4

The pinch technique in the Batalin–Vilkovisky framework

It is clear from the analysis presented until now that even though the PT Green's functions satisfy naive QED-like Ward identities, their actual derivation relies heavily on Slavnov–Taylor identities obeyed by certain subamplitudes appearing in the ordinary diagrammatic expansion such as the kernel $A_\mu A_\nu q\bar{q}$ considered in the previous chapter. Unlike QED, because of the nonlinearity of the BRST transformations, these Slavnov–Taylor identities are realized through ghost Green's functions involving composite operators such as $\langle 0|T[s\Phi(x)\cdots|0\rangle$, where s is the BRST operator and Φ is a generic QCD field. It turns out that the most efficient framework for dealing with these types of objects is the so-called Batalin-Vilkovisky formalism [1, 2, 3, 4]. In this framework, one adds to the original gauge-invariant action $\Gamma^{(0)}_{\mathrm{I}}$ the term $\mathcal{L}_{\mathrm{BRST}} = \sum_\Phi \Phi^* s\Phi$, coupling the composite operators $s\Phi$ to the BRST-invariant external sources (usually called antifields) Φ^* to obtain the new action $\Gamma^{(0)} = \Gamma^{(0)}_{\mathrm{I}} + \sum_\Phi \Phi^* s\Phi$.

From the point of view of the pinch technique, there are considerable conceptual and operational advantages to be gained from employing this formalism [5]. To begin with, the use of antifields [6], which represent a core ingredient of the BV formalism itself, streamlines the derivation of Slavnov–Taylor identities, expressing them in terms of auxiliary functions that can be constructed using a well-defined set of Feynman rules (derived from $\mathcal{L}_{\mathrm{BRST}}$). In addition, the formulation of the BFM within the BV formalism gives rise to important all-order identities, to be called *background-quantum identities* [5, 7], relating the BFM n-point functions to the corresponding conventional n-point functions in the R_ξ gauges. These identities are realized by means of unphysical Green's functions involving antifields and background sources. The prime example of such an identity is given in Eq. (4.35), the most important equation in this chapter: the conventional and PT gluon propagators, $\Delta(q)$ and $\widehat{\Delta}(q)$, respectively, are related by means of the auxiliary two-point function $G(q)$.

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The basic observation that makes the background-quantum identities so useful is that the unphysical Green's functions appearing in them (such as the $G(q)$ in Eq. (4.35)) are related to the auxiliary Green's functions appearing in the Slavnov–Taylor identities by simple expressions. Put simply, the parts of the diagrams exchanged during the pinching process, namely, the terms containing the unphysical vertices, are involved in relations connecting conventional and BFM n-point functions. In Eq. (4.44), for example, the function $G(q)$ is fully determined in terms of quantities that are defined in the context of the conventional formalism, without recourse to antifields or to the Feynman rules stemming from $\mathcal{L}_{\text{BRST}}$. This, in turn, allows for a direct comparison of the PT and BFM Green's functions: a PT Green's function is obtained from the conventional one by removing the pinching parts; but in doing so, one is practically generating the corresponding background-quantum identity, which carries over to the BFM Green's function.

The background-quantum identities play a central role in the entire PT program for one additional reason. As is already evident at the two-loop level, the two-loop PT gluon self-energy is composed of Feynman diagrams involving the conventional one-loop gluon self-energy and not the one-loop PT self-energy. This might suggest at first that one cannot arrive (eventually) at a genuine Schwinger–Dyson equation involving the same unknown quantity on both sides, i.e., either $\Pi_{\mu\nu}$ or $\widehat{\Pi}_{\mu\nu}$. There is, however, a way around this: the nonperturbative version of the background-quantum identities, and most important, that of Eq. (4.35), allows one to convert the new SD series into a dynamical equation involving either the conventional or the BFM gluon self-energy only. As we will see in Chapter 6, this is instrumental for the success of the entire approach.

In addition to Eq. (4.35), the second identity of Eq. (4.50) captures another important result of this chapter. It turns out that in the Landau gauge (only), the function $G(q)$ coincides with the so-called Kugo-Ojima function [8], $u(q)$, defined in Eq. (4.51). The latter function, and in particular its value in the deep infrared, is intimately connected with the Kugo-Ojima confinement criterion [8], which requires that $u(0) = -1$. The identity of Eq. (4.50) relates the Kugo-Ojima function with the inverse of the ghost-dressing function, $F(q)$, and an auxiliary function, $L(q)$; the latter can be shown to vanish in the deep infrared. The power of Eq. (4.50) is in that it relates the value $u(0)$ and, hence, the fulfillment or nonfulfillment of the corresponding confinement criterion, with the value of $F(0)$: the Kugo–Ojima criterion is satisfied provided that $F(0)$ diverges. However, as we will discuss briefly in Chapter 6, this is not how QCD really works. Both lattice simulations and Schwinger–Dyson equations reveal that $F(0)$ is actually finite – a fact that can ultimately be traced back to the dynamical generation of a gluon mass.

4.1 An overview of the Batalin–Vilkovisky formalism

4.1.1 Green's functions: Conventions

The 1PI Green's functions of any theory are defined in terms of the time-ordered product of interacting fields[1] as

$$\Gamma_{\Phi_1\cdots\Phi_n}(x_1,\ldots,x_n) = \langle T[\Phi_1(x_1)\cdots\Phi_n(x_n)]\rangle^{1\mathrm{PI}} \tag{4.1}$$

and can be efficiently constructed through a generating functional, which in Fourier space reads

$$\Gamma[\Phi] = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int\prod_{i=0}^{n}\mathrm{d}^4p_i\;\delta^4\left(\sum_{j=1}^{n}p_j\right)\Phi_1(p_1)\cdots\Phi_n(p_n)\Gamma_{\Phi_1\cdots\Phi_n}(p_1,\ldots,p_n). \tag{4.2}$$

In the preceding formula, the field $\Phi_i(p_i)$ represents the Fourier transform of the field $\Phi_i(x_i)$, with p_i its (in-going) momentum. Then, in terms of the generating functional $\Gamma[\Phi]$, all the (momentum-space) 1PI Green's functions can be obtained by means of functional differentiation:

$$\Gamma_{\Phi_1\cdots\Phi_n}(p_1,\ldots,p_n) = \mathrm{i}^n\left.\frac{\delta^n\Gamma}{\delta\Phi_1(p_1)\delta\Phi_2(p_2)\cdots\delta\Phi_n(p_n)}\right|_{\Phi_i=0}. \tag{4.3}$$

Our convention on the external momenta is summarized in Figure 4.1. From the definition given in Eq. (4.3), it follows that the Green's functions $\mathrm{i}^{-n}\Gamma_{\Phi_1\cdots\Phi_n}$ are simply given by the corresponding Feynman diagrams in Minkowski space.

