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FORD AND DIRICHLET REGIONS FOR FUCHSIAN GROUPS

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1. Introduction. There has recently been some interest in a class of limit points for Fuchsian groups now known as Garnett points [**5**], [**8**]. In this paper we show that such points are intimately connected with the structure of Dirichlet regions and the same ideas serve to show that the Ford and Dirichlet regions are merely examples of one single construction which also yields fundamental regions based at limit points (and which properly lies in the subject of inversive geometry). We examine in the general case how the region varies continuously with the construction. Finally, we consider the linear measure of the set of Garnett points.

2. Hyperbolic space. Let Δ be any open disc (or half-plane) in the extended complex plane $\mathbf{C}_\infty$: usually Δ will be the unit disc or the upper half-plane. We may regard Δ as the hyperbolic plane in the usual way and the conformal isometries of Δ are simply the Moebius transformations of Δ onto itself.

Each conformal isometry g of Δ may be written as $g = \sigma_2\sigma_1$ where σ_j is the reflection in a geodesic L_j in Δ. This decomposition is not unique; however there is a one-parameter family F of geodesics to which the L_j belong and either L_1 or L_2 may be chosen arbitrarily from F; the other L_j then being uniquely determined. If g is elliptic or parabolic, then g has a unique fixed point in $\bar{\Delta}$ (the closure of Δ) and F is the class of geodesics through that point (if g is elliptic) or having that point as an end-point (if g is parabolic). If g is hyperbolic, F is the set of geodesics orthogonal to the axis of g.

In general, if σ is the reflection in L, then L is called the *axis* of σ.

3. A unique representation. We now select any ξ in $\bar{\Delta}$. For each isometry g not fixing ξ we can choose L_2 so that $\xi \in L_2$. This determines L_2 (and so L_1) uniquely and we now have a unique representation $g = \sigma_2\sigma_1$. We shall denote these choices of L_1 and L_2 by L_g and L_g^* respectively and also we use σ_g, σ_g^* for σ_1, σ_2.

It is geometrically obvious that as $g(\xi) \neq \xi$ we have $\xi \notin L_g$. For an analytic proof we need only observe that if $\xi \in L_g$ then

$$g(\xi) = \sigma_g^*(\sigma_g(\xi)) = \sigma_g^*(\xi) = \xi$$

which is false by assumption.

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As $\xi \notin L_g$ we can now define the hyperbolic half-planes H_g, I_g as those half-planes which are bounded by L_g and which contain and do not contain ξ respectively: thus $\xi \in H_g$, $\xi \notin I_g$.

4. Invariance under conjugation. It is an elementary fact that if h is any Moebius transformation mapping Δ onto Σ, then $h\phi h^{-1}$ is an isometry of Σ whenever ϕ is an isometry of Δ. In particular, if σ is the reflection in a geodesic L in Δ then $h\sigma h^{-1}$ is the reflection in the geodesic $h(L)$ in Σ. These facts lead immediately to the invariance of the above concepts under conjugation: explicitly

$$\sigma_{hgh^{-1}} = h\sigma_g h^{-1},$$
$$L_{hgh^{-1}} = h(L_g),$$
$$H_{hgh^{-1}} = h(H_g)$$

and similarly for $\sigma_g{}^*$, $L_g{}^*$ and I_g, where, of course, $\sigma_{hgh^{-1}}$ (for example) refers to the point $h(\xi)$ in the same way that σ_g refers to ξ.

5. Basic properties. The following result should seem familiar to any reader who has met the concept of isometric circles.

PROPOSITION 1. *In the above notation,*

(i) $\sigma_{g^{-1}}{}^* = \sigma_g{}^*$;
(ii) $g(L_g) = L_{g^{-1}}$;
(iii) $g(H_g) = I_{g^{-1}}$, $g(I_g) = H_{g^{-1}}$.

