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MULTIPLIERS FOR THE MELLIN TRANSFORMATION

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Abstract. In this paper we generalize the Mellin multiplier theorem we proved earlier [8] to spaces with quite general weights, satisfying an A_p-type condition. Applications are made to the Hilbert transformation.

In an earlier paper [8], we proved a multiplier theorem for the Mellin transformation on weighted L_p spaces on $(0, \infty)$, where the weights were powers. This was deduced from the Mihlin multiplier theorem for the Fourier transformation, [10; Chapter IV, Theorem 3], though it could equally well have been deduced from the Marcinkiewicz multiplier theorem, [10; Chapter IV, Theorem 6]. Recently Kurtz [4; Theorem 2] has extended the Marcinkiewiez multiplier theorem to spaces with general weights satisfying the A_p condition of Muckenhoupt [5], and in this paper we shall make the corresponding extension of our Mellin multiplier theorem, which we do in Theorem 1 below.

In [7] we implicitly applied our Mellin multiplier theorem to, among other things, the conjugate Hankel operator and the even and odd Hilbert transformations. In Theorem 2 *et seq.* we shall make similar applications of Theorem 1, including obtaining some information about the boundedness of the Hilbert transformation.

Let w be a non-negative locally integrable function on $(0, \infty)$. If $\mu \in \mathbb{R}$ and $1 \le p < \infty$, we define $\mathscr{L}_{w,\mu,p}$ to consist of those complex-valued functions, measurable on $(0, \infty)$, such that $\|f\|_{w,\mu,p} < \infty$, where

$$(1) \qquad \|f\|_{w,\mu,p} = \left\{ \int_0^\infty w(x)\, |x^\mu f(x)|^p \, dx/x \right\}^{1/p}.$$

If $w \equiv 1$, then we shall denote $\mathscr{L}_{w,\mu,p}$ by $\mathscr{L}_{\mu,p}$ and $\|f\|_{w,\mu,p}$ by $\|f\|_{\mu,p}$. For further information about the spaces $\mathscr{L}_{\mu,p}$ see [8; §2], but note that the spaces $L_{\mu,p}$ of that paper are slightly differently defined and make the necessary adjustments.

As shown in [8; §2, adjusted], the Mellin transformation $\mathcal{M}$ is defined on

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$\mathcal{L}_{\mu,p}$, if $1 \le p \le 2$, by

$$(\mathcal{M}f)(\mu+it) = (\mathcal{C}_\mu f)^\wedge(t), \tag{2}$$

where

$$(\mathcal{C}_\mu f)(t) = e^{\mu t} f(e^t) \tag{3}$$

and $\hat{F}$ is the Fourier transform of F: that is if $F \in L_1(-\infty, \infty)$,

$$\hat{F}(t) = \int_{-\infty}^{\infty} e^{itx} F(x)\, dx, \tag{4}$$

and it is shown there that $\mathcal{M} \in [\mathcal{L}_{\mu,p}, L_{p'}, (-\infty, \infty)]$, if $1 \le p \le 2$, where if X and Y are Banach spaces, $[X, Y]$ denotes the bounded linear operators from X to Y, $[X, X]$ being abbreviated to $[X]$, and $p' = p/(p-1)$.

First we need a Lemma. For this, we define C_0 to consist of those continuous functions compactly supported in the topology of $(0, \infty)$.

LEMMA. *Suppose that* $f \in \mathcal{L}_{w_1,\mu_1,p_1} \cap \mathcal{L}_{w_2,\mu_2,p_2}$. *Then, given* $\varepsilon > 0$, *there is a function* $\phi \in C_0$ *such that* $\|f-\phi\|_{w_i,\mu_i,p_i} < \varepsilon$, $i = 1, 2$.

Proof. When $w_1 = w_2 \equiv 1$, the result was proved earlier [8; Lemma 2.3]. The proof for general w_1 and w_2 is a straightforward development of that proof, using the density of the step functions in certain weighted L_p spaces [9; Theorem 1.17], and Lusin's theorem [9; Theorem 6.11].