The Green's functions generated by $\Gamma[\Phi]$ can be joined together by full propagators to construct higher-point connected amplitudes, ultimately giving rise to the S-matrix elements of the theory. However, they are by no means a complete set, for the nonlinearity of the BRST transformation of NAGTs implies that auxiliary Green's functions involving ghost fields will appear in the Slavnov–Taylor identities. The latter are precisely the Green's functions with which we have always been working when applying the PT algorithm and constitute the functions we will thoroughly study in the rest of this chapter.

4.1.2 The Batalin–Vilkovisky formalism

The Batalin-Vilkovisky formalism [1, 2, 3, 4] is a powerful quantization scheme that allows us to address in an effective way several aspects of very general gauge

[1] We let Φ run over all the fields A, ψ,$\bar{\psi}$, c, $\bar{c}$, and B. Sometimes the fields appearing in the gauge-invariant Lagrangian will be collectively indicated as ϕ.

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Table 4.1. *Ghost charge, statistics (B for Bose, F for Fermi), and mass dimension of the QCD fields and antifields*

	A^m_μ	ψ^i_{f}	$\bar\psi^i_{\mathrm{f}}$	c^m	$\bar c^m$	B^m	A^{*m}_μ	ψ^{*i}_{f}	$\bar\psi^{*i}_{\mathrm{f}}$	c^{*m}	$\bar c^{*m}$
Ghost charge	0	0	0	1	-1	0	-1	-1	-1	-2	0
Statistics	B	F	F	F	F	B	F	B	B	B	B
Dimension	1	$\frac{3}{2}$	$\frac{3}{2}$	0	2	2	3	$\frac{5}{2}$	$\frac{5}{2}$	4	2

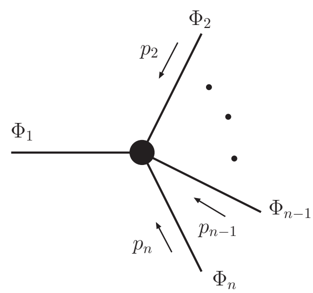

Figure 4.1. Our conventions for the (1PI) Green's functions $\Gamma_{\Phi_1\cdots\Phi_n}(p_1, \ldots, p_n)$. All momenta $p_2, \ldots, p_n$ are assumed to be incoming and are assigned to the corresponding fields starting from the rightmost one. The momentum of the leftmost field Φ_1 is determined through momentum conservation ($\sum_i p_i = 0$) and will be suppressed.

theories (e.g., their quantization, renormalization, and symmetry violation due to quantum effects), including those with open or reducible gauge symmetry algebras.

The Batalin-Vilkovisky formalism starts by introducing for each field Φ a corresponding antifield, to be denoted by Φ^*. The antifield Φ^* has opposite statistics with respect to Φ as well as a ghost charge gh(Φ^*), which is related to the ghost charge gh(Φ) of the corresponding field Φ by $\mathrm{gh}(\Phi^*) = -1 - \mathrm{gh}(\Phi)$. The ghost charges, statistics, and mass dimension of the various QCD fields and antifields are summarized in Table 4.1.

Next, one adds to the original gauge-invariant action $\Gamma^{(0)}_{\mathrm{I}}[\phi]$ (with ϕ representing the physical QCD fields A, ψ, and $\bar\psi$), a term coupling the antifields with the BRST variation of the corresponding fields; then one obtains the new action

$$\Gamma^{(0)}[\Phi, \Phi^*] = \Gamma^{(0)}_{\mathrm{I}}[\phi] + \sum_{\Phi} \Phi^* s\Phi, \tag{4.4}$$

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where

$$\sum_{\Phi} \Phi^* s\Phi = \int \mathrm{d}^4x \Big[A^{*m}_{\mu}(\partial^{\mu}c^m + gf^{mnr}A^{\mu}_{n}c^r) - \frac{1}{2}gf^{mnr}c^{*m}c^n c^r + \bar{c}^{*m}B^m$$
$$+ \mathrm{i}g\bar{\psi}^{*i}_{\mathrm{f}}c^m t^m_{ij}\psi^j_{\mathrm{f}} - \mathrm{i}gc^m\bar{\psi}^i_{\mathrm{f}}t^m_{ij}\psi^{*j}_{\mathrm{f}}\Big], \tag{4.5}$$

(f is the quark flavor index).[2]

The action (4.4) satisfies the master equation[3]

$$\int \mathrm{d}^4x \sum_{\Phi} \frac{\delta\Gamma^{(0)}}{\delta\Phi^*}\frac{\delta\Gamma^{(0)}}{\delta\Phi} = 0. \tag{4.6}$$

In fact, on one hand, the terms in Eq. (4.4) that are independent of the antifields are zero because of the gauge invariance of the action

$$\sum_{\phi} s\phi\frac{\delta\Gamma^{(0)}_{\mathrm{I}}}{\delta\phi} = \int \mathrm{d}^4x (s\Gamma^{(0)}_{\mathrm{I}}[\phi]) = 0. \tag{4.7}$$

On the other hand, the terms linear in the antifields vanish because of the nilpotency of the BRST operator:

$$\sum_{\Phi'} s\Phi'\frac{\delta(s\Phi)}{\delta\Phi'} = \int \mathrm{d}^4x \sum_{\Phi} s^2\Phi = 0. \tag{4.8}$$

The BRST symmetry is crucial for endowing a (gauge) theory with a unitary S-matrix and gauge-independent physical observables; therefore, it must be implemented to all orders. For achieving this, we establish the quantum corrected version of the master equation (4.6) in the form of the Slavnov–Taylor identity functional

$$\mathcal{S}(\Gamma)[\Phi] = \int \mathrm{d}^4x \sum_{\Phi} \frac{\delta\Gamma}{\delta\Phi^*}\frac{\delta\Gamma}{\delta\Phi} = 0, \tag{4.9}$$

where $\Gamma[\Phi, \Phi^*]$ is now the effective action. In the pure gluodynamics sector, the Slavnov–Taylor functional is given by[4]

$$\mathcal{S}(\Gamma)[\Phi] = \int \mathrm{d}^4x \left[\frac{\delta\Gamma}{\delta A^{*\mu}_{m}}\frac{\delta\Gamma}{\delta A^m_{\mu}} + \frac{\delta\Gamma}{\delta c^{*m}}\frac{\delta\Gamma}{\delta c^m} + B^m\frac{\delta\Gamma}{\delta\bar{c}^m}\right]. \tag{4.10}$$

[2] It can be easily shown that this new action is physically equivalent to the gauge-fixed QCD action because the two are related by a canonical transformation [9].

[3] Our derivatives are all left derivatives, e.g., $\delta(ab) = (\delta a)b + (-1)^{\epsilon_a}a\delta b$, with ϵ_a being the Grassmann parity of a.

