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Amicable orthogonal designs

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A powerful tool in the construction of orthogonal designs has been amicable orthogonal designs. Recent results in the construction of Hadamard matrices has led to the need to find amicable orthogonal designs A, B in order n and of types $(u_1, u_2, \ldots, u_s)$ and $(v_1, v_2, \ldots, v_r)$ respectively satisfying $A^t = -A$, $B^t = B$, and $AB^t = BA^t$ with

$$\sum_{i=1}^{s} u_i = n - 1 \quad \text{and} \quad \sum_{i=1}^{r} v_i = n .$$

For simplicity, we say A, B are amicable orthogonal designs of type $(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_r)$.

We completely answer the question in order 8 by showing $(1, 2, 2, 2; 8)$, $(1, 2, 4; 2, 2, 4)$, $(2, 2, 3; 2, 6)$, $(7, 1, 7)$ and those designs derived from the above are the only possible.

We use our results to obtain new orthogonal designs in order 32.

1. Introduction

DEFINITION. Two orthogonal designs, A, B, of the same order, are called *amicable orthogonal designs* if $AB^t = BA^t$.

In this paper we will be interested in amicable orthogonal designs A, B in order 8, and of types $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_r)$

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respectively, satisfying $A^t = -A$ and $B^t = B$ with $\sum_{i=1}^{s} u_i = n - 1$ and $\sum_{i=1}^{r} v_i = n$. We will say these are of type $(u_1, \ldots, u_s; v_1, \ldots, v_r)$.

In [4] Wolfe gives restrictions on the number of variables in the designs A and B. The following information is taken from Table 3 of [4]:

Number of variables in B	Maximum number of variables in A
1	4
2	3
3	3
4	3
> 4	0

For easy reference we summarize the main results of this paper.

The following amicable designs exist:

$$(1, 2, 2, 2; 8), \quad (2, 2, 3; 2, 6),$$
$$(1, 2, 4; 2, 2, 4), \quad (7; 1, 7),$$

and the following do not exist:

$$(1, 1, 5; 8), \quad (a, b; 1, 7), \quad a + b = 7, \quad a, b \neq 0.$$
$$(1, 3, 3; 8), \quad (7; 2, 2, 2, 2),$$
$$(7; 5), \quad (7; 1, 1, 6),$$
$$(2, 2, 3; 4, 4),$$

These results, together with Wolfe's results, completely answer the problem in order 8.

2. Some amicable orthogonal designs

The following lemmas can be used to construct all the designs of the required type in order 8.

For simplicity we replace -1 by $-$ and $-x_i$ by $\bar{x}_i$.

LEMMA 1. *There are amicable orthogonal designs of type* $(1, 2, 2, 2; 8)$ *in order* 8.

Proof. The following pair A, B are amicable orthogonal designs of

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type $(1, 2, 2, 2; 8)$:

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_2 & x_4 & x_3 & x_4 \\ \bar{x}_1 & 0 & x_3 & \bar{x}_2 & x_4 & \bar{x}_2 & x_4 & \bar{x}_3 \\ \bar{x}_2 & \bar{x}_3 & 0 & x_1 & x_3 & \bar{x}_4 & \bar{x}_2 & x_4 \\ \bar{x}_3 & x_2 & \bar{x}_1 & 0 & \bar{x}_4 & \bar{x}_3 & x_4 & x_2 \\ \bar{x}_2 & \bar{x}_4 & \bar{x}_3 & x_4 & 0 & x_1 & x_2 & \bar{x}_3 \\ \bar{x}_4 & x_2 & x_4 & x_3 & \bar{x}_1 & 0 & \bar{x}_3 & \bar{x}_2 \\ \bar{x}_3 & \bar{x}_4 & x_2 & \bar{x}_4 & \bar{x}_2 & x_3 & 0 & x_1 \\ \bar{x}_4 & x_3 & \bar{x}_4 & \bar{x}_2 & x_3 & x_2 & \bar{x}_1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & - & 1 & - & - & 1 & - \\ 1 & - & 1 & 1 & - & 1 & - & - \\ - & 1 & - & - & - & 1 & - & - \\ 1 & 1 & - & 1 & 1 & 1 & - & 1 \\ - & - & - & 1 & 1 & 1 & 1 & - \\ - & 1 & 1 & 1 & 1 & - & - & - \\ 1 & - & - & - & 1 & - & - & - \\ - & - & - & 1 & - & - & - & 1 \end{bmatrix}.$$

LEMMA 2. *There are amicable orthogonal designs of type* $(1, 2, 4; 2, 2, 4)$ *in order* 8 .