Before stating our theorem we need two definitions.

DEFINITION 1. We say $m \in \mathcal{A}$ if there are extended real numbers $\alpha(m)$ and $\beta(m)$, with $\alpha(m) < \beta(m)$, so that

(a) $m(s)$ is holomorphic in the strip $\alpha(m) < \operatorname{Re} s < \beta(m)$,

(b) in every closed substrip, $\sigma_1 \le \operatorname{Re} s \le \sigma_2$, where $\alpha(m) < \sigma_1 \le \sigma_2 < \beta(m)$, $m(s)$ is bounded, and

(c) for $\alpha(m) < \sigma < \beta(m)$, $|m'(\sigma+it)| = O(|t|^{-1})$ as $|t| \to \infty$.

DEFINITION 2. Suppose w is a non-negative locally integrable function on $(0, \infty)$ with $w(x) > 0$ a.e., and suppose $1 < p < \infty$. Then we say $w \in \mathfrak{A}_p$ if there is a constant K so that for all numbers a and b, with $0 < a < b < \infty$,

$$\left\{\int_a^b w(x)\, dx/x\right\} \cdot \left\{\int_a^b (w(x))^{-1/(p-1)}\, dx/x\right\}^{p-1} \le K (\log b/a)^p. \tag{5}$$

It should be noted that if $w(x) \equiv 1$, $w \in \mathfrak{A}_p$.

THEOREM 1. *Suppose* $m \in \mathcal{A}$. *Then there is a transformation* $H_m \in [\mathcal{L}_{w,\mu,p}]$ *for* $1 < p < \infty$, $\alpha(m) < \mu < \beta(m)$, $w \in \mathfrak{A}_p$, *so that if* $f \in \mathcal{L}_{\mu,p}$, *where* $1 \le p \le 2$, $\alpha(m) < \mu < \beta(m)$,

$$(\mathcal{M}H_m f)(s) = m(s)(\mathcal{M}f)(s), \qquad \operatorname{Re} s = \mu. \tag{6}$$

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H_m is one-to-one on $\mathscr{L}_{\mu,p}$ if $1<p\le 2$, $\alpha(m)<\mu<\beta(m)$ except when $m\equiv 0$. If $1/m\in\mathscr{A}$, then for $1<p<\infty$, $\max(\alpha(m),\alpha(1/m))<\mu<\min(\beta(m),\beta(1/m))$, $w\in\mathfrak{A}_p$, *H_m maps $\mathscr{L}_{w,\mu,p}$ one-to-one onto itself and*

$$(H_m)^{-1}=H_{1/m}. \tag{7}$$

Proof. For $w\equiv 1$, the result has been proved in [8; Theorem 1]. Suppose $\alpha<\mu<\beta$ and define m_μ by $m_\mu(t)=m(\mu+it)$. Then from (b) of Definition 1, m_μ is bounded. Also from (c) of Definition 1, there are positive constants R and M_1 so that if $|t|\ge R$, $|m'_\mu(t)|=|m'(\mu+it)|\le M_1/|t|$. Further, since from (a) of Definition 1, $m(s)$ is holomorphic in $\alpha<\operatorname{Re} s<\beta$, $|t|\,|m'_\mu(t)|$ is continuous in $[-R,R]$, and hence is bounded there, say by M_2, and hence if $M=\max(M_1,M_2)$, for $t\in\mathbb{R}$, $|m'_\mu(t)|\le M/|t|$. Hence if I is any dyadic interval in $\mathbb{R}$,

$$\int_I |dm_\mu(t)|=\int_I |m'_\mu(t)|\,dt\le M\int_I dt/|t|=M\log 2.$$

Thus m_μ satisfies the hypotheses of "m" of [4; Theorem 2], with $B=\max(M\log 2,\|m_\mu\|_\infty)$ and hence there is a transformation T_μ such that if $W\in A_p$, where $1<p<\infty$, then