Proof. As every reflection is its own inverse we have

$$\sigma_{g^{-1}}{}^*\sigma_{g^{-1}} = g^{-1} = \sigma_g\sigma_g{}^* = \sigma_g{}^*(\sigma_g{}^*\sigma_g\sigma_g{}^*).$$

By the uniqueness of the decomposition this yields (i) together with the identity

$$\sigma_{g^{-1}} = \sigma_g{}^*\sigma_g\sigma_g{}^*.$$

This identity shows that the axis of $\sigma_{g^{-1}}$ is precisely $\sigma_g{}^*(L_g)$ and this proves (ii) as

$$g(L_g) = \sigma_g{}^*\sigma_g(L_g) = \sigma_g{}^*(L_g).$$

Finally, (ii) shows that the pair of half-planes $g(H_g)$, $g(I_g)$ is the same as the pair of half-planes $H_{g^{-1}}$, $I_{g^{-1}}$. As

$$\xi = \sigma_g{}^*(\xi) \notin \sigma_g{}^*\sigma_g(H_g) = g(H_g)$$

we see that (iii) follows.

In order to relate these ideas to the more familiar ideas concerning Ford and Dirichlet regions, we now identify L_g in certain cases.

PROPOSITION 2. (i) *If* $\xi \in \Delta$, *then* L_g *is the hyperbolic bisector of* ξ *and* $g^{-1}(\xi)$.

(ii) *If* $\xi = \infty$, *then* L_g *is part of the isometric circle of* g.

Proof. To prove (i) observe first that $\sigma_g^*(\xi) = \xi$. Next, if $z \in L_g$ then $\sigma_g(z) = z$ and so (using ρ for the hyperbolic metric)

$$\rho(z, g^{-1}(\xi)) = \rho(z, \sigma_g\sigma_g^*(\xi)) = \rho(z, \sigma_g(\xi)) = \rho(\sigma_g(z), \xi) = \rho(z, \xi).$$

This proves (i).

To prove (ii), observe first that when $\xi = \infty$, σ_g^* is a Euclidean reflection, hence a Euclidean isometry. As σ_g acts as a Euclidean isometry (indeed, it is the identity) on L_g, we see that g acts as a Euclidean isometry on L_g. Then $|g'(z)| = 1$ on L_g and (ii) follows.

Remark 1. If Δ is the unit disc, and $\xi = 0$ then every extended L_g^* also goes through ∞ and so the construction of the L_g^* for $\xi = 0$ and $\xi = \infty$ yield the same line. This shows that the isometric circle of g coincides with the bisector of 0 and $g^{-1}(0)$ and so gives a geometric insight into this well known fact [**4**, p. 151].

Remark 2. It is a simple matter to compute the equation of L_g when Δ is the unit disc. If $\xi \in \Delta$, the equation is

$$|z - g(\xi)|^2 \, |z - \xi|^{-2} = |g'(\xi)|$$

and (by continuity considerations or by taking conjugates) this remains true when $|\xi| = 1$. We shall not need this formula and we omit the proof.

In view of Proposition 2, it is natural to consider a Fuchsian group Γ acting on Δ and to write

$$D_\xi = \bigcap_{g \in \Gamma, g \neq I} H_g$$

under the assumption that no non-trivial element of Γ fixes ξ. Indeed, from the standard theory of Fuchsian groups, we know that D_ξ is a fundamental polygon for Γ when $\xi \in \Delta$ (D_ξ is then the Dirichlet region with center ξ) or when $\xi \in \partial\Delta$ and is an ordinary point of Γ (D_ξ is then the image of the Ford polygon for a conjugate group). We now make an observation which will be used repeatedly throughout the paper. If Γ acts in the upper half-plane Δ and $w \in \Delta$, we say that w is of *strictly maximal height* if for every $g \in \Gamma$, $g \neq I$ we have $\operatorname{Im} g(w) < \operatorname{Im}(w)$. We say that w is of *maximal height* if for every $g \in \Gamma$ we have $\operatorname{Im} g(w) \leqq \operatorname{Im}(w)$.

PROPOSITION 3. *If* Γ *is a Fuchsian group in the upper half-plane* Δ *then*

(i) $w \in \Delta$ *is of strictly maximal height if and only if* $w \in D_\infty$;

(ii) $w \in \Delta$ *is of maximal height if and only if* $w \in \bar{D}_\infty$.

