$ for all $\lambda > 0$ and all $v \in T_xM$, having also the *strong convexity property* that the Hessian matrix, called *the fundamental tensor*, $g_{ij}(x, y) := 1/2(\partial^2 F^2(x, y)/(\partial y^i \partial y^j))$ is positive definite at every point of $\widetilde{TM} = TM \setminus \{0\}$.

The restriction of a Finsler norm to a tangent plane T_xM gives a Minkowski norm on T_xM. For an arbitrary fixed $x \in M$, this Minkowski norm induces a flat Riemannian metric on the punctured tangent plane $\widetilde{T_xM}$ by

$$\hat{g} := g_{ij}(y)\, dy^i \otimes dy^j, \tag{2.1}$$

where $y = (y^i)$ are the global coordinates in T_xM.

A Finsler surface (M, F) is equivalent to giving a smooth hypersurface $\Sigma \subset TM$ for which the canonical projection $\pi : \Sigma \to M$ is a surjective submersion and having the property that for each $x \in M$, the π-fiber $\Sigma_x = \pi^{-1}(x)$ is a strictly convex smooth curve including the origin $O_x \in T_xM$.

In order to study the geometry of the Finsler surface (M, F), we consider the pull-back bundle π^*TM with base manifold Σ and fibers $(T_xM)|_u$, where $u \in \Sigma$ is such that $\pi(u) = x$ (see [3, Ch. 2] for details). It is known (see [3, page 30]) that the vector bundle π^*TM has a distinguished global section $l := (y^i/F(y))(\partial/\partial x^i)$. Using this section, one can construct a positively oriented g-orthonormal frame $\{e_1, e_2\}$ for π^*TM, where $g := g_{ij}(x, y)\, dx^i \otimes dx^j$ is the naturally induced Riemannian metric on the fibers of π^*TM by means of the fundamental tensor g_{ij}. The frame $\{u; e_1, e_2\}$ for any $u \in \Sigma$ is a globally defined g-orthonormal frame field for π^*TM called the *Berwald frame*.

We introduce the following notation. For $N(t) \in T_{\gamma(t)}M$, we denote by

$$g_N := g_{ij}(\gamma(t), N(t))\, dx^i \otimes dx^j$$

the inner product induced on the tangent space $T_{\gamma(t)}M$ by the fundamental tensor g_{ij} evaluated in the direction N. A concrete choice for this N will be made later (see Section 4).

By duality, one defines a moving coframe $(u; \omega^1, \omega^2, \omega^3)$ on Σ, orthonormal with respect to the Riemannian metric on Σ induced by the Finsler metric F, where $u \in \Sigma$ and $\{\omega^1, \omega^2, \omega^3\} \in T^*\Sigma$. The moving equations on this frame lead to the so-called Chern connection. This is an almost metric compatible, torsion-free connection of the vector bundle (π^*TM, π, Σ).

Indeed, by a theorem of Cartan, it follows that the coframe $(\omega^1, \omega^2, \omega^3)$ must satisfy the following structure equations:

$$\begin{aligned} d\omega^1 &= -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \\ d\omega^2 &= -\omega^1 \wedge \omega^3, \\ d\omega^3 &= K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3. \end{aligned} \tag{2.2}$$

The functions I, J, K are smooth functions on Σ called the *invariants* of the Finsler surface (M, F) in the sense of Cartan's equivalence problem [3, 5].

The scalar functions I and K are called the *Cartan scalar* and the *Gauss curvature* of the Finsler surface, respectively. In the case when F is Riemannian, $I = J = 0$ and K coincides with the usual Gauss curvature of a Riemannian surface.

Differentiating again (2.2), one obtains the *Bianchi identities*

$$J = I_2, \quad K_3 + KI + J_2 = 0,$$

where the indices in I_2, K_3, J_2 etc indicate differential terms with respect to ω_1, ω_2, ω_3. For example, $dK = K_1\omega^1 + K_2\omega^2 + K_3\omega^3$. The scalars K_1, K_2, K_3 are called the directional derivatives of K.

More generally, given any function $f : \Sigma \to \mathbb{R}$, one can write its differential in the form $df = f_1\omega_1 + f_2\omega_2 + f_3\omega_3$.