$$\int_{-\infty}^{\infty} W(x)\,|(T_\mu f)(x)|\,dx\le N\int_{-\infty}^{\infty} W(x)\,|F(x)|^p\,dx, \tag{8}$$

for every measurable function F for which the right hand side of (8) is finite, N being a constant independent of F, and if $F\in L_2(-\infty,\infty)$,

$$(T_\mu F)^\wedge(t)=m_\mu(t)\hat{F}(t)=m(\mu+it)\hat{F}(t). \tag{9}$$

We define

$$H_m=\mathscr{C}_\mu^{-1}T_\mu\mathscr{C}_\mu. \tag{10}$$

Note that if $w\in\mathfrak{A}_p$ and $W(t)=w(e^t)$, then $W\in A_p$. For if $-\infty<\alpha<\beta<\infty$, then from (5),

$$\begin{aligned}
&\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta W(t)\,dt\right\}\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta (W(t))^{-1/(p-1)}\,dt\right\}^{p-1}\\
&=\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta w(e^t)\,dt\right\}\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta (w(e^t))^{-1/(p-1)}\,dt\right\}^{p-1}\\
&=(\beta-\alpha)^{-p}\left\{\int_{e^\alpha}^{e^\beta} w(x)\,dx/x\right\}\left\{\int_{e^\alpha}^{e^\beta} (w(x))^{-1/(p-1)}\,dx/x\right\}^{p-1}\\
&\le(\beta-\alpha)^{-p}\cdot K(\log(e^\beta/e^\alpha))^p=K.
\end{aligned}$$

Hence if $f\in\mathscr{L}_{w,\mu,p}$, where $1<p<\infty$, and if $w\in\mathfrak{A}_p$, then from (10) and (5),

$$\begin{aligned}\|H_mf\|_{w,\mu,p}&=\left\{\int_0^\infty w(x)\,|x^\mu(H_mf)(x)|^p\,dx/x\right\}^{1/p}\\&=\left\{\int_{-\infty}^\infty w(e^t)\,|(\mathscr{C}_\mu H_mf)(t)|^p\,dt\right\}^{1/p}\\&=\left\{\int_{-\infty}^\infty W(t)\,|(T_\mu\mathscr{C}_\mu f)(t)|^p\,dt\right\}^{1/p}\le N^{1/p}\left\{\int_{-\infty}^\infty W(t)\,|(\mathscr{C}_\mu f)(t)|^p\,dt\right\}^{1/p}\\&=N^{1/p}\,\|f\|_{w,\mu,p},\end{aligned}$$

so that $H_m\in[\mathscr{L}_{w,\mu,p}]$.

H_m, as defined by (10), seems to depend on μ. But, as proved in [8; Lemma 3.2], on C_0, H_m is independent of μ for $\alpha(m)<\mu<\beta(m)$, and then using our lemma, the fact that it is independent of μ on $\mathscr{L}_{w,\mu,p}$ follows as in the proof of [8; Lemma 3.2], while the remainder of the theorem now follows as in the case for $w\equiv 1$ in [8; Theorem 1].

In [7; Theorems 6.1 and 7.1] we studied operators $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ and $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ where $I_{\nu,\alpha,\xi}$ and $J_{\nu,\beta,\eta}$ are defined by [7; (1.2) and (1.3)]. The operators were studied using implicitly [8; Theorem 1], that is Theorem 1 above for $w(x)\equiv 1$. Using Theorem 1 for general $w\in\mathfrak{A}_p$, it is immediate that all the results of [7; Theorems 6.7 and 7.1] extend to $\mathscr{L}_{w,\mu,p}$ for $w\in\mathfrak{A}_p$, except the unitariness statements, provided the following theorem is proved.

