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Resultants of Chebyshev Polynomials: the First, Second, Third, and Fourth Kinds

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Abstract. We give an explicit formula for the resultant of Chebyshev polynomials of the first, second, third, and fourth kinds. We also compute the resultant of modified cyclotomic polynomials.

1 Introduction

In [4], Jacobs, Rayes, and Trevisan obtained explicit formulas for the resultants of Chebyshev polynomials of the first and second kinds, and Louboutin gave a short proof in [8]. As there are four (first, second, third, and fourth) kinds of Chebyshev polynomials, it is the purpose of this note to compute the resultant of two Chebyshev polynomials of any kinds. It is intriguing to notice that the Jacobi symbol is involved in the result. For the proof, we use the roots of Chebyshev polynomials, basic properties of sine and cosine values, and basic properties of the Jacobi symbol, including the reciprocity law. When restricted to the first or second kinds, our proof is different from both [4] and [8]. As an application, we also compute the resultant of modified cyclotomic polynomials. Our result is a refinement of a well-known formula due to Diederichsen [2] (see also [1,3,5,7,9]) for the resultants of cyclotomic polynomials.

2 Resultant of Chebyshev Polynomials

The Chebyshev polynomials T_n, U_n, V_n, and W_n of the first, second, third, and fourth kind, respectively, are characterized by

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$
$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, \qquad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin\theta/2},$$

where n is an integer (*cf.* [11,12]). The normalized Chebyshev polynomials of the first and second kinds are defined by $C_n(x) = 2T_n(x/2), S_n(x) = U_n(x/2)$. We adopt Schur's notation $\mathscr{S}_n(x) = S_{n-1}(x)$. For odd n we define

$$\mathscr{V}_n(x) = V_{(n-1)/2}(x/2), \quad \mathscr{W}_n(x) = W_{(n-1)/2}(x/2).$$

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In this note, we always assume odd indices for the polynomials $\mathscr{V}_n$ and $\mathscr{W}_n$. For $n \geq 1$ we have

$$C_n(x) = \prod_{\substack{0<j<2n,\\ j:\text{odd}}} \left(x - 2\cos\frac{j\pi}{2n}\right), \qquad \mathscr{S}_n(x) = \prod_{\substack{0<j<2n,\\ j:\text{even}}} \left(x - 2\cos\frac{j\pi}{2n}\right),$$

$$\mathscr{V}_n(x) = \prod_{\substack{0<j<n,\\ j:\text{odd}}} \left(x - 2\cos\frac{j\pi}{n}\right), \qquad \mathscr{W}_n(x) = \prod_{\substack{0<j<n,\\ j:\text{even}}} \left(x - 2\cos\frac{j\pi}{n}\right).$$

We will use the following identities:

(2.1) $$W_n(-x) = (-1)^n V_n(x),$$

(2.2) $$\mathscr{W}_m(C_n(x)) = \begin{cases} \mathscr{W}_{mn}(x)/\mathscr{W}_n(x) & \text{if } n \text{ is odd,} \\ \mathscr{S}_{mn/2}(x)/\mathscr{S}_{n/2}(x) & \text{if } n \text{ is even.} \end{cases}$$

Let $\operatorname{res}(f,g)$ denote the resultant of two polynomials f and g. For the definition and properties of the resultant, see [4]. In particluar, we note that

(2.3) $$\operatorname{res}(g,f) = (-1)^{\deg(f)\deg(g)} \operatorname{res}(f,g).$$

In addition to those properties listed in [4], we also quote the following from [10]. If h is a polynomial with leading coefficient c, then

(2.4) $$\operatorname{res}\big(f(h(x)), g\big((h(x))\big)\big) = c^{\deg(f)\deg(g)\deg(h)} \operatorname{res}(f,g)^{\deg(h)}.$$

Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorization of a positive odd integer n, and a an integer. The Jacobi symbol is defined by

(2.5) $$\left(\frac{a}{n}\right) = \prod_{i=1}^r \left(\frac{a}{p_i}\right)^{e_i},$$

where (a/p_i) is the Legendre symbol. For a prime p and an integer n we write $\operatorname{ord}_p(n) = k$ when p^k is the highest power of p dividing n.