Recall that a Finsler surface is called *Landsberg* if the invariant J vanishes. Bianchi identities imply that in this case $I_2 = 0$ and $K_3 = -KI$. A Finsler surface having $I_1 = 0$ and $I_2 = 0$ is called a *Berwald* surface (see [3, Lemma 10.3.1, page 267] for details). It is known that a Berwald surface is in fact Riemannian if $K \neq 0$ or locally Minkowski flat if $K = 0$ (see [16] and [3, page 278]).

3. The variational problem in Griffiths' formulation

We will review and fix the notation for the variational problem in Griffiths' formulation. This is a natural generalization of the classical variational problem formulated in the language of exterior differential systems (see [4, 7, 8, 11, 12]).

A *variational problem* $(\mathcal{I}, \omega; \varphi)$ is the study of the functional

$$\Phi : \mathcal{V}(\mathcal{I}, \omega) \to \mathbb{R}, \quad \Phi(\gamma_{|(a,b)}) = \int_\gamma \varphi = \int_a^b \gamma^*\varphi, \tag{3.1}$$

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where $(\mathcal{I}, \omega)$ is a Pfaffian differential system of rank s with independence condition ω on a manifold X, γ is a typical integral manifold of $(\mathcal{I}, \omega)$, that is,

$$\gamma_{|(a,b)} \in \mathcal{V}(\mathcal{I}, \omega) := \{\gamma : (a, b) \to X \mid \gamma^*(\mathcal{I}) = 0, \gamma^*(\omega) \neq 0\},$$

and φ is a one-form on X (here the curves that differ only by parametrization will be identified).

The main problem of the calculus of variations is the same as in the classical case: *describe the extremals of the functional* Φ*, that is, determine the Euler–Lagrange equations of* Φ.

More precisely, if we denote by $T_\gamma \mathcal{V}(\mathcal{I}, \omega)$ the 'tangent space' of $\mathcal{V}(\mathcal{I}, \omega)$ at γ, then we can consider the differential of (3.1), that is,

$$\delta\Phi_\gamma : T_\gamma \mathcal{V}(\mathcal{I}, \omega) \to \mathbb{R}, \quad \delta\Phi_\gamma(v) = \frac{d}{du}\Big(\int_{\gamma_u} \varphi\Big)\Big|_{u=0},$$

where $\gamma_u \in \mathcal{V}(\mathcal{I}, \omega)$ is any compactly supported variation of γ with $\gamma_0 = \gamma$ and $v \in T_\gamma \mathcal{V}(\mathcal{I}, \omega)$ is the associated infinitesimal variation vector field defined along γ corresponding to the variation $u \mapsto \gamma_u$.

With this notation, *the Euler–Lagrange equations* of Φ are

$$\delta\Phi_\gamma(v) = 0, \quad \forall v \in T_\gamma \mathcal{V}(\mathcal{I}, \omega)$$

and the integral curves γ satisfying these equations are called *the extremals* of Φ.

The 'tangent space' $T_\gamma \mathcal{V}(\mathcal{I}, \omega)$, that is, the space of smooth variation vector fields of γ, can be described to first order by the variational equations of an integral curve of the Pfaffian system $(\mathcal{I}, \omega)$ (see [8, 9] for details). The variational equations of the integral curves of $(\mathcal{I}, \omega)$ are given by

$$\mathcal{D}_\gamma(v) = 0, \quad v \in T_{\gamma(t)} X \setminus \langle \gamma'(t) \rangle,$$

where

$$\mathcal{D}_\gamma(v) := e_\alpha \otimes (v \lrcorner d\theta^\alpha + d(v \lrcorner \theta^\alpha))|_\gamma.$$

Here $\{\theta^1, \theta^2, \ldots, \theta^s\}$ is a local basis for $\mathcal{I}$ and $\{e_1, e_2, \ldots, e_s\}$ its dual frame field along γ. Locally, the tangent space $T_\gamma \mathcal{V}(\mathcal{I}, \omega)$ is described by the equations

$$(v \lrcorner d\theta^\alpha + d(v \lrcorner \theta^\alpha))|_\gamma = 0, \quad \alpha = 1, 2, \ldots, s,$$

called *the variational equations* of $(\mathcal{I}, \omega)$.

For the variational problem $(\mathcal{I}, \omega; \varphi)$, rank $\mathcal{I} = s > 0$, on a manifold X, one can associate the Euler–Lagrange Pfaffian differential system $(\mathcal{J}, \omega)$ on a new manifold Y such that its variational equations coincide with the Euler–Lagrange equations for Φ. The integral manifolds of this system give the extremals of Φ.