Proof. Part (i) follows easily from Proposition 2 and the fact that for a Moebius transform g preserving Δ

$$\operatorname{Im} g(z) = |g'(z)|^2 \operatorname{Im} z.$$

To prove part (ii) suppose first that $w \in \Delta$ is of maximal height and let L denote the open half-geodesic joining w to ∞. No isometric circle of a transform in Γ can meet L (otherwise w is interior to an isometric circle and so has an image of greater height) thus $L \subset D_\infty$ and w $\in \bar{D}_\infty$. Conversely, if w is not of maximal height then for some $g \in \Gamma$, $\operatorname{Im} g(w) > \operatorname{Im}(w)$ and from our comments above we see that $|g'(w)| > 1$. From Proposition 2 we see that $w \in I_g$. As I_g is open and $\bar{D}_\infty \subset \bar{H}_g = H_g \cup L_g$ we see that $w \notin \bar{D}_\infty$.

We aim to consider the continuity of D_ξ as a function of ξ and also to characterize those limit points ξ for which D_ξ is a fundamental region.

6. Continuity. We recall the definition of convergence of a sequence of sets $\{A_i\}$. Defining

$$\liminf A_i = \bigcup_{i=1}^{\infty} \bigcap_{k \geqq i} A_k \quad \text{and} \quad \limsup A_i = \bigcap_{i=1}^{\infty} \bigcup_{k \geqq i} A_k$$

we say that $\{A_i\}$ converges to A if $\liminf A_i = \limsup A_i = A$. Given a sequence ξ_n in Δ which converges to ξ in $\bar{\Delta}$ we ask whether the regions D_{ξ_n} converge to D_ξ. If $\xi \in \Delta$ this result is almost trivial to prove. In the case where $\xi \in \partial\Delta$ we have the following result.

THEOREM 1. *Let Γ be a Fuchsian group acting in Δ and $\xi \in \partial\Delta$. If ξ is not fixed by any element of Γ (except the identity) and if ξ_n is a sequence of points converging to ξ in a cusp then*

$$D_\xi \subset \liminf D_{\xi_n} \subset \limsup D_{\xi_n} \subset \bar{D}_\xi.$$

Proof. By conjugation we assume Δ is the upper half-plane and $\xi = \infty$. Thus ξ_n converge to ∞ in some strip $\{z: a < \operatorname{Re} z < b\}$. Now choose w in Δ and define

$$A_n(w) = \{z \in \Delta: \rho(z, \xi_n) < \rho(w, \xi_n)\}$$

where ρ is the non-euclidean metric in Δ; note that $A_n(w)$ is a disc in Δ. We note that

$$A_n(w) = \left\{z \in \Delta: \frac{|z - \xi_n|^2}{\operatorname{Im} z} < \frac{|w - \xi_n|^2}{\operatorname{Im} w}\right\}.$$

If $z \in \limsup A_n(w)$ then, for infinitely many n,

$$\frac{|z - \xi_n|^2}{\operatorname{Im} z} < \frac{|w - \xi_n|^2}{\operatorname{Im} w}$$

and, since ξ_n converges to ∞ in a cusp, we see that $\operatorname{Im} w \leqq \operatorname{Im} z$. Similarly, if $\operatorname{Im} w < \operatorname{Im} z$, then for all n large enough

$$\frac{|z - \xi_n|^2}{\operatorname{Im} z} < \frac{|w - \xi_n|^2}{\operatorname{Im} w}$$

and so $z \in \liminf A_n(w)$. We have proved

$$(1) \quad \{z: \operatorname{Im} z > \operatorname{Im} w\} \subset \liminf A_n(w) \subset \limsup A_n(w) \subset \{z: \operatorname{Im} z \geqq \operatorname{Im} w\}.$$

Suppose that $w \notin \bar{D}_\infty$. Then for some $g \in \Gamma$, $\operatorname{Im} g(w) > \operatorname{Im} w$ and so from (1),

$$g(w) \in \liminf A_n(w).$$

As D_{ξ_n} is the Dirichlet region centered at ξ_n we see that $w \notin D_{\xi_n}$ for all $n \geqq n(w)$ say and hence that $w \notin \limsup D_{\xi_n}$. This shows that

$$\limsup D_{\xi_n} \subset \bar{D}_\infty.$$

To complete the proof we suppose $w \in D_\infty$ and so w is of strictly maximal height. If $w \notin \liminf D_{\xi_n}$ then, for infinitely many n, $w \notin D_{\xi_n}$ and there exists a sequence g_n of elements of Γ with $g_n(w) \in \overline{A_n(w)}$. However, since

$$\{z: 0 < \operatorname{Im} z < \operatorname{Im} w\} \cap \left(\bigcup_{n=1}^{\infty} \overline{A_n(w)}\right)$$

is a compact subset of Δ it can contain only finitely many Γ images of w. Thus the sequence g_n contains only finitely many distinct transforms. We see then that for some $g \in \Gamma$, $g \neq I$, $g(w) \in \overline{A_n(w)}$ for infinitely many n and we obtain a contradiction with (1) since $\operatorname{Im} g(w) < \operatorname{Im} w$. We have shown that

$$D_\infty \subset \liminf D_{\xi_n}$$

and this completes the proof of Theorem 1.