Theorem 2.1 *Let m, n be positive integers and let $g = \gcd(m,n)$.*

(i) $\operatorname{res}(C_m, C_n) = \begin{cases} (-1)^{mn/2} 2^g & \textit{if } \operatorname{ord}_2(m) \neq \operatorname{ord}_2(n), \\ 0 & \textit{if } \operatorname{ord}_2(m) = \operatorname{ord}_2(n). \end{cases}$

(ii) $\operatorname{res}(\mathscr{S}_m, \mathscr{S}_n) = \begin{cases} (-1)^{(m-1)(n-1)/2} & \textit{if } g = 1, \\ 0 & \textit{if } g > 1. \end{cases}$

(iii) $\operatorname{res}(\mathscr{V}_m, \mathscr{V}_n) = \left(\frac{m}{n}\right)$.

(iv) $\operatorname{res}(\mathscr{W}_m, \mathscr{W}_n) = \left(\frac{n}{m}\right)$.

(v) $\operatorname{res}(C_m, \mathscr{S}_n) = \begin{cases} (-1)^{m(n-1)/2} 2^{g-1} & \textit{if } \operatorname{ord}_2(m) \geq \operatorname{ord}_2(n), \\ 0 & \textit{if } \operatorname{ord}_2(m) < \operatorname{ord}_2(n). \end{cases}$

(vi) $\operatorname{res}(C_m, \mathscr{V}_n) = \operatorname{res}(\mathscr{W}_n, C_m) = \left(\frac{2}{n/g}\right) 2^{(g-1)/2}$.

(vii) $\operatorname{res}(\mathscr{S}_m, \mathscr{V}_n) = \operatorname{res}(\mathscr{W}_n, \mathscr{S}_m) = \left(\frac{m}{n}\right)$.

(viii) $\operatorname{res}(\mathscr{W}_m, \mathscr{V}_n) = \left(\frac{2}{g}\right) 2^{(g-1)/2}$.

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Proof For positive integers n and a we introduce the following products:

$$S(n,a) = \prod_{0<j<n} 2\sin\frac{aj\pi}{n}, \qquad C(n,a) = \prod_{0<j<n} 2\cos\frac{aj\pi}{n},$$
$$s(n,a) = \prod_{0<j<n/2} 2\sin\frac{aj\pi}{n}, \qquad c(n,a) = \prod_{0<j<n/2} 2\cos\frac{aj\pi}{n}.$$

Then we have

$$\operatorname{res}(\mathscr{S}_m,\mathscr{S}_n) = \prod_{0<j<m} \mathscr{S}_n\Big(2\cos\frac{j\pi}{m}\Big) = \frac{S(m,n)}{S(m,1)},$$
$$\operatorname{res}(\mathscr{S}_m,C_n) = \prod_{0<j<m} C_n\Big(2\cos\frac{j\pi}{m}\Big) = C(m,n),$$
$$\operatorname{res}(\mathscr{W}_m,\mathscr{W}_n) = \prod_{\substack{0<j<m,\\ j:\text{even}}} \mathscr{W}_n\Big(2\cos\frac{j\pi}{m}\Big) = \frac{s(m,n)}{s(m,1)},$$
$$\operatorname{res}(\mathscr{W}_m,C_n) = \prod_{\substack{0<j<m,\\ j:\text{even}}} C_n\Big(2\cos\frac{j\pi}{m}\Big) = c(m,2n),$$
$$\operatorname{res}(\mathscr{W}_m,\mathscr{S}_n) = \prod_{\substack{0<j<m,\\ j:\text{even}}} \mathscr{S}_n\Big(2\cos\frac{j\pi}{m}\Big) = \frac{s(m,2n)}{s(m,1)},$$
$$\operatorname{res}(\mathscr{W}_m,\mathscr{V}_n) = \prod_{\substack{0<j<m,\\ j:\text{even}}} \mathscr{V}_n\Big(2\cos\frac{j\pi}{m}\Big) = \frac{c(m,n)}{c(m,1)}.$$

Note that the denominators are nonzero. For these resultants the results are immediate from Lemma 2.3. Then changing the sign of x in (iv) and using (2.1), (2.4), and the reciprocity law of the Jacobi symbol, we obtain (iii). This also applies to the first equalities in (vi) and (vii). Finally,

$$\operatorname{res}(C_m,C_n) = \prod_{\substack{0<j<2m,\\ j:\text{odd}}} C_n\Big(2\cos\frac{j\pi}{2m}\Big) = \prod_{\substack{0<j<2m,\\ j:\text{odd}}} 2\cos\frac{nj\pi}{2m}$$

is nonzero if and only if $\operatorname{ord}_2(m) \neq \operatorname{ord}_2(n)$. If this is the case, then

$$\operatorname{res}(C_m,C_n) = \frac{C(2m,n)}{C(m,n)},$$

and (i) follows from Lemma 2.3. ∎

Remark 2.2 In [4,8], explicit formulas for $\operatorname{res}(T_m,T_n)$ and $\operatorname{res}(U_m,U_n)$ were obtained. They are identical to our (i) and (ii), respectively, in view of (2.4).