For this we follow [8, 9] and construct the affine sub-bundle

$$Z := \mathcal{I} + \varphi \subset T^*X,$$

that is, $Z_x = \mathcal{I}_x + \varphi_x$ is an affine subspace of $T^*_x X$ for any $x \in X$.

Locally, $Z_U \simeq U \times \mathbb{R}^s$, where $U \subset X$ is an open set; in other words, we identify the pair $(x, \lambda) \in U \times \mathbb{R}^s$ with the one-form

$$\psi_x := \varphi_x + \sum_{\alpha=1}^{s} \lambda_i \theta^i_x \in T^*_x X,$$

where $\{\theta^1, \theta^2, \dots, \theta^s\}$ is a local basis for $\mathcal{I}$ over U. By this identification, it results that ψ is the canonical one-form on Z obtained by the restriction of the canonical one-form on T^*X to Z.

If we denote $\Psi := d\psi$ and $C(\Psi) := \{v \lrcorner \Psi : v \text{ vector field on } Z\}$ the *Cartan system* of Ψ, then the Pfaffian system $(\mathcal{J} := C(\Psi), \omega)$, typically restricted to a submanifold $Y \subset Z$, is called *the Euler–Lagrange system* associated to the variational problem $(\mathcal{I}, \omega; \varphi)$ on X. Any variational problem for curves can be formulated in this setting.

The solutions of the Euler–Lagrange equations are in natural one-to-one correspondence with the integral manifolds of the Euler–Lagrange differential system $(\mathcal{J}, \omega)$ on Y.

A variational problem $(\mathcal{I}, \omega; \varphi)$ is called *nondegenerate* if there exists a positive integer m such that

$$\begin{cases} \dim \ Y = 2m + 1, \\ \psi \wedge \Psi^m \neq 0, \end{cases}$$

where $\Psi^m = \underbrace{\Psi \wedge \cdots \wedge \Psi}_{m \text{ times}}$. Here we denote $\psi|_Y$ and $\Psi|_Y$ with the same letters ψ and Ψ, respectively, for simplicity.

Remark 3.1. The characteristic direction of Ψ living on Y generates a global foliation $\varpi : Y \to Q := Y/_{\Psi^\perp}$.

Here Q is a $(2m)$-dimensional symplectic manifold with the two-form $\bar{\Omega}$ satisfying $\varpi^* \bar{\Omega} = \Psi$.

For a nondegenerate variational problem, it is easy to formulate the end-point conditions (see [7] for a general theory).

Remark 3.2. However, one must pay attention to the following problems when working in this formalism.

(1) In general, the differential system $(C(\Psi), \omega)$ is not a Pfaffian system with independence condition on Z, so we need to construct an involutive prolongation of $(C(\Psi), \omega)$ on a submanifold $Y \subset Z$ [7, 8]. The integral elements of $(C(\Psi), \omega)$ are the lines $E \subset T_z Z$ such that $\eta(z)|_E = 0$ for all $\eta \in C(\Psi)$ and $\omega(z)|_E \neq 0$. The set of integral elements form the set $V(C(\Psi), \omega) \in PT(Z)$, where $\pi : PT(Z) \to Z$ is the projectivized tangent bundle of Z. Inductively define

$$\begin{cases} Z_1 = \pi(V(C(\Psi), \omega)), \\ V_1(C(\Psi), \omega) = \{E \in V(C(\Psi), \omega) : E \text{ is tangent to } Z_1\}, \\ Z_2 = \pi(V_1(C(\Psi), \omega)), \\ V_2(C(\Psi), \omega) = \{E \in V_1(C(\Psi), \omega) : E \text{ is tangent to } Z_2\}, \\ \vdots \end{cases}$$

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We obtain $Z_1 \supset Z_2 \supset \cdots$ and, under reasonable assumptions, that there exists a positive integer k such that $Z_k = Z_{k+1} = \cdots = Y$ and

$$\mathcal{J} := \{\eta(z) \in T^*_z(Y) : \eta \in C(\Psi)|_Y\}$$

gives a sub-bundle of $T^*(Y)$, and obviously the integral manifolds of $(C(\Psi), \omega)$ and $(\mathcal{J}, \omega)$ coincide. The Pfaffian system $(\mathcal{J}, \omega)$ on $Y \subset Z$ is called *the involutive prolongation* of $(C(\Psi), \omega)$.