[4] Quarks can be easily taken into account by adding to the Slavnov–Taylor functional (4.10) the term

$$\int \mathrm{d}^4x \left[\frac{\delta\Gamma}{\delta\psi^{*i}_{\mathrm{f}}}\frac{\delta\Gamma}{\delta\bar{\psi}^i_{\mathrm{f}}} + \frac{\delta\Gamma}{\delta\psi^i_{\mathrm{f}}}\frac{\delta\Gamma}{\delta\bar{\psi}^{*i}_{\mathrm{f}}}\right].$$

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The structure of the preceding master equation can be simplified by noticing that the antighost $\bar{c}^a$ and the multiplier B^a have *linear* BRST transformations and therefore do not present with the usual complications of the other QCD fields. Together with their corresponding antifields, they enter bilinearly in the action, which can be then decomposed in the sum of a *minimal* and *nonminimal* sector:

$$\Gamma_{\mathrm{C}}^{(0)}[\Phi, \Phi^*] = \Gamma^{(0)}[A_\mu^m, A_\mu^{*m}, \psi, \psi_{\mathrm{f}}^{*i}, \bar{\psi}_{\mathrm{f}}^i, \bar{\psi}_{\mathrm{f}}^{*i}, c^m, c^{*m}] + \bar{c}^{*m} B^m. \tag{4.11}$$

The last term has no effect on the master equation (4.6), which in fact is satisfied by $\Gamma^{(0)}$ alone. The fields $\{A_\mu^m, A_\mu^{*m}, \psi, \psi_{\mathrm{f}}^{*i}, \bar{\psi}_{\mathrm{f}}^i, \bar{\psi}_{\mathrm{f}}^{*i}, c^m, c^{*m}\}$ are then often called *minimal variables*, whereas $\{\bar{c}^m, B^m\}$ are referred to as *trivial* or *contractible pairs*.[5] Then, in the minimal sector, the *reduced* Slavnov–Taylor functional is given by the complete functional of Eq. (4.10) once the last term $B^m \delta\Gamma/\delta\bar{c}^m$ is left out.

Taking functional derivatives of $\mathcal{S}(\Gamma)[\Phi]$ and setting afterward all fields and antifields to zero will generate the complete set of the all-order Slavnov–Taylor identities of the theory.[6] This is an exact analogy (see Eq. (4.3)) to what happens with the generating functional, where taking functional derivatives of $\Gamma[\Phi]$ and setting afterward all fields to zero generates the Green's functions of the theory. However, to reach meaningful expressions, one needs to keep in mind that (1) $\mathcal{S}(\Gamma)$ has ghost charge $+1$ and (2) functions with nonzero ghost charge vanish, for the ghost charge is a conserved quantity. Thus, to extract nonzero identities from Eq. (4.10), one needs to differentiate the latter with respect to a combination of fields containing either one ghost field or two ghost fields and one antifield. The only exception to this rule is when differentiating with respect to a ghost antifield, which needs to be compensated by three ghost fields. Specifically, identities involving one or more gauge fields are obtained by differentiating Eq. (4.10) with respect to the set of fields in which one gauge boson has been replaced by the corresponding ghost field. This is because the linear part of the BRST transformation of the gauge field is proportional to the ghost field: $sA_\mu^m|_{\mathrm{linear}} = \partial_\mu c^m$. Finally, for obtaining Slavnov–Taylor identities involving Green's functions that contain ghost fields, one ghost field must be replaced by two ghost fields because of the quadratic nature of the BRST ghost field transformation ($sc^m \propto f^{mnr} c^n c^r$).

The last technical point to be clarified is the dependence of the Slavnov–Taylor identities on the (external) momenta. One should notice that the integral over d^4x present in Eq. (4.10), together with the conservation of momentum flow of the

[5] Equivalently, the minimal sector action can be obtained by subtracting from the complete action the local term corresponding to the gauge–fixing Lagrangian $\mathcal{L}_{\mathrm{GF}}$.

[6] In practice, the Slavnov–Taylor identities obtained from the reduced functional coincide with the ones obtained by the complete functional (1) after implementing the Faddeev–Popov equation described in the next section [10] and (2) taking into account that Green's functions involving unphysical fields coincide only up to constant terms proportional to the gauge-fixing parameter, e.g., $\Gamma_{A_\mu^m A_\nu^n}(q) = \Gamma^{\mathrm{C}}_{A_\mu A_\nu}(q) - \mathrm{i}\delta^{mn}\xi^{-1} q_\mu q_\nu$.

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Table 4.2. *Ghost charge, statistics (B for Bose, F for Fermi), and mass dimension of the QCD background fields and sources*

	$\widehat{A}^m_\mu$	Ω^{*m}_μ
Ghost charge	0	1
Statistics	B	F
Dimension	1	1

Green's functions, implies that no momentum integration is left over. As a result, the Slavnov–Taylor identities will be expressed as a sum of products of (at most two) Green's functions.

The (complete) Slavnov–Taylor functional is absolutely general and need not be modified if one changes the way of gauge fixing the Lagrangian, e.g., switching from a general R_ξ gauge to the BFM. In this latter case, however, to control the dependence of the Green's functions on the background fields, some new terms, implementing the equation of motion of the background fields at the quantum level, are conventionally added to the Slavnov-Taylor functional. Specifically, one extends the BRST symmetry to the background gluon field through the relations

$$s\widehat{A}^m_\mu = \Omega^m_\mu \qquad s\Omega^m_\mu = 0. \tag{4.12}$$

The expression Ω^m_μ represents a (classical) vector field with the same quantum numbers as the gluon but with ghost charge $+1$ and Fermi statistics (see also Table 4.2). The dependence of the Green's functions on the background fields is then controlled by the modified Slavnov–Taylor functional

$$\mathcal{S}'(\Gamma')[\Phi] = \mathcal{S}(\Gamma')[\Phi] + \int \mathrm{d}^4x \; \Omega^\mu_m \left[\frac{\delta\Gamma'}{\delta\widehat{A}^m_\mu} - \frac{\delta\Gamma'}{\delta A^m_\mu}\right] = 0, \tag{4.13}$$

where Γ' denotes the effective action that depends on the background sources Ω^m_μ (with $\Gamma \equiv \Gamma'|_{\Omega=0}$), and $\mathcal{S}(\Gamma')[\Phi]$ is the Slavnov–Taylor identity functional of Eq. (4.10). Differentiation of the Slavnov–Taylor functional (4.13) with respect to the background source and background or quantum fields will then provide the background-quantum identities relating 1PI Green's functions involving background fields to the ones involving quantum fields and already briefly discussed.[7]

Finally, the background gauge invariance of the BFM effective action implies that Green's functions involving background fields satisfy linear Ward identities when

[7] As it happens, for Slavnov–Taylor identities, background-quantum identities are not deformed by the renormalization procedure. The new background variables enter, in fact, as BRST doublets, and they cannot change the cohomology of the linearized Slavnov–Taylor operator [11].