Lemma 2.3 *Let n, a be positive integers and $g = \gcd(n,a)$.*

(i) $$S(n,a) = \begin{cases} (-1)^{(n-1)(a-1)/2}n & \text{if } g = 1,\\ 0 & \text{otherwise.}\end{cases}$$

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(ii) $C(n,a) = \begin{cases} (-1)^{(n-1)a/2} 2^{g-1} & \textit{if } \operatorname{ord}_2(n) \le \operatorname{ord}_2(a), \\ 0 & \textit{otherwise.} \end{cases}$

(iii) *If n is odd, then*

$$s(n,a) = \begin{cases} \left(\frac{a}{n}\right)\sqrt{n} & \textit{if } a \textit{ is odd,} \\ \left(\frac{a/2}{n}\right)\sqrt{n} & \textit{if } a \textit{ is even.} \end{cases}$$

(iv) *If n is odd, then*

$$c(n,a) = \begin{cases} \left(\frac{2}{g}\right) 2^{(g-1)/2} & \textit{if } a \textit{ is odd,} \\ \left(\frac{2}{n/g}\right) 2^{(g-1)/2} & \textit{if } a \textit{ is even.} \end{cases}$$

Proof We put $n' = n/g$, $a' = a/g$.

(i) If $g > 1$, then $S(n,a) = 0$, since $\sin(an'\pi/n) = 0$. Suppose $g = 1$ and let $\zeta_N = e^{2\pi i/N}$ (N is a positive integer). Since $2\sin(aj\pi/n) = i\zeta_{2n}^{-aj}(1-\zeta_n^{aj})$, we have

$$S(n,a) = i^{n-1}\zeta_{2n}^{-an(n-1)/2}\prod_{j=1}^{n-1}\left(1-\zeta_n^j\right) = (-1)^{(n-1)(a-1)/2} n.$$

(ii) If $\operatorname{ord}_2(n) > \operatorname{ord}_2(a)$, then $C(n,a) = 0$, since $\cos(a(n'/2)\pi/n) = 0$. Suppose $\operatorname{ord}_2(n) \le \operatorname{ord}_2(a)$; in particular, n' is odd. If $g = 1$, then by (i) we have $S(n,a) \ne 0$, $S(n,2a) \ne 0$, and

$$C(n,a) = \frac{S(n,2a)}{S(n,a)} = (-1)^{(n-1)a/2}.$$

If $g > 1$, then

$$\begin{aligned} C(n,a) &= \prod_{k=0}^{g-1}\prod_{j=1}^{n'-1} 2\cos\frac{a(kn'+j)\pi}{n} \times \prod_{k=1}^{g-1} 2\cos\frac{a(kn')\pi}{n} \\ &= \prod_{k=0}^{g-1}\prod_{j=1}^{n'-1}(-1)^{a'k} 2\cos\frac{a'j\pi}{n'} \times \prod_{k=1}^{g-1} 2(-1)^{a'k} \\ &= (-1)^{(n-n')a/2} 2^{g-1} C(n',a')^g, \end{aligned}$$

and we are reduced to the case $g = 1$.

(iii) If $g > 2$, then $S(n,a) = 0$ since $\sin(an'\pi/n) = 0$. Suppose $g = 1$. It follows from (i) and the identity $S(n,a) = (-1)^{(n-1)(a-1)/2} s(n,a)^2$ that $|s(n,a)| = \sqrt{n}$. By Gauss' Lemma, the sign of $s(a,n)$ is equal to $\left(\frac{a/2}{n}\right)$ if a is even (*cf.* [6, Proposition 8.1]). If a is odd, then, counting the number of odd j's in the interval $0 < j < n/2$, we see that $s(n,a) = \left(\frac{2}{n}\right) s(n,a+n)$, so the sign of $s(n,a)$ is $\left(\frac{a}{n}\right)$.