We conclude this section with an example to show that Theorem 1 is essentially best possible. For each positive integer n let $r_n = (n!)^2$ and set

$$x_n = 2 \sum_{i=1}^{n-1} r_i + r_n.$$

Let C_n be the circle of radius r_n centered at x_n on the real axis and let C_n' be the reflection of C_n across the imaginary axis. Let P be a parabolic transform preserving the upper half-plane with isometric circles C_1, C_1'. For each $n \geqq 2$ let H_n be a hyperbolic transform preserving the upper half-plane with isometric circles C_n, C_n'. Now Γ given by

$$\Gamma = \langle P, H_2, H_3, \ldots \rangle$$

is a Fuchsian group preserving the upper half-plane. This group was considered in [**5**] where it was shown that i is the unique point of maximal height in its orbit and that, for $n \geqq 2$,

(2) $\quad 1 > \operatorname{Im} H_n(i) > (1 + 2/n)^{-2}.$

Now D_∞ is the open region in the upper half-plane exterior to all the circles C_n, C_n' and clearly contains i. We set, for $n \geqq 2$,

$$w_n = \operatorname{Re} H_n(i) + iy_n$$

where y_n is chosen so that $y_n/\operatorname{Re} H_n(i) \to +\infty$. Thus the points w_n approach ∞ in arbitrarily thin Stolz angles. An easy calculation shows that for n large enough

$$\rho(w_n, i) > \rho(w_n, V_n(i))$$

so i is exterior to the Dirichlet region D_{w_n} for n large enough. Thus, in this example, D_∞ is not the limit of D_{w_n}.

7. Garnett points. The open hyperbolic disc with center w in Δ and radius r is

$$D(w, r) = \{z \in \Delta: \rho(z, w) < r\}.$$

We shall now extend this terminology to include cases where $w \in \partial\Delta$.

A *horocyclic region* Q in Δ is an open Euclidean disc contained in Δ and internally tangent to $\partial\Delta$ at some point η in $\partial\Delta$: we say Q is *based at* η. The Euclidean circle ∂Q is called a *horocycle* at η and it has equation

$$P(z, \eta) = k$$

where $P(z, \eta)$ is the Poisson kernel for Δ and k is some positive constant.

When $\eta \in \partial\Delta$, we use $D(\eta, r)$ to denote the set $Q \cup \{\eta\}$ where Q is the horocyclic region based at η and given by $P(z, \eta) > r^{-1}$.

For the remainder of this section we will assume that Γ is a Fuchsian group acting on Δ, that $\xi \in \bar{\Delta}$ and that $g(\xi) \neq \xi$ whenever $g \in \Gamma$ and $g \neq I$. We denote the set of elliptic fixed points of Γ by E and the set of parabolic fixed points of Γ by P: thus

$$\xi \notin P \cup E.$$

We now distinguish certain types of limit points of Γ.

Definition. (i) We say that w in $\partial\Delta$ is a *horocyclic limit point for* Γ if for some (and hence all) a in Δ, the orbit $\Gamma(a)$ of a meets every horocyclic region based at w.

(ii) We say that w in $\partial\Delta$ is a *Garnett point for* Γ if there is some $a \in \Delta$ and some positive r such that $\bar{D}(w, r)$ does not meet $\Gamma(a)$ whereas for every positive ϵ, $D(w, r + \epsilon)$ does meet $\Gamma(a)$.

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We denote the set of horocyclic limit points by H and the set of Garnett points by G. The existence of Garnett points has recently been established by Nicholls [**5**]. If Γ is a finitely generated group then $G = \emptyset$ and the limit set is the disjoint union of P and H [**2**].

8. Fundamental polygons. In this section we characterize those points ξ for which D_ξ is a fundamental region. Throughout we assume that Γ is a Fuchsian group acting on Δ and $\xi \in \bar{\Delta}$. To begin with we record the following trivial result.