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contracted with the momentum corresponding to a background leg. These Ward identities are generated by taking functional differentiations of the Ward identity functional

$$\mathcal{W}_\vartheta[\Gamma'] = \int \mathrm{d}^4x \sum_\Phi (\delta_\vartheta \Phi) \frac{\delta\Gamma'}{\delta\Phi} = 0, \tag{4.14}$$

where $\delta_\vartheta \Phi$ are given by the BRST transformation of the corresponding fields when replacing ghosts with the local infinitesimal parameters $\vartheta^a(x)$ corresponding to the $SU(3)$ generators t^a; the background transformations of the antifields $\delta_\vartheta \Phi^*$ coincide with the gauge transformations of the corresponding quantum fields according to their specific representation. To obtain the Ward identity satisfied by the Green's functions involving background gluons $\widehat{A}$, one has then to differentiate the functional (4.14) with respect to the corresponding parameter ϑ.

4.2 Examples

4.2.1 Slavnov–Taylor identities

One of the most useful Slavnov–Taylor identities in a PT context is definitely the one satisfied by the three-gluon vertex. The textbook derivation of this identity has been sketched in Chapter 1 (see Section 1.5.1); here we derive the same identity within the Batalin-Vilkovisky formalism.

According to the rules stated in the previous section, the three-gluon Slavnov–Taylor identity can be obtained by considering the following functional differentiation:

$$\left.\frac{\delta^3 \mathcal{S}(\Gamma)}{\delta c^a(q)\delta A^m_\mu(k_1)\delta A^n_\nu(k_2)}\right|_{\Phi,\Phi^*=0} = 0, \tag{4.15}$$

which gives the result

$$\begin{aligned} -\Gamma_{c^a A^{*\alpha}_{a'}}(-q)\Gamma_{A^{a'}_\alpha A^m_\mu A^n_\nu}(k_1,k_2) = {}& \Gamma_{c^a A^n_\nu A^{*\gamma}_d}(k_2,k_1)\Gamma_{A^d_\gamma A^m_\mu}(k_1) \\ &+ \Gamma_{c^a A^m_\mu A^{*\gamma}_d}(k_1,k_2)\Gamma_{A^d_\gamma A^n_\nu}(k_2). \end{aligned} \tag{4.16}$$

To further simplify the preceding identity, we need to resort to the so-called Faddeev–Popov (or ghost) equation, which describes the action of longitudinal momenta when acting on auxiliary Green's functions. To derive this equation in the R_ξ gauges, one observes that in the QCD action, the only term proportional to the antighost fields comes from the Faddeev–Popov Lagrangian density, which can be rewritten as

$$\mathcal{L}^{R_\xi}_{\mathrm{FPG}} = -\bar{c}^m \partial^\mu (s A^m_\mu) = -\bar{c}^m \partial^\mu \frac{\delta\Gamma}{\delta A^{*m}_\mu}. \tag{4.17}$$

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Differentiation of the action with respect to $\bar{c}^m$ then yields the Faddeev–Popov equation in the form of the identity

$$\frac{\delta\Gamma}{\delta\bar{c}^m} + \partial^\mu \frac{\delta\Gamma}{\delta A^{*m}_\mu} = 0, \tag{4.18}$$

so that, taking the Fourier transform, we arrive at

$$\frac{\delta\Gamma}{\delta\bar{c}^m} + \mathrm{i}q^\mu \frac{\delta\Gamma}{\delta A^{*m}_\mu} = 0. \tag{4.19}$$

Thus, in the R_ξ case, Eq. (4.19) amounts to the simple statement that the contraction of a leg corresponding to a gluon antifield (A^{*m}_μ) by its own momentum (q^μ) converts it to an antighost leg ($\bar{c}^m$). Notice that the Faddeev–Popov equation depends crucially on the form of the ghost Lagrangian, which in turn depends on the gauge-fixing function. In the presence of background gluons and sources, the presence of extra terms in the BFM gauge-fixing function will modify Eq. (4.18), which will read, in this case,

$$\frac{\delta\Gamma'}{\delta\bar{c}^m} + \left(\widehat{\mathcal{D}}^\mu \frac{\delta\Gamma'}{\delta A^*_\mu}\right)^m - \left(\mathcal{D}^\mu \Omega_\mu\right)^m = 0. \tag{4.20}$$

Notice that by setting the background field and source to zero, one correctly recovers the R_ξ equation (4.18).

Let us now differentiate Eq. (4.19) with respect to a ghost field c; after setting the fields and antifields to zero, we get

$$\Gamma_{c^m\bar{c}^n}(q) + \mathrm{i}q^\nu \Gamma_{c^m A^{*n}_\nu}(q) = 0, \tag{4.21}$$

which can be used to relate the auxiliary function $\Gamma_{c^m A^{*n}_\nu}(q)$ with the full ghost propagator $D^{mn}(q)$. Owing to Lorentz invariance, we can in fact write $\Gamma_{c^m A^{*n}_\nu}(q) = q_\nu \Gamma_{c^m A^{*n}}(q)$, and therefore

$$\Gamma_{c^m\bar{c}^n}(q) = -\mathrm{i}q^\nu \Gamma_{c^m A^{*n}_\nu}(q) = -\mathrm{i}q^2 \Gamma_{c^m A^{*n}}(q). \tag{4.22}$$

On the other hand, observing that $\mathrm{i}D^{mr}(q)\Gamma_{c^r\bar{c}^n}(q) = \delta^{mn}$, we get the announced relation

$$\Gamma_{c^m A^{*n}_\nu}(q) = q_\nu \Gamma_{c^m A^{*n}}(q) = q_\nu [q^2 D^{mn}(q)]^{-1}, \tag{4.23}$$

which, inserted back into Eq. (4.16), gives

$$\begin{aligned} q^\alpha \Gamma_{A^a_\alpha A^m_\mu A^n_\nu}(k_1, k_2) = [q^2 D^{aa'}(q)] \Big\{ & \Gamma_{c^{a'} A^n_\nu A^{*\gamma}_d}(k_2, k_1) \Gamma_{A^d_\gamma A^m_\mu}(k_1) \\ & + \Gamma_{c^{a'} A^m_\mu A^{*\gamma}_d}(k_1, k_2) \Gamma_{A^d_\gamma A^n_\nu}(k_2) \Big\}. \end{aligned} \tag{4.24}$$