(iv) If $g = 1$, then by (iii) we have $s(n,a) \ne 0$ and

$$c(n,a) = \frac{s(n,2a)}{s(n,a)} = \begin{cases} 1 & \text{if } a \text{ is odd,} \\ \left(\frac{2}{n}\right) & \text{if } a \text{ is even.} \end{cases}$$

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Suppose $g > 1$. We compute

$$\begin{aligned}
c(n,a) &= \prod_{k=0}^{(g-1)/2} \prod_{0<j<n'/2} 2\cos\frac{a(kn'+j)\pi}{n} \\
&\quad \times \prod_{k=1}^{(g-1)/2} \prod_{0<j<n'/2} 2\cos\frac{a(kn'-j)\pi}{n} \times \prod_{k=1}^{(g-1)/2} 2\cos\frac{a(kn')\pi}{n} \\
&= \prod_{k=0}^{(g-1)/2} \prod_{0<j<n'/2} (-1)^{a'k} 2\cos\frac{a' j\pi}{n'} \\
&\quad \times \prod_{k=1}^{(g-1)/2} \prod_{0<j<n'/2} (-1)^{a'k} 2\cos\frac{a' j\pi}{n'} \times \prod_{k=1}^{(g-1)/2} 2(-1)^{a'k} \\
&= \varepsilon 2^{(g-1)/2} c(n',a')^g,
\end{aligned}$$

where $\varepsilon = 1$ if a' is even, and $\varepsilon = (-1)^{(g^2-1)/8} = \left(\frac{2}{g}\right)$ if a' is odd. Thus we are reduced to the case $g = 1$. ∎

3 Resultant of Modified Cyclotomic Polynomials

For $n \geq 3$ let Ψ_n denote the minimal polynomial of $2\cos(2\pi/n)$ over $\mathbb{Q}$. Then $\Psi_n(x) \in \mathbb{Z}[x]$ and $\deg(\Psi_n) = \phi(n)/2$. These are what we called the modified cyclotomic polynomials in the introduction, as we have the identity

$$\Psi_n(x + x^{-1}) = x^{-\phi(n)/2}\Phi_n(x), \tag{3.1}$$

where Φ_n is the n-th cyclotomic polynomial. Here are some properties of Ψ_n. For n odd we have

$$\Psi_{2n}(x) = \Psi_n(-x). \tag{3.2}$$

By [13, Proposition 2.5] we have

$$\Psi_n(x) = \begin{cases} \prod_{d\mid n} \mathscr{W}_d(x)^{\mu(n/d)} & \text{if } n \equiv 1 \pmod 2, \\ \prod_{d\mid n/2} \mathscr{V}_d(x)^{\mu(n/2d)} & \text{if } n \equiv 2 \pmod 4, \\ \prod_{d\mid n/2} \mathscr{S}_d(x)^{\mu(n/2d)} & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{3.3}$$

Combining this with (2.2), for p a prime we have

$$\Psi_n(C_p(x)) = \begin{cases} \Psi_{pn}(x)\Psi_n(x) & \text{if } p \nmid n, \\ \Psi_{pn}(x) & \text{if } p \mid n. \end{cases} \tag{3.4}$$

We need to generalize the Jacobi symbol to the Kronecker symbol; it is defined by (2.5) for any positive integer n by requiring further that

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod 2, \\ 1 & \text{if } a \equiv \pm 1 \pmod 8, \\ -1 & \text{if } a \equiv \pm 3 \pmod 8. \end{cases}$$

We introduce one more notation. For a positive integer n let $L(n) = p$ if n is a power of some prime p, and $L(n) = 1$ otherwise. The notation $\Lambda(n) = \log L(n)$ is

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often used in analytic number theory. We note the following identity:

$$L(n) = \prod_{d|n} d^{\mu(n/d)}. \tag{3.5}$$

Lemma 3.1 *Let $n \geq 3$.*

(i) $\Psi_n(2) = L(n)$.