(2) The Euler–Lagrange equations for the variational problem $(\mathcal{I}, \omega; \varphi)$, that we describe above, are sufficient conditions for γ to be extremals. However, in the case rank $\mathcal{I} > 0$ these are not always necessary conditions. Indeed, it is known that any regular extremal curve of Φ is a solution of the Euler–Lagrange equations [9], where γ regular means that it is a generic integral curve of a bracket-generating differential system.

4. The normal lift of a curve

In this section we recall the normal lift of a curve from [10, 14].

Let us consider a smooth (or piecewise C^∞) curve $\gamma : [a, b] \to M$ with the tangent vector $\dot\gamma(t) = T(t)$, parametrized such that $F(\gamma(t), \dot\gamma(t)) = 1$.

We construct the normal vector field N along γ, that is,

$$g_N(N, N) = 1, \quad g_N(N, T) = 0, \quad g_N(T, T) = \sigma^2(t), \tag{4.1}$$

where $\sigma(t)$ is a scalar function nonconstant along γ and g_N is the Riemannian metric of $T_{\gamma(t)}M$ induced by the Finsler fundamental tensor evaluated in the normal direction (see (2.1)). Obviously, $\{T, N\}$ is a g-orthonormal frame of $T_{\gamma(t)}M$ and, since we are in the two-dimensional case, the existence of such an N is guaranteed.

This leads us to the *normal lift* $\hat\gamma^\perp$ of γ to Σ defined by

$$\hat\gamma^\perp : [a, b] \to \Sigma, \quad t \mapsto \hat\gamma^\perp(t) = (\gamma(t), N(t)). \tag{4.2}$$

It follows that

$$\begin{aligned}
&g_N(D^{(N)}_T N, N) = 0,\\
&g_N(D^{(N)}_T T, N) + g_N(T, D^{(N)}_T N) = 0,\\
&g_N(D^{(N)}_T T, T) = \sigma(t)\frac{d\sigma}{dt} - A^{(N)}(T, T, D^{(N)}_T N),
\end{aligned}$$

where

$$D^{(N)}_T U = (D^{(N)}_T U)^i \cdot \frac{\partial}{\partial x^i}_{|_{\gamma(t)}} = \left[\frac{dU^i}{dt} + T^j U^k \Gamma^i_{jk}(x, N)\right] \cdot \frac{\partial}{\partial x^i}_{|_{\gamma(t)}}$$

for any $U = U^i(x)(\partial/\partial x^i)$ vector field along γ; Γ^i_{jk} are the Chern connection coefficients, that is, $\omega_i^j = \Gamma^j_{ik}\, dx^k$, that is, the covariant derivative along γ with reference vector N, and $A^{(N)}(U, V, W) := A_{ijk}(x, N)U^i V^j W^k$, for $V = V^i(x)(\partial/\partial x^i)$ and $W = W^i(x)(\partial/\partial x^i)$, are vector fields along γ. Here $A_{ijk} := (1/4)(\partial^3 F^2/\partial y^i \partial y^j \partial y^k)$.

We can define the notion of an N-parallel of a Finsler surface.

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DEFINITION 4.1. A curve γ on the surface M, in Finsler natural parametrization, is called an *N-parallel* of the Finsler surface (M, F) if and only if the normal vector field is parallel along γ, namely,

$$D_T^{(N)} N = 0.$$

If γ is an N-parallel, then

$$g_N(D_T^{(N)} T, N) = 0, \quad g_N(D_T^{(N)} T, T) = \sigma(t)\frac{d\sigma}{dt}.$$

In case of an arbitrary curve γ on M, from

$$g_N(D_T^{(N)} N, N) = 0, \quad g_N(T, N) = 0,$$

it follows that the vector $D_T^{(N)} N$ is proportional to T, that is, there exists a nonvanishing function $k_T^{(N)}(t)$ such that

$$D_T^{(N)} N = -\frac{k_T^{(N)}(t)}{\sigma^2(t)} T, \quad \sigma(t) \neq 0.$$

The function $k_T^{(N)}(t)$ will be called the *N-parallel curvature of* γ. The minus sign is put only in order to obtain the same formulas as in the classical theory of Riemannian surfaces.