To get the Slavnov–Taylor identity in the same form of Eq. (4.16), one factors out the color structure $\mathrm{i}gf^{amn}$, uses the relation

$$\Gamma_{A^a_\alpha A^b_\beta}(q) = (\Delta^{-1})^{ab}_{\alpha\beta}(q) - \mathrm{i}\delta^{ab} q_\alpha q_\beta = \mathrm{i}\delta^{ab} P_{\alpha\beta}(q)\Delta^{-1}(q^2), \tag{4.25}$$

and identifies $H_{\mu\gamma}(k_1, k_2)$ with $\Gamma_{cA_\mu A^*_\gamma}(k_1, k_2)$. Notice then that the relation between H and the gluon-ghost vertex is automatic, being a manifestation of the Faddeev-Popov equation; in fact, by differentiating Eq. (4.19) with respect to a gluon and a ghost field, we get the identity

$$\Gamma_{c^r A^n_\nu \bar{c}^m}(k, q) + \mathrm{i}q^\mu \Gamma_{c^r A^n_\nu A^{*m}_\mu}(k, q) = 0. \tag{4.26}$$

We conclude by observing that within the Batalin–Vilkovisky formalism, one can also obtain Slavnov–Taylor identities for kernels appearing e.g., in the usual skeleton expansion of QCD Green's functions. To do so, one decomposes the kernel under scrutiny in terms of 1PI Green's functions, calculates the corresponding Slavnov–Taylor identities by taking functional differentiation of the functional (4.10) with respect to suitable fields' combinations, and then puts together all the pieces. For example, the Batalin-Vilkovisky formalism version of the Slavnov–Taylor identity satisfied by the fundamental PT kernel $\mathcal{K}_{AA\psi\bar{\psi}}$, identified in the previous chapter, reads (suppressing the quark flavor and color indices)

$$\begin{aligned} k_1^\mu \mathcal{K}_{A^m_\mu A^n_\nu \psi\bar{\psi}}(k_2, p_2, -p_1) = [k_1^2 D^{mm'}(k_1)] & \\ \times \Big\{ & \Gamma_{c^{m'} A^n_\nu A^{*\gamma}_d}(k_2, -k_1 - k_2)\Gamma_{A^d_\gamma \psi\bar{\psi}}(p_2, -p_1) \\ & + \Gamma_{\psi\bar{\psi}}(p_1)\mathcal{K}_{A^n_\nu \psi c^{m'} \bar{\psi}^*}(p_2, k_1, -p_1) \\ & + \mathcal{K}_{A^n_\nu \psi^* \bar{\psi} c^{m'}}(p_2, -p_1, k_1)\Gamma_{\psi\bar{\psi}}(p_2) \\ & + \Gamma_{c^{m'} A^{*\gamma}_d \psi\bar{\psi}}(k_2, p_2, -p_1)\Gamma_{A^d_\gamma A^n_\nu}(k_2) \Big\}, \end{aligned} \tag{4.27}$$

where we have defined the auxiliary kernels

$$\begin{aligned} \mathcal{K}_{A^n_\nu \psi c^{m'} \bar{\psi}^*}(p_2, k_1, -p_1) = {} & \Gamma_{A^n_\nu \psi c^{m'} \bar{\psi}^*}(p_2, k_1, -p_1) \\ & - \mathrm{i}\Gamma_{\psi c^{m'} \bar{\psi}^*}(k_1, -p_1) S(\ell) \Gamma_{A^n_\nu \psi\bar{\psi}}(p_2, -\ell) \end{aligned} \tag{4.28}$$

$$\begin{aligned} \mathcal{K}_{A^n_\nu \psi^* \bar{\psi} c^{m'}}(p_2, -p_1, k_1) = {} & \Gamma_{A^n_\nu \psi^* \bar{\psi} c^{m'}}(p_2, -p_1, k_1) \\ & - \mathrm{i}\Gamma_{A^n_\nu \psi\bar{\psi}}(\ell, -p_1) S(\ell) \Gamma_{\psi^* \bar{\psi} c^{m'}}(-\ell, k_1). \end{aligned} \tag{4.29}$$

4.2.2 *Background-quantum identities*

The first background-quantum identity we can construct is the one relating the conventional with the BFM gluon self-energies. To this end, consider the following

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functional differentiations ($q+p=0$):

$$\left.\frac{\delta^2\mathcal{S}'\left(\Gamma'\right)}{\delta\Omega^a_\alpha(p)\delta A^b_\beta(q)}\right|_{\Phi,\Phi^*,\Omega=0}=0;\qquad \left.\frac{\delta^2\mathcal{S}'\left(\Gamma'\right)}{\delta\Omega^a_\alpha(p)\delta \widehat{A}^b_\beta(q)}\right|_{\Phi,\Phi^*,\Omega=0}=0, \tag{4.30}$$

which give the relations

$$\mathrm{i}\Gamma_{\widehat{A}^a_\alpha A^b_\beta}(q)=\left[\mathrm{i}g^\gamma_\alpha\delta^{ad}+\Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)\right]\Gamma_{A^d_\gamma A^b_\beta}(q) \tag{4.31}$$

$$\mathrm{i}\Gamma_{\widehat{A}^a_\alpha \widehat{A}^b_\beta}(q)=\left[\mathrm{i}g^\gamma_\alpha\delta^{ad}+\Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)\right]\Gamma_{A^d_\gamma \widehat{A}^b_\beta}(q). \tag{4.32}$$

We can now combine Eqs. (4.31) and (4.32) such that the two-point function mixing background and quantum fields drop out; then, using the transversality of the gluon two-point function Γ_{AA}, we get the background-quantum identity

$$\begin{aligned}\mathrm{i}\Gamma_{\widehat{A}^a_\alpha \widehat{A}^b_\beta}(q)&=\mathrm{i}\Gamma_{A^a_\alpha A^b_\beta}(q)+2\Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)\Gamma_{A^d_\gamma A^b_\beta}(q)\\&\quad-\mathrm{i}\Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)\Gamma_{A^d_\gamma A^e_\epsilon}(q)\Gamma_{\Omega^b_\beta A^{*\epsilon}_e}(q).\end{aligned} \tag{4.33}$$

This identity can be rewritten in a more suggestive form by trading the two-point functions Γ_{AA} and $\Gamma_{\widehat{A}\widehat{A}}$ for the corresponding (inverse) propagators and setting[8]

$$\Gamma_{\Omega^a_\alpha A^{*d}_\gamma}(q)=\mathrm{i}\delta^{ad}\left[g_{\alpha\gamma}G(q^2)+\frac{q_\alpha q_\gamma}{q^2}L(q^2)\right]. \tag{4.34}$$

One then gets

$$\widehat{\Delta}^{-1}(q^2)=\left[1+G(q^2)\right]^2\Delta^{-1}(q^2). \tag{4.35}$$