(ii) *Except for $\Psi_4(-2) = -2$, we have*

$$\Psi_n(-2) = \begin{cases} \left(\frac{-1}{L(n)}\right) & \text{if } n \text{ is odd,} \\ \left(\frac{-1}{L(n/2)}\right) L(n/2) & \text{if } n \text{ is even, } n > 4. \end{cases}$$

(iii) *Except for $\Psi_4(0) = 0, \Psi_8(0) = -2$, we have*

$$\Psi_n(0) = \begin{cases} \left(\frac{-2}{L(n)}\right) & \text{if } n \equiv 1 \pmod 2, \\ \left(\frac{2}{L(n/2)}\right) & \text{if } n \equiv 2 \pmod 4, \\ \left(\frac{-1}{L(n/4)}\right) L(n/4) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof The exceptional cases are clear as $\Psi_4(x) = x, \Psi_8(x) = x^2 - 2$. For odd n the claims follow from (3.3), (3.5), and the facts $\mathscr{W}_n(2) = n, \mathscr{W}_n(-2) = \left(\frac{-1}{n}\right), \mathscr{W}_n(0) = \left(\frac{-2}{n}\right)$. Then applying (3.4) for $p = 2$ (so $C_2(x) = x^2 - 2$), we complete the proof. ■

Now we compute $\operatorname{res}(\Psi_m, \Psi_n)$. In view of (2.3), we may make some additional restrictions on m and n.

Theorem 3.2 *Let $m, n \geq 3, m \neq n$.*

(i) *If $m \nmid n, n \nmid m$, and m is odd, then*

$$\operatorname{res}(\Psi_m, \Psi_n) = \left(\frac{L(n)}{L(m)}\right).$$

(ii) *If $m \nmid n, n \nmid m, m < n$, and m, n are even, then*

$$\operatorname{res}(\Psi_m, \Psi_n) = \left(\frac{L(m/2)}{L(n/2)}\right).$$

(iii) *If $m \mid n$ and m is odd, then, putting $L_1 = L(m), L_2 = L(n/m)$, we have*

$$\frac{\operatorname{res}(\Psi_m, \Psi_n)}{L_2^{\phi(m)/2}} = \begin{cases} \left(\frac{L_2}{L_1}\right) & \text{if } L_1 \neq L_2, \\ 1 & \text{if } L_1 = L_2. \end{cases}$$

(iv) *If $m \mid n$ and $\operatorname{ord}_2(m) = 1$, then, putting $L_1 = L(m/2), L_2 = L(n/m)$, we have*

$$\frac{\operatorname{res}(\Psi_m, \Psi_n)}{L_2^{\phi(m)/2}} = \begin{cases} \left(\frac{L_2}{L_1}\right) & \text{if } L_1 \neq L_2, \\ \left(\frac{-1}{L_1}\right) & \text{if } L_1 = L_2. \end{cases}$$

(v) *If $4 \mid m \mid n$, then, putting $L_2 = L(n/m)$, we have*

$$\frac{\operatorname{res}(\Psi_m, \Psi_n)}{L_2^{\phi(m)/2}} = \begin{cases} -1 & \text{if } m = 4, n = 8, \\ \left(\frac{-1}{L_2}\right) & \text{if } m = 4, n \neq 8, \\ 1 & \text{otherwise.} \end{cases}$$

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Proof (i) We use induction on the number of prime divisors (multiplicity taken into account) of $g = \gcd(m, n)$.

If $g = 1$, then the claim follows from (3.3), Theorem 2.1, and (3.5); note that $L(n) = 1$ if $n \equiv 2 \pmod 4$ and that $L(n/2) = L(n)$ if $n \equiv 0 \pmod 4$.

Suppose $g > 1$. Since the assumption implies $L(n) = 1$ or $L(m) = 1$, we have to show that $\operatorname{res}(\Psi_m, \Psi_n) = 1$. Let p be a prime divisor of g. We have $m/p \geq 3$, $n/p \geq 2$ by the assumption. If $n/p = 2$, then by (2.3), (2.4), and (3.2) we have

$$\operatorname{res}(\Psi_m, \Psi_n) = \operatorname{res}\big(\Psi_{2m}(-x), \Psi_p(-x)\big) = \operatorname{res}(\Psi_p, \Psi_{2m}),$$

whose computation will be postponed until (iii). Suppose $n/p \geq 3$. By (2.4) we have

$$\operatorname{res}(\Psi_{m/p}, \Psi_{n/p})^p = \operatorname{res}(\Psi_{m/p}(C_p(x)), \Psi_{n/p}(C_p(x)),$$