PROPOSITION 4. *Let Γ be a Fuchsian group acting in Δ. Then $D_\xi \cap g(D_\xi) = \emptyset$ whenever $g \in \Gamma$, $g(\xi) \neq \xi$. Further, D_ξ is convex.*

Proof. As D_ξ is an intersection of half-planes, it is convex. If $g \in \Gamma$ and $g \neq I$ then, by Proposition 1,

$$g(D_\xi) \cap D_\xi \subset g(H_g) \cap H_{g^{-1}} = I_{g^{-1}} \cap H_{g^{-1}} = \emptyset.$$

With the notation of the previous section we state our theorem.

THEOREM 2. *The interior of D_ξ is a convex fundamental region for Γ if and only if $\xi \notin H \cup G \cup P \cup E$.*

Proof. Suppose first that $\xi \in H \cup G \cup P \cup E$. If $\xi \in P \cup E$ then D_ξ is not defined and so cannot be a fundamental region. In the other cases, $\xi \in \partial\Delta$ and by conjugation we may assume that Δ is the upper half-plane and that $\xi = \infty$. Thus we assume that $\infty \in H \cup G$ and that no element of Γ (except the identity) fixes ∞. Suppose next that $D_\infty \neq \emptyset$; any $w \in D_\infty$ is of strictly maximal height and so $\infty \notin H$. Thus if $D_\infty \neq \emptyset$ then $\infty \notin H$ and we have only to consider the case $\infty \in G$. In this case there is a point a in Δ and a positive real r such that

$$\{z: \operatorname{Im} z \geqq r\} \cap \Gamma(a) = \emptyset$$

and, for any $\delta > 0$,

$$\{z: \operatorname{Im} z > r - \delta\} \cap \Gamma(a) \neq \emptyset.$$

This implies that no point in $\Gamma(a)$ is of maximal height, hence $\Gamma(a) \cap \bar{D}_\infty$ is empty and so $\Gamma(a)$ does not meet $\overline{\operatorname{Int} D_\infty}$. Thus in all cases $\operatorname{Int} D_\infty$ is not a fundamental region.

Now suppose that $\xi \notin H \cup P \cup G \cup E$. The set D_ξ is defined and, in view of Proposition 4 we have only to show that each z in Δ has an image in $\overline{\operatorname{Int} D_\xi}$. The argument is well known when $\xi \in \Delta$ (this is simply the case of the Dirichlet region) and we omit the details: even so, the following argument for the case $\xi \in \partial\Delta$ can easily be modified to cover the case $\xi \in \Delta$ (independently of the well known method).

We suppose then that $\xi \in \partial\Delta$ and, as before, we may assume that Δ is the upper half-plane and that $\xi = \infty$. The assumption that $\xi \notin P \cup H$

implies that no element of Γ fixes ∞ (except the identity) so every non-trivial g in Γ possesses a corresponding geodesic L_g which is a Euclidean semi-circle with center on the real axis (the isometric circle of g). As $\xi \notin H \cup G$, for each z in Δ there is a maximal closed disc $\bar{D}(\infty, r)$ such that z has an equivalent point z' satisfying

$$z' \in \partial\bar{D}(\infty, r),\ \Gamma(z) \cap D(\infty, r) = \emptyset.$$

Thus z' is of maximal height.

Now select two distinct orbits and select from them representatives, say z_1, z_2 of maximal height. We see, as in the proof of Proposition 3 that no L_g can meet the half-geodesic joining ∞ to z_1 or the half-geodesic joining ∞ to z_2. We deduce from elementary geometry that no L_g can penetrate the interior of the triangle with vertices ∞, z_1, z_2. This interior is therefore a subset of Int D_∞ and, in particular, $z_1 \in \overline{\text{Int } D_\infty}$. As this is true for any z_1 we have shown that any z in Δ has a representative in the closure of the interior of D_ξ and the proof is complete.

We conclude this section with several remarks.

Remark 1. If $\xi \notin H \cup G \cup P \cup E$ then the interior of D_ξ is a convex fundamental region and its boundary is polygonal. This is well known when $\xi \in \Delta$ and in the case when $\xi \in \partial\Delta$ it may be deduced from the methods developed by Nicholls and Zarrow [**6**, Lemma 1 and Theorem 1].