As we know from Chapter 1, the quantity $\widehat{\Delta}(q^2)$ appearing on the left-hand side (lhs) of the preceding equation captures the running of the QCD beta function, exactly as happens with the QED vacuum polarization.[9] For example, to lowest order, one can use the closed expression (4.44) to get (in the Landau gauge)

$$\begin{aligned}1+G(q^2)&=1+\frac{9}{4}\frac{C_{\mathrm{A}}g^2}{48\pi^2}\ln\left(\frac{q^2}{\mu^2}\right)\\ \Delta^{-1}(q^2)&=q^2\left[1+\frac{13}{2}\frac{C_{\mathrm{A}}g^2}{48\pi^2}\ln\left(\frac{q^2}{\mu^2}\right)\right],\end{aligned} \tag{4.36}$$

thus recovering the well-known result

$$\widehat{\Delta}^{-1}(q^2)=q^2\left[1+bg^2\ln\left(\frac{q^2}{\mu^2}\right)\right], \tag{4.37}$$

[8] From Tables 4.1 and 4.2, one sees that the dimensions of the gluon antifield A^* and background source Ω are, respectively, 3 and 1; then simple power counting shows that the (logarithmically) divergent part of $\Gamma_{\Omega^a_\alpha A^{*d}_\gamma}(q)$ can be proportional to $g_{\alpha\gamma}$ only, whereas the longitudinal form factor $L(q^2)$ is ultraviolet finite.

[9] Recall that this is a fundamental property of the BFM gluon self-energy, valid for every value of the (quantum) gauge-fixing parameter [12].

where b is the usual one-loop beta function coefficient of Eq. (1.69). As we will see in Chapter 6, Eq. (4.35) plays a pivotal role in the derivation of a new set of QCD Schwinger–Dyson equations [13, 14] that can be truncated in a manifestly gauge-invariant way [15].

Other background-quantum identities involving, e.g., the quark-gluon vertex and the three-gluon vertex read

$$\begin{aligned} \mathrm{i}\Gamma_{\widehat{A}^a_\alpha \psi\bar{\psi}}(p_2,-p_1) &= [\mathrm{i}g^\gamma_\alpha \delta^{ad} + \Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)]\Gamma_{A^d_\gamma \psi\bar{\psi}}(p_2,-p_1) \\ &\quad + \Gamma_{\psi^*\bar{\psi}\Omega^a_\alpha}(-p_1,q)\Gamma_{\psi\bar{\psi}}(p_2) \\ &\quad + \Gamma_{\psi\bar{\psi}}(p_1)\Gamma_{\psi\Omega^a_\alpha\bar{\psi}^*}(q,-p_1) \end{aligned} \tag{4.38}$$

$$\begin{aligned} \mathrm{i}\Gamma_{\widehat{A}^a_\alpha A^r_\rho A^s_\sigma}(p_2,-p_1) &= [\mathrm{i}g^\gamma_\alpha \delta^{ad} + \Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q)]\Gamma_{A^d_\gamma A^r_\rho A^s_\sigma}(p_2,-p_1) \\ &\quad + \Gamma_{\Omega^a_\alpha A^s_\sigma A^{*\gamma}_d}(-p_1,p_2)\Gamma_{A^d_\gamma A^r_\rho}(p_2) \\ &\quad + \Gamma_{\Omega^a_\alpha A^r_\rho A^{*\gamma}_d}(p_2,-p_1)\Gamma_{A^d_\gamma A^s_\sigma}(p_1). \end{aligned} \tag{4.39}$$

Notice first that the auxiliary function appearing in square brackets on the right-hand side (rhs) of the preceding identities is always $\Gamma_{\Omega A^*}$: this is at the root of the process independence of the PT algorithm. Second, observe that in more general identities other than the two-point one, the form factor $L(q^2)$ is also relevant.

4.2.3 Closed expressions for auxiliary functions

From the PT point of view, it would be not enough to be able to derive the Slavnov–Taylor identities and the background-quantum identities in the form given earlier. In fact, one is really striving for a formal link between the Slavnov–Taylor identities, which are triggered by the action of the longitudinal momenta, and the background-quantum identities, which relate Green's functions written in the conventional (R_ξ) and BFM gauges.

The key observation that makes this link possible is that one can always replace an antifield or BFM source with the corresponding BRST composite operator to which it is coupled. This means that we can use the replacements[10] (see Figure 4.2)

$$A^{*a}_\alpha(q) \to -\mathrm{i}\Gamma^{(0)}_{c^{e'} A^{n'}_{\nu'} A^{*a}_\alpha} \int_{k_1} \Delta^{\nu'\nu}_{n'n}(k_2) D^{e'e}(k_1)\cdots \tag{4.40}$$

$$\Omega^a_\alpha(q) \to -\mathrm{i}\Gamma^{(0)}_{\Omega^a_\alpha A^{n'}_{\nu'} \bar{c}^{e'}} \int_{k_1} \Delta^{\nu'\nu}_{n'n}(k_2) D^{e'e}(k_1)\cdots, \tag{4.41}$$

[10] For consistency with the definition (4.3), we use here (and later in Chapter 6) a definition of the full gluon propagator in which the rhs of Eq. (1.25) corresponds to $-\Delta_{\alpha\beta}$; this will not affect the inverse propagator, which will now be determined by the equation $\mathrm{i}\Delta_{\alpha\mu}(\Delta^{-1})^{\mu\beta} = g^\beta_\alpha$. Full gluon lines will then contribute a factor of $\mathrm{i}\Delta_{\alpha\beta}$ to the corresponding amplitude (ghost lines will contribute an $\mathrm{i}D$ factor).

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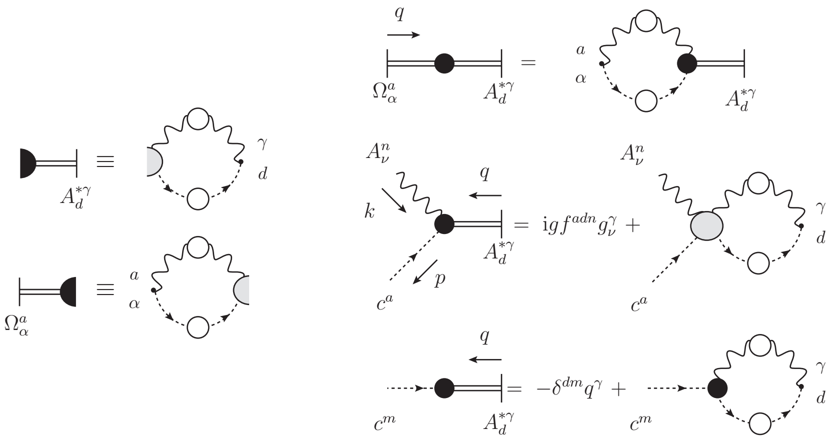