then by (3.4) we find that

$$\operatorname{res}(\Psi_m, \Psi_n) = \begin{cases} \dfrac{\operatorname{res}(\Psi_{m/p}, \Psi_{n/p})^{p-1}}{\operatorname{res}(\Psi_m, \Psi_{n/p}) \operatorname{res}(\Psi_{m/p}, \Psi_n)}, & \text{if } \mu = \nu = 1, \\ \dfrac{\operatorname{res}(\Psi_{m/p}, \Psi_{n/p})^p}{\operatorname{res}(\Psi_{m/p}, \Psi_n)}, & \text{if } \mu = 1, \nu \geq 2, \\ \dfrac{\operatorname{res}(\Psi_{m/p}, \Psi_{n/p})^p}{\operatorname{res}(\Psi_m, \Psi_{n/p})}, & \text{if } \mu \geq 2, \nu = 1, \\ \operatorname{res}(\Psi_{m/p}, \Psi_{n/p})^p, & \text{if } \mu \geq 2, \nu \geq 2, \end{cases} \tag{3.6}$$

where we put $\mu = \operatorname{ord}_p(m)$, $\nu = \operatorname{ord}_p(n)$. The idea of using this identity is borrowed from [3]. In each case, it follows from the induction hypothesis that $\operatorname{res}(\Psi_m, \Psi_n) = 1$.

(ii) The case $m = 4$ is immediate from Lemma 3.1, since $\operatorname{res}(\Psi_4, \Psi_n) = \Psi_n(0)$. We suppose that $m \geq 6$ and use the identity (3.6), which is valid also for $p = 2$.

The case $\operatorname{ord}_2(m) = \operatorname{ord}_2(n) = 1$. By (2.3),(3.6), and (i) we have

$$\operatorname{res}(\Psi_m, \Psi_n) = (-1)^{\phi(m/2)\phi(n/2)/4}\Big(\frac{L(n/2)}{L(m/2)}\Big).$$

Since we have $\phi(k) \equiv L(k) - 1 \pmod 4$, if $k \geq 3$ is odd, we have

$$\operatorname{res}(\Psi_m, \Psi_n) = \Big(\frac{L(m/2)}{L(n/2)}\Big)$$

by the reciprocity law.

The case $\operatorname{ord}_2(m) = 1, \operatorname{ord}_2(n) \geq 2$. Similarly, we have

$$\operatorname{res}(\Psi_m, \Psi_n) = \Big(\frac{L(n)}{L(m/2)}\Big).$$

We have either $L(n) = L(n/2) = 1$ or $L(n) = L(n/2) = 2$, so, in any case,

$$\Big(\frac{L(n)}{L(m/2)}\Big) = \Big(\frac{L(m/2)}{L(n/2)}\Big).$$

The case $\operatorname{ord}_2(m) \geq 2, \operatorname{ord}_2(n) = 1$. Similarly, we have

$$\operatorname{res}(\Psi_m, \Psi_n) = (-1)^{\phi(m)\phi(n/2)/4}\Big(\frac{L(m)}{L(n/2)}\Big).$$

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Since we have $4 \mid \phi(m), 2 \mid \phi(n/2)$, and $L(m) = L(m/2)$, we are done.

The case $\operatorname{ord}_2(m) \geq 2, \operatorname{ord}_2(n) \geq 2$. Similarly, we have $\operatorname{res}(\Psi_m, \Psi_n) = 1$. Since at least one of $m/2, n/2$ is not a prime power, we are done.

(iii) We use induction on the number of prime divisors (multiplicity taken into account) of m. Let p be a prime divisor of m.

The case $m = p$. If $n = 2p$, then by Theorem 2.1 we have

$$\operatorname{res}(\Psi_m, \Psi_n) = \operatorname{res}(\mathscr{W}_p, \mathscr{V}_p) = \Big(\frac{2}{p}\Big) 2^{(p-1)/2},$$

as desired. Suppose $n/p \geq 3$. First we compute

$$\operatorname{res}(\Psi_p, \Psi_{n/p}(C_p)) = \prod_{j=1}^{(p-1)/2} \Psi_{n/p}\Big(C_p\Big(2\cos\frac{2j\pi}{p}\Big)\Big) = \Psi_{n/p}(2)^{(p-1)/2},$$

so that by Lemma 3.1 we have

$$\operatorname{res}\big(\Psi_p, \Psi_{n/p}(C_p)\big) = L(n/p)^{\phi(p)/2}. \tag{3.7}$$

Now, if $\operatorname{ord}_p(n) = 1$, then we have

$$\operatorname{res}(\Psi_p, \Psi_n) = \frac{\operatorname{res}(\Psi_p, \Psi_{n/p}(C_p))}{\operatorname{res}(\Psi_p, \Psi_{n/p})} = \Big(\frac{L(n/p)}{L(p)}\Big) L(n/p)^{\phi(p)/2}$$

by (i) and (3.7). If $\operatorname{ord}_p(n) \geq 2$, then we have

$$\operatorname{res}(\Psi_p, \Psi_n) = \operatorname{res}\big(\Psi_p, \Psi_{n/p}(C_p)\big) = L(n/p)^{\phi(p)/2}.$$

Since either $L(n/m) = 1$ or $L(n/m) = L(m) = p$ holds, we are done.

