A NOTE ON SEPARATION AXIOMS WEAKER THAN T_1

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Introduction

In [1], Aull and Thron introduce several separation axioms between T_0 and T_1. In particular, they define a space X to be T_D if for each $x \in X$, $\{x\}'$ is a closed set.

On page 37 of their paper, Aull and Thron ask whether there is a separation axiom weaker than T_1 which when combined with normality implies T_4. In [2], Thron asks whether a product of T_D spaces is again a T_D space. In the first section of this note, we answer both of these questions: the first question admits a positive solution, while the second conjecture is never true in the infinite case. In the second section of this paper, we investigate a heirarchy of separation axioms weaker than T_D.

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The answer to the second question is contained in the following theorem.

THEOREM 1. *If $\{X_i : i \in I\}$ is an infinite collection of T_D spaces which are not T_1, then $\Pi\{X_i : i \in I\}$ is not a T_D space.*

PROOF. Since each X_i is not a T_1 space, there exists an $\alpha \in \Pi\{X_i : i \in I\}$ such that for each i, the derived set of the i-th projection of α, $\{\pi_i(\alpha)\}' \neq \emptyset$. Let $Y = \Pi\{\overline{\pi_i(\alpha)} = i \in I\}$ and let $\{\alpha\}'^*$ denote the derived set of α in the relative topology. Since the property of being T_D is inherited by subspaces, it suffices to show that $\{\alpha\}'^*$ is not closed in Y. We first show that $\{\alpha\}$ is not open in Y. To do this, we observe first that if 0 is a basic open neighborhood of y in Y, $0 = Y \cap \Pi\{0_i : i \in I\}$, where each 0_i is an open neighborhood of $\pi_i(y)$ in X_i and $0_i = X_i$ for almost all $i \in I$. Since I is infinite, there exists a non-empty subset $I' \subset I$ such that $0_i = X_i$ for all $i \in I'$. Let 0 be such a basic neighborhood of α. Because our choice of α implies that $\overline{\pi_i(\alpha)} - \pi_i(\alpha) \neq \emptyset$ for $i \in I$, we may select $\beta \in Y$ such that $\pi_i(\beta) = \pi_i(\alpha)$ for $i \in I - I'$ and $\pi_i(\beta) \neq \pi_i(\alpha)$ for $i \in I'$. Clearly, $\beta \in 0$; hence $\{\alpha\}$ is not open.

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Let $\rho \in Y - \{\alpha\}$ and let 0 be a basic open neighborhood of ρ. The observation that $\pi_i(\rho) \in \overline{\pi_i(\alpha)}$ yields that $\pi_i(\alpha) \in \pi_i(0)$ for $i \in I$. Therefore $\rho \in \{\alpha\}'^*$. Hence $\{\alpha\}'^* = Y - \{\alpha\}$ and so $\{\alpha\}'^*$ is not closed.

The following corollary shows that there is no possibility of introducing a separation axiom between T_D and T_1 that is inherited by arbitrary products.

COROLLARY. *Let X be a T_D space. Then, an arbitrary product of X is a T_D space if and only if X is T_1.*

We shall show subsequently that there is no separation axiom between a much weaker axiom than T_D and T_1 that is inherited by arbitrary products.

DEFINITION. X is a *strong* T_D space if for each $x \in X$, $\{x\}'$ is either empty or is a union of a finite family of non-empty closed sets, such that the intersection of this family is empty.

For example, $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ is a strong T_D space which is not T_1.

DEFINITION. X is a *strong* T_0 space provided that for each $x \in X$, $\{x\}'$ is either empty or a union of non-empty closed sets, such that the intersection of this union is empty and at least one of the non-empty members is compact.

THEOREM 2. *A normal space X which is either strong T_0 or strong T_D is T_4.*

PROOF. It suffices to show that X is T_1. Suppose X is strong T_0 and normal, and consider $x \in X$. We claim that $\{x\}' = \emptyset$. For if not, $\{x\}' = \bigcup\{C_i : i \in I\}$, where each C_i is non-empty. Let C_1 be a compact set in this collection and let $\mathscr{I} = \{J \subset I : \cap \{C_i : i \in J\} \cap C_1 \neq \emptyset\}$. $\mathscr{I}$ is a non-empty collection and is partially ordered by set inclusion; furthermore, the compactness of C_1 guarantees that $\mathscr{I}$ is inductive. Thus, Zorn's lemma assures a maximal element I^* of $\mathscr{I}$. Let $F = \cap \{C_i : i \in I^*\}$; clearly, there exists an $i_0 \in I$ such that $C_{i_0} \cap F = \emptyset$. But normality implies C_{i_0} and F can be separated by disjoint open sets, 0_1 and 0_2. Because C_{i_0} and F are non-empty subsets of $\{x\}'$, 0_1 and 0_2 meet $\{x\}'$ and hence x. This is impossible since $0_1 \cap 0_2 = \emptyset$.