Proof. Let

$$X = \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

Put

$$A = \begin{bmatrix} x_1X & x_2Y & x_3Y & x_3Y \\ \bar{x}_2Y & x_1X & x_3Y & \bar{x}_3Y \\ \bar{x}_3Y & \bar{x}_3Y & x_1X & x_2Y \\ \bar{x}_3Y & x_3Y & \bar{x}_2Y & x_1X \end{bmatrix}$$

and

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$$B = \begin{bmatrix} y_1Y & y_2Y & y_3Y & y_3Y \\ y_2Y & \bar{y}_1Y & y_3Y & \bar{y}_3Y \\ y_3Y & y_3Y & \bar{y}_2Y & \bar{y}_1Y \\ y_3Y & \bar{y}_3Y & \bar{y}_1Y & y_2Y \end{bmatrix} ;$$

then the pair A, B are amicable orthogonal designs of the required form.

LEMMA 3. *There are amicable orthogonal designs of type* $(2, 2, 3; 2, 6)$.

Proof. Put

$$A = \begin{bmatrix} 0 & x_2 & x_3 & x_3 & x_3 & x_2 & \bar{x}_1 & \bar{x}_1 \\ \bar{x}_2 & 0 & x_3 & \bar{x}_3 & x_2 & \bar{x}_3 & \bar{x}_1 & x_1 \\ \bar{x}_3 & \bar{x}_3 & 0 & x_2 & \bar{x}_1 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{x}_3 & x_3 & \bar{x}_2 & 0 & \bar{x}_1 & x_1 & \bar{x}_3 & x_2 \\ \bar{x}_3 & \bar{x}_2 & x_1 & x_1 & 0 & x_2 & x_3 & x_3 \\ \bar{x}_2 & x_3 & x_1 & \bar{x}_1 & \bar{x}_2 & 0 & x_3 & \bar{x}_3 \\ x_1 & x_1 & x_2 & x_3 & \bar{x}_3 & \bar{x}_3 & 0 & x_2 \\ x_1 & \bar{x}_1 & x_3 & \bar{x}_2 & \bar{x}_3 & x_3 & \bar{x}_2 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} y_2 & y_1 & \bar{y}_2 & \bar{y}_2 & \bar{y}_2 & y_1 & \bar{y}_2 & \bar{y}_2 \\ y_1 & \bar{y}_2 & y_2 & \bar{y}_2 & \bar{y}_1 & \bar{y}_2 & \bar{y}_2 & y_2 \\ \bar{y}_2 & y_2 & \bar{y}_1 & y_2 & \bar{y}_2 & \bar{y}_2 & \bar{y}_2 & y_1 \\ \bar{y}_2 & \bar{y}_2 & y_2 & y_1 & \bar{y}_2 & y_2 & \bar{y}_1 & \bar{y}_2 \\ \bar{y}_2 & \bar{y}_1 & \bar{y}_2 & \bar{y}_2 & \bar{y}_2 & y_1 & y_2 & y_2 \\ y_1 & \bar{y}_2 & \bar{y}_2 & y_2 & y_1 & y_2 & \bar{y}_2 & y_2 \\ \bar{y}_2 & \bar{y}_2 & \bar{y}_2 & \bar{y}_1 & y_2 & \bar{y}_2 & \bar{y}_1 & \bar{y}_2 \\ \bar{y}_2 & y_2 & y_1 & \bar{y}_2 & y_2 & y_2 & \bar{y}_2 & y_1 \end{bmatrix} ;$$

then the pair A, B are amicable orthogonal designs of the required form.