Figure 4.2. (*Left*) Expansion of the gluon antifield and BFM source in terms of the corresponding composite operators. Notice that if the antifield or the BFM sources are attached to a 1PI vertex, such an expansion will in general convert the 1PI vertex into a (connected) Schwinger–Dyson kernel. (*Right*) The corresponding expansion of the two-point function $\Gamma_{\Omega A^*}$ and the three-point function Γ_{cAA^*}.

to write, e.g.,[11]

$$-\Gamma_{c^m A^{*\gamma}_d}(q) = -\delta^{dm}q_\gamma + gf^{dne}\int_{k_1} D(k_1)\Delta^{\gamma\nu}(k_2)\Gamma_{c^m A^n_\nu \bar{c}^e}(k_2,k_1), \qquad (4.42)$$

$$\mathrm{i}\Gamma_{c^a A^n_\nu A^{*\gamma}_d}(k,q) = \mathrm{i}gf^{adn}g^\gamma_\nu - \mathrm{i}gf^{edr}\int_{k_1} D(k_1)\Delta^{\gamma\rho}(k_2)\mathcal{K}_{c^a A^n_\nu A^r_\rho \bar{c}^e}(k,k_2,k_1), \qquad (4.43)$$

$$-\Gamma_{\Omega^a_\alpha A^{*\gamma}_d}(q) = gf^{aen}\int_{k_1} D(k_1)\Delta^\nu_\alpha(k_2)\Gamma_{c^e A^n_\nu A^{*\gamma}_d}(k_2,-q). \qquad (4.44)$$

Equation (4.43) then shows explicitly the equivalence between Γ_{cAA^*} and the function H introduced earlier (compare also Figures 1.11 and 4.2).

The systematic use of this expansion to write closed expressions for the auxiliary functions appearing in the Slavnov–Taylor identities as well as in the background-quantum identities allows one to unveil a pattern that will be exploited when applying the pinch technique to the Schwinger–Dyson equations of QCD: the auxiliary functions appearing in the background-quantum identity satisfied by a particular Green's function can be written in terms of kernels appearing in the Slavnov–Taylor identities triggered when the PT procedure is applied to that same Green's function.

[11] The expansion of Eqs. (4.40) and (4.41) is, of course, not valid at tree level, which must be explicitly accounted for when present.

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4.2.4 A special case: The (background) Landau gauge

When choosing to quantize the theory in the background Landau gauge $(\widehat{\mathcal{D}}^{\mu} A_{\mu})^m = 0$, a new local equation (called the *antighost equation*) appears [16]:

$$\frac{\delta \Gamma}{\delta c^m} - \left(\widehat{\mathcal{D}}^{\mu} \frac{\delta \Gamma}{\delta \Omega_{\mu}}\right)^m - \left(\mathcal{D}^{\mu} A^*_{\mu}\right)^m - f^{mnr} c^{*n} c^r + f^{mnr} \frac{\delta \Gamma}{\delta B^n} \bar{c}^r = 0. \quad (4.45)$$

This equation fully constrains the dynamics of the ghost field c and implies that the latter will not get an independent renormalization constant. To see this, let us differentiate Eq. (4.20) with respect to a ghost field and a background source to get (after a Fourier transform)

$$\begin{aligned} \Gamma_{c^m \bar{c}^n}(q) &= -\mathrm{i} q^{\nu} \Gamma_{c^m A^{*n}_{\nu}}(q) \\ \Gamma_{\bar{c}^n \Omega^m_{\mu}}(q) &= q_{\mu} \delta^{mn} - \mathrm{i} q^{\nu} \Gamma_{\Omega^m_{\mu} A^{*n}_{\nu}}(q). \end{aligned} \quad (4.46)$$

On the other hand, differentiating the antighost equation (4.45) with respect to a gluon antifield and an antighost, one gets

$$\begin{aligned} \Gamma_{c^m A^{*n}_{\nu}}(q) &= q_{\nu} \delta^{mn} - \mathrm{i} q^{\mu} \Gamma_{\Omega^m_{\mu} A^{*n}_{\nu}}(q) \\ \Gamma_{c^m \bar{c}^n}(q) &= -\mathrm{i} q^{\mu} \Gamma_{\bar{c}^a \Omega^m_{\mu}}(q). \end{aligned} \quad (4.47)$$

Next, contracting the first equation in Eq. (4.47) with q^{ν}, and making use of the first equation in Eq. (4.46), we see that the dynamics of the ghost sector are entirely captured by the $\Gamma_{\Omega_{\mu} A^*_{\nu}}$ auxiliary function because

$$\Gamma_{c\bar{c}}(q) = -\mathrm{i} q^2 - q^{\mu} q^{\nu} \Gamma_{\Omega_{\mu} A^*_{\nu}}(q). \quad (4.48)$$

Introducing the Lorentz decompositions

$$\Gamma_{cA^*_{\mu}}(q) = q_{\mu} C(q^2); \qquad \Gamma_{\bar{c}\Omega_{\mu}}(q) = q_{\mu} E(q^2), \quad (4.49)$$

we find that Eq. (4.48), together with the last equations of Eqs. (4.46) and (4.47), gives the identities [16, 17]

$$\begin{aligned} C(q^2) = E(q^2) \;&=\; F^{-1}(q^2) \\ F^{-1}(q^2) &= 1 + G(q^2) + L(q^2), \end{aligned} \quad (4.50)$$

where $F(q^2)$ is the so-called ghost dressing function (with $D(q^2) = \mathrm{i}F(q^2)/q^2$ being the ghost propagator).

In addition, in this gauge, one can prove that the form factor G coincides with the well-known Kugo–Ojima function $u(q^2)$ [8], defined (in Euclidean space) through the two-point composite operator function

$$\int \mathrm{d}^4 x \, \mathrm{e}^{-iq\cdot(x-y)} \langle T\left[\left(\mathcal{D}_{\mu} c\right)^m_x \left(\mathcal{D}_{\mu} \bar{c}\right)^n_y \right] \rangle = -\frac{q_{\mu} q_{\nu}}{q^2} \delta^{mn} + P_{\mu\nu}(q) \delta^{mn} u(q^2). \quad (4.51)$$

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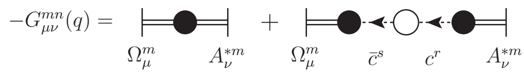

Figure 4.3. Connected components contributing to the function $\mathcal{G}^{mn}_{\mu\nu}(q)$.