THEOREM 2. *Suppose* $1<p<\infty$, *and* $w\in\mathfrak{A}_p$. *Then:* (i) *if* $\mu<\nu\operatorname{Re}\xi$ *and* $\operatorname{Re}\alpha>0$, $I_{\nu,\alpha,\xi}\in[\mathscr{L}_{w,\mu,p}]$; (ii) *if* $\mu>-\nu\operatorname{Re}\eta$ *and* $\operatorname{Re}\beta>0$, $J_{\nu,\beta,\eta}\in[\mathscr{L}_{w,\mu,p}]$.

Proof. It is shown in [7; Corollary 4.1] that if $f\in[\mathscr{L}_{\mu,p}]$, $1\le p\le 2$, $\mu<\nu\operatorname{Re}\xi$, then $(\mathscr{M}I_{\nu,\alpha,\xi}f)(s)=m(s)(\mathscr{M}f)(s)$, $\operatorname{Re}s=\mu$, where $m(s)=\Gamma(\xi-(s/\nu))/\Gamma(\xi+\alpha-(s/\nu))$. Now $m\in\mathscr{A}$, with $\alpha(m)=-\infty$, $\beta(m)=\nu\operatorname{Re}\xi$; for clearly $m(s)$ is holomorphic in the strip $-\infty<\operatorname{Re}s<\nu\operatorname{Re}\xi$; also, since from [2; 1.18(6)], $\Gamma(x+iy)\sim\sqrt{2\pi}\,|y|^{x-1/2}e^{-\pi|y|/2}$ as $|y|\to\infty$, uniformly in x for x in any bounded interval, then as $|t|\to\infty$, $m(\sigma+it)\sim|t/\nu|^{-\operatorname{Re}\alpha}$ uniformly in σ for σ in any bounded interval, and thus if $\alpha<\sigma_1\le\sigma_2<\beta$, $m(s)$ is bounded in the strip $\sigma_1\le\operatorname{Re}s\le\sigma_2$; further, from [2; 1.18(7)], if $\Psi(z)=\Gamma'(z)/\Gamma(z)$, then $\Psi(z)=\log z-1/2z+O(|z|^{-2})$ as $|z|\to\infty$ in $|\arg z|\le\pi-\delta$, where $0<\delta\le\pi$, and thus $m'(\sigma+it)=m(\sigma+it)\{\log(\xi+\alpha-(\sigma+it)/\nu)-1/(2(\xi+\alpha-(\sigma+it)/\nu))-\log(\xi-(\sigma+it/\nu)+1/(2(\xi-(\sigma+it)/\nu))+O(|t|^{-2})=m(\sigma+it)\{(i\nu\alpha)/t+O(|t|^{-2})\}=O(|t|^{-1})$ as $|t|\to\infty$, and thus $m\in\mathscr{A}$. Hence from Theorem 1, $I_{\nu,\alpha,\xi}\in[\mathscr{L}_{w,\mu,p}]$ if $1<p<\infty$, $w\in\mathfrak{A}_p$, $\mu<\nu\operatorname{Re}\xi$, and $\operatorname{Re}\alpha>0$. The results for $J_{\nu,\beta,\eta}$ follows similarly.

The results about $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ and $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ in [7] were applied in [7; Theorem 8.1] to an operator $H_{\rho,\lambda,\gamma}$, which is the product of two Hankel transformations, and it now follows that all the results of [7; Theorem 8.1] extend to $\mathcal{L}_{w,\mu,p}$, for $w \in \mathfrak{A}_p$, except again the unitariness results. In particular, since $H_{\lambda+1/2,\lambda-1/2,1}$ is Muckenhoupt and Stein's Hankel conjugate operator $\mathcal{H}_\lambda$ [6; §16], it follows that if $1<p<\infty$ and $w \in \mathfrak{A}_p$, then $\mathcal{H}_\lambda \in [\mathcal{L}_{w,\mu,p}]$ for $-1<\mu<2\lambda+1$.