LEMMA 4. *If there is a pair of amicable orthogonal designs in order* n *and of types* $(1, u_1, u_2, \ldots, u_s)$ *and* $(v_1, v_2, \ldots, v_t)$ *, then there are amicable orthogonal designs of type*

$$(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) .$$

Proof. Let X be the design of type $(1, u_1, u_2, \ldots, u_s)$, in the variables $(x_0, x_1, \ldots, x_s)$, and Y be the design of type $(v_1, \ldots, v_t)$. We can find matrices P and Q with $PP^t = QQ^t = I$ such that

$$PXQ = x_0 I + A$$

and

$$PYQ = B ,$$

where A is an orthogonal design of type $(u_1, u_2, \ldots, u_s)$ and B is a design of type $(v_1, v_2, \ldots, v_t)$. Since

$$XX^t = \left(x_0^2 + \sum_{i=1}^{s} u_i^2 x_i^2\right) I ,$$

then A is skew. We also have $XY^t = YX^t$, and hence B is symmetric and $AB^t = BA^t$.

Therefore, A, B are amicable orthogonal designs of type $(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t)$.

LEMMA 5 (Wallis [3]). *Let* $q \equiv 3 \pmod 4$ *be a prime power. Then there exists a pair of amicable orthogonal designs of order* $q + 1$ *and both of type* $(1, q)$.

The above two lemmas give the following result.

COROLLARY 6. *There are amicable orthogonal designs of type* $(7; 1, 7)$ *in order* 8.

Other amicable orthogonal designs can be constructed from the above designs by equating variables.

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3. Non-existence results

THEOREM. *There are no amicable orthogonal designs of type* $(7; 5)$ *in order* 8 .

In order to prove this theorem, we need the following two lemmas.

LEMMA 7. *If* a *and* b *are* ± 1 , *then*

(1) $a + b \equiv ab + 1 \pmod 4$,

(2) $-a - b \equiv a + b \pmod 4$.

LEMMA 8. *Let*

$$\mathcal{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{B}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix} .$$

If B *is a symmetric* $(0, 1, -1)$ *matrix in order* 8 *such that* $BB^t = 5I$, *then we can find a monomial matrix,* R , *such that*

$$RBR^t = \begin{bmatrix} P & \mathcal{B} \\ \mathcal{B} & Q \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathcal{B}_1 & I \\ I & -\mathcal{B}_1 \end{bmatrix}$$

where P *and* Q *are monomials and* $\mathcal{B}_1 = \mathcal{B}$ *or* $\mathcal{B}'$.

Proof. Let row i of B be $b_{i1}, b_{i2}, \ldots, b_{i8}$ and define

$$O_B(i, j) = \sum_{k=1}^{8} |b_{ik}||b_{jk}| .$$

Since $BB^T = 0$, we have $O_B(i, j) = 2$ or 4 for all i, j , $i \neq j$.

If $O_B(i, j) = 4$ for at most one j , then any column containing a zero from row i has at least six ± 1's . Therefore, for all i , there exists j_1 and j_2 such tnat

$$(*) \qquad O_B(i, j_1) = O_B(i, j_2) = 4 .$$

We also note that $O_B(j_1, j_2) = 4$ for the j_1 and j_2 given in (*).

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Now we define an equivalence relation, $\sim$, on the rows of B as follows:

$$\text{row } i \sim \text{row } j \text{ if and only if } O_B(i, j) = 4 ,$$

and consider the equivalence classes of $\sim$.

Since each equivalence class contains at least three rows, it can be seen that there are at most two equivalences classes, each with 4 rows.

If all the rows were in the same equivalence class then $O_B(i, j) = 4$ for all i and j , which is clearly impossible.

We now consider a permutation matrix R_1, such that the first four rows of R, B are in the same equivalence class. Now let $R_1BR_1^t = B_1$.

Clearly, the first four rows of B_1 are in the same equivalence class and B_1 is symmetric, and hence it can be shown that

$$B_1 = \begin{bmatrix} A_1 & A_2 \\ A_2^t & A_3 \end{bmatrix} ,$$

where either A_1 and A_3 are symmetric Hadamard matrices in order 4 and A_2 is a monomial matrix, or A_1 and A_3 are symmetric monomial matrices and A_2 is an Hadamard matrix.