In other words,

$$g_N(D_T^{(N)} N, T) = -k_T^{(N)}(t).$$

PROPOSITION 4.2. *A curve γ on M is N-parallel if and only if its N-parallel curvature $k_T^{(N)}$ vanishes.*

By making use of the cotangent map of $\hat{\gamma}^{\perp *}$,

$$\hat{\gamma}^{\perp *}\omega^1 = \sigma(t)\, dt, \quad \hat{\gamma}^{\perp *}\omega^2 = 0, \quad \hat{\gamma}^{\perp *}\omega^3 = -\frac{k_T^{(N)}}{\sigma(t)}\, dt \tag{4.3}$$

(see [10] for details).

5. Variational problem for the *N*-lift

We will formulate a variational problem in Griffiths' formalism for our setting by specifying the manifold X and the Pfaffian with independence condition $(\mathcal{I}, \omega)$ by means of the 3-manifold Σ with the coframe $\{\omega^1, \omega^2, \omega^3\}$ generated by the Finsler surface (M, F).

First, we remark that Equations (4.3) imply the following result.

PROPOSITION 5.1. *Let $\gamma : [a, b] \to M$ be a smooth curve on M. Then its normal lift $\hat{\gamma}^{\perp}$ to Σ given by (4.2) is an integral manifold of the Pfaffian system with independence condition $(\mathcal{I}, \omega^1)$ on the manifold Σ, where $\mathcal{I} = \{\omega^2\}$.*

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Clearly, the projection to M of any integral curve of $(\mathcal{I}, \omega^1)$ is a curve on M.

We will consider the variational problem in Griffiths' formalism $(\mathcal{I}, \omega^1; \varphi)$ on the manifold Σ, with

$$\varphi = \omega^1.$$

More precisely, we consider the functional

$$\Phi : \mathcal{V}(\mathcal{I}, \omega^1) \to \mathbb{R}, \quad \Phi(\hat{\gamma}^{\perp}) = \int_{\hat{\gamma}^{\perp}} \omega^1, \tag{5.1}$$

where $\hat{\gamma}^{\perp} : (a, b) \to \Sigma$ is a typical integral manifold of the rank-one Pfaffian system $(\mathcal{I}, \omega^1)$.

The extremals of this functional are called the *N-extremals* of the Finsler surface (M, F).

In order to compute the Euler–Lagrange equations of this variational problem, we follow Griffiths' recipe in Section 3 and consider the manifold $Z := \Sigma \times \mathbb{R}$, where $\mathbb{R}$ has the coordinate λ, and put

$$\psi := \varphi + \lambda\omega^2$$

on Z.

The exterior derivative $\Psi = d\psi$ is given by

$$\Psi = d\omega^1 + d\lambda \wedge \omega^2 + \lambda d\omega^2 = -(I + \lambda)\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 + d\lambda \wedge \omega^2.$$

A coframe on Z is given by

$$\{\omega^1, \omega^2, \omega^3; d\lambda\}$$

and the corresponding frame is

$$\left\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \frac{\partial}{\partial\lambda}\right\},$$

where we use $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ for the dual frame of $\{\omega^1, \omega^2, \omega^3\}$ on Σ.

It follows that the Cartan system $C(\Psi)$ is given by

$$\begin{aligned} C(\Psi) &:= \left\{\frac{\partial}{\partial\lambda} \lrcorner \Psi, \hat{e}_2 \lrcorner \Psi, \hat{e}_3 \lrcorner \Psi\right\} = \{\omega^2, -d\lambda + \omega^3, (\lambda + I)\omega^1 - \omega^2\} \\ &= \{\omega^2, d\lambda - \omega^3, (\lambda + I)\omega^1\}. \end{aligned}$$

However, we remark that $(C(\Psi), \omega^1)$ is not a Pfaffian system with independence condition on Σ because $(\lambda + I)\omega^1 \in C(\Psi)$.

Hence, we need an involutive prolongation of $(C(\Psi), \omega^1)$ on a submanifold on Z. Following the first part of Remark 3.2, we consider the submanifold $Z_1 := \{\lambda = -I\} \subset Z$, and therefore

$$C(\Psi)_{|Z_1} = \{\omega^2, dI + \omega^3\}. \tag{5.2}$$

We can see that the iterative construction stops for $k = 1$ and, therefore, if we put $Y := \{\lambda = -I\}$, then $(C(\Psi)_{|Y}, \omega_{|Y})$ is a Pfaffian system with independence condition on $Z_1 = Z_2 = \cdots = Y$.