Remark 2. The set D_ξ may be empty and we can say precisely when this will happen. Recalling the definition of a Garnett point for a Fuchsian group Γ, we say that w in $\partial\Delta$ is a *universal Garnett point* (written $w \in G^*$) if for every $a \in \Delta$ there exists a positive r such that $\bar{D}(w, r)$ does not meet $\Gamma(a)$ whereas for every positive ϵ, $D(w, r + \epsilon)$ does meet $\Gamma(a)$. The set D_ξ is empty if and only if $\xi \in H \cup G^* \cup P \cup E$. The sufficiency of this condition follows immediately from the definition of D_ξ. Now suppose

$$\xi \notin H \cup G^* \cup P \cup E.$$

If $\xi \in \Delta$ then D_ξ is a Dirichlet region and consequently non-empty. Thus we suppose that $\xi \in \partial\Delta$ and we conjugate ξ to infinity. Since $\infty \notin H \cup G^* \cup P \cup E$ we see that there exists a point of Δ, say a, which is of maximal height. It follows immediately from Proposition 3 that D_∞ is not empty.

Remark 3. We observe that our construction yields new examples of convex fundamental regions (apart from the Dirichlet and Ford regions). See, for example, the region D_∞ for the group Γ mentioned at the end of Section 6.

Remark 4. The region D_ξ need not be open or locally finite (group images of D_ξ may accumulate in Δ). To see this consider the group Γ

defined at the end of Section 6. We may find a proper subgroup Γ_1 of Γ for which $\infty \in G$ and the region D_∞ for Γ_1 is the closure of the region D_∞ for Γ. The Γ_1 images of D_∞ accumulate in Δ (see [**5**] for more details).

Note that in this example the interior of D_∞ is not a fundamental region for Γ_1. It may be the case that if the interior of D_ξ is a fundamental region for a group Γ then D_ξ will always be open and locally finite. However we are unable to prove this.

Remark 5. Observe that the family

$$\{D(w, r): w \in \bar{\Delta}, r \geqq 0\}$$

is a base for a Hausdorff topology on $\bar{\Delta}$ and the induced topology on Δ is the Euclidean (and hyperbolic) topology. With this in mind we see that for $\xi \in \partial\Delta$ the region D_ξ has a natural interpretation as a Dirichlet region centered at ξ.

9. The measure of G. It is clearly desirable to have some information on the size of the set G and we conclude that paper by showing that G has zero linear measure. This result is implicit both in the work of Pommerenke [**7**] and of Sullivan [**8**]. We will follow Pommerenke's approach.

THEOREM 3. *Let* Γ *be a Fuchsian group acting in* Δ *then* $m(G) = 0$.

Proof. Following Pommerenke [**7**] we define Q, the set of *oricyclic points* for Γ, to be the set of points of $\partial\Delta$ with the property that every horocyclic region at the point contains only finitely many Γ images of some point (and hence all points) of Δ. We see from this definition that

(3) $\quad (\partial\Delta \backslash G) \supset H \cup Q.$

If $w \in \Delta$ we define

$$E_w = \{z \in \partial\Delta: V(z) \in \partial D_w \text{ for some } V \in \Gamma\}.$$

We need one more definition. If $\xi \in \partial\Delta$ and $w \in \Delta$ we say that ξ is a *Garnett point with respect to the orbit of* w if there is a positive r so that $\bar{D}(\xi, r)$ does not meet $\Gamma(w)$ whereas for every positive ϵ, $D(\xi, r + \epsilon)$ does meet $\Gamma(w)$. We denote the set of such points ξ by g_w and note that

$$G = \bigcup_{w \in \Delta} g_w.$$

Nicholls [**5**] has shown that, for any $w \in \Delta$,

$$\partial\Delta = H \cup E_w \cup g_w$$

and Sullivan [**8**, Section 4] has shown that, for any $w \in \Delta$

$$m(g_w) = 0.$$

Thus, for any w in Δ, $H \cup E_w$ is a set of full measure on $\partial\Delta$. We now use Pommerenke's result [**7**, Theorem 1] that $m(Q) = m(E_w)$ for any w in Δ to deduce that $H \cup Q$ is a set of full measure on $\partial\Delta$. In view of (2) the proof of Theorem 3 is complete.

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