In fact, in the background Landau gauge, the function appearing on the lhs of the preceding equation is precisely given by

$$-\,\mathcal{G}^{mn}_{\mu\nu}(q) = \frac{\delta^2 W}{\delta\Omega^m_\mu \delta A^{*n}_\nu}, \tag{4.52}$$

where W is the generator of the connected Green's functions and the two connected diagrams contributing to $\mathcal{G}_{\mu\nu}$ are shown in Figure 4.3. Factoring out the color structure and making use of the identities (4.50), one has

$$\begin{aligned}\mathcal{G}_{\mu\nu}(q) &= \Gamma_{\Omega_\mu A^*_\nu}(q) + \mathrm{i}\Gamma_{\Omega_\mu \bar{c}}(q) D(q^2)\Gamma_{A^*_\nu c}(q) \\ &= -\mathrm{i}\frac{q_\mu q_\nu}{q^2} + \mathrm{i}P_{\mu\nu}(q)G(q^2).\end{aligned} \tag{4.53}$$

Passing to the Euclidean formulation, and comparing with Eq. (4.51), we then arrive at the announced equality[12]

$$u(q^2) = G(q^2). \tag{4.54}$$

4.3 Pinching in the Batalin–Vilkovisky framework

It is now important to make contact between the PT algorithm and the Batalin-Vilkovisky formalism. This is, of course, best done at the one-loop level, where all calculations are straightforward and it is relatively easy to compare the standard diagrammatic results with those we will be finding. Not only will this comparison help us in identifying the pieces that will be generated when applying the PT algorithm but it will also be useful for establishing the rules to distribute them among the different Green's functions appearing in the calculation.

The starting point is the embedding of the (one-loop) gluon propagator into an S-matrix element (Figure 4.4), exactly as done in Chapter 1. Then, carrying out the PT decomposition $\Gamma = \Gamma^{\mathrm{P}} + \Gamma^{\mathrm{F}}$ on the tree-level three-gluon vertex of diagram (b), we get for the pinching part

$$(b)^{\mathrm{P}} = -gf^{amn}g^\nu_\alpha \int_{k_1} \frac{1}{k_1^2}\frac{1}{k_2^2} k_1^\mu \mathcal{K}^{(0)}_{A^m_\mu A^n_\nu \psi\bar{\psi}}(k_2, p_2, -p_1). \tag{4.55}$$

[12] Many of the results of this section turn out to be valid also in the conventional R_ξ Landau gauge [17, 18], where, however, only an integrated version of the ghost equation (4.45) is available [19].

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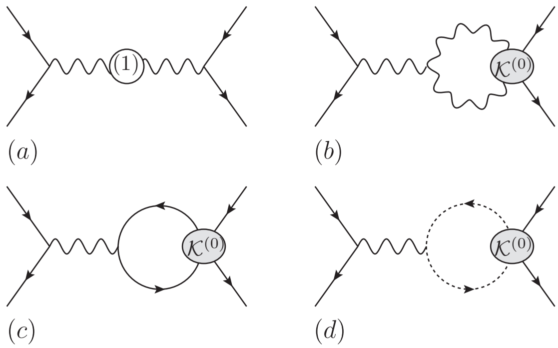

Figure 4.4. The S-matrix one-loop PT setting for constructing the gluon propagator.

On the other hand, observing that $\Gamma_{cA^*\psi\bar{\psi}}$ is zero at tree level, we find that the Slavnov–Taylor identity of Eq. (4.27) reduces to

$$\begin{aligned} k_1^\mu \mathcal{K}^{(0)}_{A^m_\mu A^n_\nu \psi\bar{\psi}}(k_2, p_2, -p_1) = &-gg^\gamma_\nu f^{dmn}\Gamma^{(0)}_{A^d_\gamma \psi\bar{\psi}}(p_2, -p_1) \\ &+ \Gamma^{(0)}_{\psi\bar{\psi}}(p_1)\mathcal{K}^{(0)}_{A^n_\nu \psi c^m \bar{\psi}^*}(p_2, k_1, -p_1) \\ &+ \mathcal{K}^{(0)}_{A^n_\nu \psi^* \bar{\psi} c^m}(p_2, -p_1, k_1)\Gamma^{(0)}_{\psi\bar{\psi}}(p_2). \end{aligned} \tag{4.56}$$

Now, notice that when the external legs are on shell, the last two terms of the preceding Slavnov–Taylor identity drop out by virtue of the quark equations of motion; thus, making use of Eq. (4.44), we are left with the final result

$$\begin{aligned} (b)^{\mathrm{P}} &= g^2 C_A \delta^{ad} g^\gamma_\alpha \int_{k_1} \frac{1}{k_1^2}\frac{1}{k_2^2}\Gamma^{(0)}_{A^d_\gamma \psi\bar{\psi}}(p_2, -p_1) \\ &= -\Gamma^{(1)}_{\Omega^a_\alpha A^{*\gamma}_d}(-q)\Gamma^{(0)}_{A^d_\gamma \psi\bar{\psi}}(p_2, -p_1). \end{aligned} \tag{4.57}$$

At this point, the calculation is over, and one needs to reshuffle the pieces generated. On one hand, to get the PT (on-shell) quark-gluon vertex, one adds to the Abelian diagram (c) the Γ^{F} part of diagram (b); thus one is left with the combination $(b) + (c) - (b)^{\mathrm{P}}$ or

$$\mathrm{i}\widehat{\Gamma}^{(1)}_{A^a_\alpha \psi\bar{\psi}}(p_2, -p_1) = \mathrm{i}\Gamma^{(1)}_{A^a_\alpha \psi\bar{\psi}}(p_2, -p_1) + \Gamma^{(1)}_{\Omega^a_\alpha A^{*\gamma}_d}(-q)\Gamma^{(0)}_{A^d_\gamma \psi\bar{\psi}}(p_2, -p_1). \tag{4.58}$$

On the other hand, the PT self-energy will be given by adding to the diagram (a) twice the pinching contribution $(b)^{\mathrm{P}}$ (one for each vertex), i.e.,

$$\widehat{\Pi}^{(1)}_{\alpha\beta}(q) = \Pi^{(1)}_{\alpha\beta}(q) + 2\mathrm{i}\Gamma^{(1)}_{\Omega^a_\alpha A^{*\gamma}_d}(q)\Gamma^{(0)}_{A^d_\gamma A^b_\beta}(q). \tag{4.59}$$

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The comparison of the PT Green's functions with those of the background Feynman gauge is now immediate by virtue of the background-quantum identities. Equation (4.58) represents the one-loop (on-shell) version of the background-quantum identity (4.38), and, recalling that $-\Gamma_{A^m_\mu A^n_\nu} = \delta^{mn}\Pi_{\mu\nu}$, we find that Eq. (4.59) correctly reproduces the background-quantum identity (4.33). Thus we have (once again) proved the PT background Feynman gauge correspondence at one loop.

The procedure just described goes through almost unaltered when choosing the external legs of the embedding process to be gluons, rendering the (one-loop) proof of the PT process's independence effortless.

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