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We remark that by identifying the graph of a function with its domain of definition, we can see that actually $Y = \Sigma$.

We obtain the following results.

THEOREM 5.2.

(1) *The Euler–Lagrange differential system of the variational problem* $(I, \varphi; \omega)$ *is*

$$(dI + \omega^3)|_{\hat{\gamma}^\perp} = 0.$$

(2) *The corresponding Euler–Lagrange equation is*

$$(I_3 \circ \hat{\gamma}^\perp + 1)k_T^{(N)} = (I_1 \circ \hat{\gamma}^\perp)\sigma^2. \tag{5.3}$$

PROOF. From (5.2), we obtain (1). Moreover, using now the Equations (4.3), the Euler–Lagrange equation (2) follows. □

Moreover, we have the following result.

THEOREM 5.3. *The Euler–Lagrange equation (5.3) is a necessary and sufficient condition for the extremals of the functional (5.1).*

PROOF. The sufficiency is obvious from construction. In order to prove the necessity, since any curve on a contact manifold is regular, it is enough to show that the distribution $D = \mathcal{I}^\perp$ is bracket generating. Since $\mathcal{I} = \{\omega^2\}$ on Σ, it follows that $D = \langle \hat{e}_1, \hat{e}_3 \rangle$. The Cartan formula $d\omega(X, Y) = X(\omega(Y)) - \omega([X, Y]) - Y(\omega(X))$ implies that

$$[\hat{e}_1, \hat{e}_2] = -K\hat{e}_3, \quad [\hat{e}_2, \hat{e}_3] = -\hat{e}_1, \quad [\hat{e}_3, \hat{e}_1] = -I\hat{e}_1 - \hat{e}_2 - J\hat{e}_3.$$

It follows immediately that $D_1 := [D, D] = \langle \hat{e}_1, \hat{e}_2, \hat{e}_3 \rangle = T\Sigma$ and therefore D is bracket generating. □

We remark that

$$\psi_{|Y} = \omega^1 - I\omega^2, \quad \Psi_{|Y} = -I_1\omega^1 \wedge \omega^2 + (I_3 + 1)\omega^2 \wedge \omega^3.$$

Since $\dim Y = 2 \times 1 + 1 = 3$, it follows that $m = 1$ and hence

$$\psi_{|Y} \wedge \Psi^m_{|Y} = (I_3 + 1)\omega^1 \wedge \omega^2 \wedge \omega^3 \neq 0$$

for $I_3 \neq 1$. That is, in this case, the variational problem $(\mathcal{I}, \omega^1; \varphi)$ is *nondegenerate*.

Following Remark 3.1, the end-point conditions are given by

$$K = \{\omega^1, \omega^2\} \tag{5.4}$$

and the variational problem is well posed with reduced momentum space

$$Q = \Sigma/_{\{\omega^1=0, \omega^2=0\}} = M.$$

In our case an admissible variation is a map

$$\hat{\gamma}^\perp : [a, b] \times [0, \varepsilon] \to \Sigma, (t, u) \mapsto \hat{\gamma}^\perp(t, u)$$

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such that each u-curve in the variation, namely $\hat{\gamma}^{\perp}_u : [a,b] \to \Sigma$, is an integral manifold of $(\mathcal{I}, \omega)$.

Then the admissible variations satisfying the end-point conditions (5.4) mean that

$$\hat{\gamma}^{\perp *}(\omega^1) = \hat{\gamma}^{\perp *}(\omega^2) = 0 \quad \text{on } \{a,b\} \times [0,\varepsilon].$$

That is, this corresponds to varying a curve γ in $Q = M$ keeping its end points fixed in the usual sense.

In other words, we have the following results.

THEOREM 5.4.

(1) *If $I_1 \circ \hat{\gamma}^{\perp} = 0$, then the Euler–Lagrange equation implies that $k_T^{(N)} = 0$, provided $I_3 \circ \hat{\gamma}^{\perp} + 1 \neq 0$, that is, in this case the N-extremals are the N-parallels. Conversely, if the N-parallels are the N-extremals, then $I_1 \circ \hat{\gamma}^{\perp} = 0$.*

(2) *If $I_1 \circ \hat{\gamma}^{\perp} \neq 0$, then the Euler–Lagrange equation implies that*

$$k_T^{(N)} = \frac{I_1 \circ \hat{\gamma}^{\perp}}{I_3 \circ \hat{\gamma}^{\perp} + 1}, \tag{5.5}$$

provided $I_3 \circ \hat{\gamma}^{\perp} + 1 \neq 0$, that is, in this case the N-extremals are those curves on Σ whose geodesic curvature is given above.