A direct application of Theorem 1 to $\mathcal{H}_\lambda$ yields slightly more. For, it is easy to see from [7, §8] that if $f \in \mathcal{L}_{\mu,p}$, where $1<p\le 2, -1<\mu<2\lambda+1$, $(\mathcal{M}\mathcal{H}_\lambda f)(s) = m_\lambda(s)(\mathcal{M}f)(s)$, where $m_\lambda(s) = (\Gamma(\frac{1}{2}(1+s))\Gamma(\frac{1}{2}(2\lambda+1-s)))/(\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}(2\lambda+2-s)))$, and it follows from the asymptotic behaviour of $\Gamma(z)$ and $\Psi(z)$, in much the same way as in the proof of Theorem 2, that if $\lambda>-1$, $m_\lambda \in \mathcal{A}$, with $\alpha(m_\lambda)=-1$, $\beta(m_\lambda)=2\lambda+1$, and that $1/m_\lambda \in \mathcal{A}$ with either $\alpha(1/m_\lambda)=0, \beta(1/m_\lambda)=2\lambda+2$ or $\alpha(1/m_\lambda)=-1, \beta(1/m_\lambda)=0$. Thus except for $\mu=0, \mathcal{H}_\lambda(\mathcal{L}_{w,\mu,p})=\mathcal{L}_{w,\mu,p}$.

Since the even Hilbert transformation, H_+, is $\mathcal{H}_0$, it follows that if $1<p<\infty$, $w \in \mathfrak{A}_p$, $H_+ \in [\mathcal{L}_{w,\mu,p}]$ for $-1<\mu<1$, and except when $\mu=0, H_+(\mathcal{L}_{w,\mu,p})=\mathcal{L}_{w,\mu,p}$. Similar analysis for the odd Hilbert transformation H_- yields that if $1<p<\infty$, $w \in \mathfrak{A}_p$, $H_- \in [\mathcal{L}_{w,\mu,p}]$ for $0<\mu<2$, and except for $\mu=1, H_-(\mathcal{L}_{w,\mu,p})=\mathcal{L}_{w,\mu,p}$. These results should be contrasted with those of Andersen [1] who gave necessary and sufficient conditions on a weight W that $H_\pm$ be bounded on the $L_p(0,\infty)$ space with weight W. Applying Andersen's conditions on W to the weight $x^{p\mu-1}w(x)$ that we are using here, it follows that if $w \in \mathfrak{A}_p$, then for $0<\nu<2p, 0<a<b$,

$$\left\{\int_a^b x^\nu w(x)\,dx/x\right\}\left\{\int_a^b (x^{\nu-2p}w(x))^{-1/(p-1)}\,dx/x\right\}^{p-1} \le K(b^2-a^2)^p.$$

The Hilbert transform H of a function f can be constructed from the even Hilbert transform of the even part of f and the odd Hilbert transform of the odd part of f. Putting things together in this way yields that for $0<\mu<1$,

$$\int_{-\infty}^{\infty} w(|x|)\,|x|^{p\mu-1}|\,(Hf)(x)|^p\,dx \le K\int_{-\infty}^{\infty} w(|x|)\,|x|^{p\mu-1}\,|f(x)|^p\,dx$$

for all f measurable on $\mathbb{R}$ for which the right hand side is finite. Necessary and sufficient conditions that the Hilbert transformation be bounded on a weighted $L_p(-\infty,\infty)$ with weight W have been given by Hunt, Muckenhoupt and Wheeden [3], and applying these here, it follows that if $w \in \mathfrak{A}_p$, then for $0<\nu<p, 0\le a<b$,

$$\left\{\int_a^b x^\nu w(x)\,dx/x\right\}\left\{\int_a^b (x^{\nu-p}w(x))^{-1/(p-1)}\,dx/x\right\}^{p-1} \le K(b-a)^p.$$

In particular, with $\nu=1$, if $w \in \mathfrak{A}_p$, $w(|x|) \in A_p$.

Thus Theorem 1 produces significant results about well known classes of functions, and about important operators.

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