The proof for the strong T_D case is similar and does not invoke any form of the axiom of choice.

REMARK. In the definition of strong T_0 space, we cannot omit the compactness condition and still obtain the theorem. For example, let N be the natural numbers and let the topology be

$$\{\emptyset, N, \{1\}, \{1, 2\}, \{1, 2, 3\}, \cdots, \{1, 2, 3, \cdots, n\}, \cdots\}.$$

For each

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$$\begin{aligned} n \in N,\ \{n\}' &= \{n+1, n+2, \cdots\} \\ &= \cup\, \{\{n+j, n+j+1, n+j+2, \cdots\} : j = 1, 2, \cdots\} \\ &= \cup\, \{C_{n,j} : j = 1, 2, \cdots\}, \end{aligned}$$

and we notice that $\cap\, \{C_{n,j} : j = 1, 2, \cdots\} = \emptyset$. However, this space is normal and not T_4.

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In this section we shall discuss separation axioms weaker than T_D and characterize spaces enjoying one of these separation properties.

DEFINITION. Let m be an infinite cardinal. X is a $T^{(m)}$ space provided that for each $x \in X$, $\{x\} = F \cap (\cap\, \{0_i : i \in I\})$, where F is closed, each 0_i is open, and card $(I) = m$.

THEOREM 3. *The following statements are equivalent.*

i. *X is a $T^{(m)}$ space.*

ii. *For each $x \in X$, $\{x\}'$ is an m-fold union of closed sets.*

iii. *For each $A \subseteqq X$, such that card $(A) \leqq m$, A' is an m-fold union of closed set.*

PROOF. i $\to$ ii. $\{x\}' = \overline{\{x\}} - \{x\} = \overline{\{x\}} - (F \cap (\cap\, \{0_i : i \in I\}))$. Upon noting that $\overline{\{x\}} \subset F$ and applying the De Morgan laws, we obtain $\{x\}' = \cup\, \{X - 0_i : i \in I\}$.

ii $\to$ iii. Let $A = \{x_i : \in I\}$, where card $(I) \leqq m$. Let C denote the set of ω-limit points of A; clearly, C is closed. Thus,

$$A' = C \cup (\cup\, \{\{x_i\}' : x_i \in A\}),$$

for A' certainly contains the right hand side and if $x \in A' - C$, then $x \in \{x_i\}'$ for some $x_i \in A$.

iii $\to$ i. Obviously, iii $\cdot \to$ ii $\cdot$. And, if $\{x\}' = \cup\, \{F_i : i \in I\}$,

$$x = \overline{\{x\}} - \{x\}' = \overline{\{x\}} - \cup\, \{F_i : i \in I\} = \overline{\{x\}} \cap \{X - F_i : i \in I\}.$$

COROLLARY. *Each $T^{(m)}$ space is a T_0 space, and each T_0 space is a $T^{(m)}$ space for some m.*

THEOREM 4. *An m product of $T^{(m)}$ spaces is again a $T^{(m)}$ space.*

PROOF. Let $Y = \Pi\{X_i : i \in I\}$ be an m-product of $T^{(m)}$ spaces. Then, for each $x \in Y$, $x = \cap\, \{\pi_i^{-1}(\pi_i(x)) : i \in I\} = \cap\, \{\pi_i^{-1}[G_{m,i} \cap F_i] : i \in I\}$, where each F_i is closed in X_i and each $G_{m,i}$ is an intersection of m open sets in X_i. Thus, $x = \cap\, \{\pi_i^{-1}[G_{m,i}] : i \in I\} \cap \{\pi_i^{-1}[F_i] : i \in I\}$, and the continuity of each π_i yields the desired conclusion.

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THEOREM 5. *An n product of $T^{(m)}$ spaces (none of which is T_1) is again a $T^{(m)}$ space if and only if $n \leqq m$.*

PROOF. The sufficiency follows from the preceeding corollary, while the necessity follows from arguments analogous to those used in the proof of Theorem 1.

REMARK. We see thus that there is no $T^{(m)}$ space which inherits this property under arbitrary products with itself. This observation may be employed to construct, for any cardinal m, a $T^{(m)}$ space which is not a $T^{(k)}$ space for any $k \leqq m$; we need only take an m fold product of T_D spaces, none of which is T_1.

References

[1] C. E. Aull and W. J. Thron, Separation axioms between T_0 and T_1, *Indag. Math.* 23 (1962), 26–37

[2] W. J. Thron, *Topological Structures* (Holt, Rinehart, and Winston, New York, 1966).

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