6. Special Finsler surfaces

6.1. Berwald surfaces. We discuss the *Berwald surfaces* case. It is known that there are only two cases.

(1) $K \neq 0$, that is, (M, F) is a Riemannian surface.

(2) $K = 0$, that is, (M, F) is a locally Minkowski plane.

We discuss first the Riemannian case (M, a). Indeed, in this case, $I = 0$ and the Euler–Lagrange equations (5.3) read $k_T^{(N)} = 0$. In other words, the solutions of the Euler–Lagrange equations (5.3) are the N-parallel curves or, equivalently, the solutions of the following second order differential equation:

$$\frac{d^2\gamma^i}{dt^2} + \gamma^i_{jk}(\gamma(t))\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = \frac{d}{dt}[\log \alpha(t)]\frac{d\gamma^i}{dt},$$

where $\alpha(t) = \sqrt{a(T,T)}$, that is, the usual equation of Riemannian geodesics in arbitrary parametrization.

In the locally Minkowski case, we have $I_1 = 0, I_2 = 0, K = 0$.

LEMMA 6.1. *Let (M, F) be a Finsler surface. If I_3 = constant everywhere on Σ, then I_3 must vanish on Σ.*

PROOF. If I_3 = constant everywhere on Σ, it follows that the scalar I must be constant along every indicatrix curve $\Sigma_x \subset T_xM$. On the other hand, it is known that the average

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value of the Cartan scalar over the indicatrix Σ_x must be zero (see for example [3, page 85]), that is,

$$\int_0^L I(t)\,dt = 0, \tag{6.1}$$

where $I(t)$ is the Cartan scalar evaluated over the indicatrix and L is the Riemannian length of Σ_x.

If $I(t) = c =$ constant, then (6.1) implies $cL = 0$, that is, $c = 0$ since the indicatrix length L cannot be zero. □

In this case, we have the following result.

Proposition 6.2. *If (M, F) is a Minkowski surface, then the solutions of the Euler–Lagrange equations (5.3), that is, the N-extremals, coincide with the N-parallel curves and they differ from the usual geodesics.*

Proof. From Theorem 5.4 and Lemma 6.1, it follows that N-extremals and N-parallels must coincide for a Minkowski surface.

One can see that Finsler geodesics and the N-extremals cannot coincide on Σ because their tangent vectors $\hat{e}_2$ and $\hat{e}_1$, respectively, are linearly independent. □

6.2. A new class of Finsler spaces. We define a new class of special Finsler spaces as follows.

Definition 6.3. A Finsler surface (M, F) that satisfies the following conditions:

$$I_1 = 0, \quad I_3 \neq 0$$

is called an *S-Finsler surface*.

Remark 6.4. Any Berwald manifold is an S-Finsler surface.

Let us remark that on an S-Finsler surface we have the equations

$$\begin{aligned} dI &= I_2\omega^2 + I_3\omega^3, \\ dK &= K_1\omega^1 + K_2\omega^2 - (KI + I_{22})\omega^3, \\ dI_2 &= -KI_3\omega^1 + I_{22}\omega^2 + I_{23}\omega^3. \end{aligned} \tag{6.2}$$

The tableau of the free derivatives has Cartan characters $s_1 = 3$, $s_2 = 2$, $s_3 = 0$; the Cartan test yields $7 = s_1 + 2s_2 + 3s_3$ and therefore the system is involutive. The Cartan–Kähler theory implies that, modulo diffeomorphism, such structures depend on two functions of two variables.

Returning to our variational problem, we have the following result.

Proposition 6.5. *If (M, F) is an S-Finsler surface, then the integral lines of the codimension-one foliation $\{\omega^2 = 0, \omega^3 = 0\}$ coincide with the N-extremals.*

Remark 6.6. The geometrical meaning of the S-Finsler surfaces it is now clear. These are those Finsler surfaces where the N-parallels coincide with the N-extremals.

Finding more examples and applications of the present theory is the subject of forthcoming research.

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