The case $\operatorname{ord}_p(m) = 1, m > p$. In this case $L(m) = 1$, so we have to show that $\operatorname{res}(\Psi_m, \Psi_n) = L(n/m)^{\phi(m)/2}$. First suppose $\operatorname{ord}_p(n) = 1$. By (3.6) and the induction hypothesis we have

$$\operatorname{res}(\Psi_{m/p}, \Psi_{n/p}) = \pm L(n/m)^{\phi(m/p)/2}, \quad \operatorname{res}(\Psi_{m/p}, \Psi_n) = 1.$$

Since $m \nmid (n/p), (n/p) \nmid m$, and $L(m) = 1$, we have $\operatorname{res}(\Psi_m, \Psi_{n/p}) = 1$ by (i). Thus we have $\operatorname{res}(\Psi_m, \Psi_n) = L(n/m)^{\phi(m)/2}$. We next suppose $\operatorname{ord}_p(n) \geq 2$, and use (3.6) and the induction hypothesis. If $L_2 = 1$, then $L(n/(m/p)) = 1$, so $\operatorname{res}(\Psi_m, \Psi_n) = 1$. Otherwise, $L_2 = L(n/(m/p)) = p$, so

$$\operatorname{res}(\Psi_m, \Psi_n) = \Big(\Big(\frac{p}{L(m/p)}\Big) p^{\phi(m/p)/2}\Big)^{p-1} = p^{\phi(m)/2}.$$

The case $\operatorname{ord}_p(m) \geq 2$. By equation (3.6) and the induction hypothesis, and noting that $L(m/p) = L(m)$, we obtain the desired result.

(iv) *The case* $\operatorname{ord}_2(n) = 1$.

By (2.4) and (3.2) we have

$$\operatorname{res}(\Psi_m, \Psi_n) = (-1)^{\phi(m)\phi(n)/4} \operatorname{res}(\Psi_{m/2}, \Psi_{n/2}).$$

We could use (3.6) for $p = 2$ to deduce this. As is easily seen, we have

$$(-1)^{\phi(m)\phi(n)/4} = \begin{cases} 1 & \text{if } L_1 \neq L_2, \\ \big(\frac{-1}{L_1}\big) & \text{if } L_1 = L_2, \end{cases}$$

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so the claim follows from (iii).

The case $\mathrm{ord}_2(n) \geq 2$. We use equation (3.6) for $p = 2$ and (iii). If $L_2 = 1$, then $L(n/(m/2)) = 1$, so $\mathrm{res}(\Psi_m, \Psi_n) = 1$. Otherwise, $L_2 = L(n/(m/2)) = 2$ and $L_1 \neq L_2$, so

$$\mathrm{res}(\Psi_m, \Psi_n) = \left(\frac{2}{L_1}\right) 2^{\phi(m/2)/2}.$$

(v) This is immediate from Lemma 3.1 if $m = 4$. Otherwise, using (3.6) for $p = 2$ and induction, we complete the proof. ■

Corollary 3.3 ([1–3, 5, 7, 9]) *If* $3 \leq m < n$, *then*

$$\mathrm{res}(\Phi_m, \Phi_n) = \begin{cases} 1 & \text{if } m \nmid n, \\ L(n/m)^{\phi(m)} & \text{if } m \mid n. \end{cases}$$

Proof Let $\zeta = e^{2\pi i/m}$. By (3.1) we have

$$\mathrm{res}(\Psi_m, \Psi_n)^2 = \prod_{j \in (\mathbb{Z}/m)^\times} \Psi_n(\zeta^j + \zeta^{-j}) = \prod_{j \in (\mathbb{Z}/m)^\times} (\zeta^j)^{-\phi(n)/2} \Phi_n(\zeta^j) = \mathrm{res}(\Phi_m, \Phi_n),$$

since $\sum_{j \in (\mathbb{Z}/m)^\times} j \equiv 0 \pmod{m}$. So the claim follows from Theorem 3.2. ■

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