If A_2 is an Hadamard matrix, then there exist monomial matrices D and D' such that $DA_2D'^t = \mathcal{B}$.

Now let

$$R_2 = \begin{bmatrix} D & 0 \\ 0 & D' \end{bmatrix} ,$$

and therefore

$$R_2B_1R_2^t = \begin{bmatrix} P & \mathcal{B} \\ \mathcal{B} & Q \end{bmatrix} .$$

Hence

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$$RBR^t = \begin{bmatrix} P & B \\ B & Q \end{bmatrix},$$

where $R = R_2R_1$. Now, if A_1 and A_3 are symmetric Hadamard matrices, then there exist monomial matrices C and D such that

$$CA_1C^t = B_1 \quad\text{and}\quad CA_2D^t = I .$$

Now, let

$$R_2' = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix};$$

then

$$\begin{aligned} R_2'B_1R_2^t &= \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2^t & A_3 \end{bmatrix} \begin{bmatrix} C^t & 0 \\ 0 & D^t \end{bmatrix} \\ &= \begin{bmatrix} B_1 & I \\ I & DA_3D^t \end{bmatrix}. \end{aligned}$$

But $BB^t = 0$; so $DA_3D^t = -B_1$. Therefore, on putting $R = R_2'R_1$, we have the required result.

Proof of Theorem. Assume there exists amicable orthogonal designs, A and B, of type $(7; 5)$.

We may assume B is one of the forms given in Lemma 8.

Firstly we assume

$$B = \begin{bmatrix} P & B \\ B & Q \end{bmatrix}.$$

We can further assume

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix} \quad\text{or}\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \quad\text{or}\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix},$$

where $a = \pm 1$, since, given any other P, either no Q can be found, or

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we can apply permutations to B which leave B fixed but transform P into one of the above forms.

Assume

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix}$$

and let the first five rows of A be

$$\begin{matrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \bar{a}_1 & 0 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ \bar{a}_2 & \bar{b}_2 & 0 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & 0 & d_4 & d_5 & d_6 & d_7 \\ \bar{a}_4 & \bar{b}_4 & \bar{c}_4 & \bar{d}_4 & 0 & e_5 & e_6 & e_7 \end{matrix} .$$

For (A, B) to be amicable orthogonal designs, AB must be symmetric.

Consider positions $(1, 4)$ and $(4, 1)$ in AB.

$$aa_2 + a_4 - a_5 - a_6 + a_7 = -a_3 + d_4 + d_5 + d_6 + d_7 .$$

Hence, by Lemma 7,

(1) $$aa_2a_3a_4a_5a_6a_7d_4d_5d_6d_7 = -1 .$$

But $-a_1b_3 - a_2c_3 + a_4d_4 + a_5d_5 + a_6d_6 + a_7d_7 = 0$ (by the orthogonality of A); that is

(2) $$a_1a_2a_4a_5a_6a_7b_3c_3d_4d_5d_6d_7 = -1 .$$

On multiplying (1) and (2) we get

(3) $$aa_1a_3b_3c_3 = 1 .$$

Now we consider positions $(2, 4)$ and $(4, 2)$ of AB. By reasoning as above, we obtain $aa_1a_3b_3c_3 = -1$ which contradicts (3).

By using similar reasoning to that of the above case, it can be shown that none of the possible B's can be used to produce amicable orthogonal designs of type $(7; 5)$ in order 8.

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LEMMA 9. *There are no amicable orthogonal designs of type* $(1, 1, 5; 8)$.

Proof. Let A, B be amicable orthogonal designs of type $(1, 1, 5; 8)$ and let A be the $(1, 1, 5)$ design in variables (x_1, x_2, x_3).

By applying various permutations on A (and B) we can assume the top left hand 4×4 block of A is

$$\begin{matrix} 0 & x_1 & x_2 & x_3 \\ \bar{x}_1 & 0 & x_3 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 & 0 & x_1 \\ \bar{x}_3 & x_2 & \bar{x}_1 & 0 \end{matrix} .$$

Let this block, with $x_3 = 0$, be Y and let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^t & B_3 \end{bmatrix}$$

with B_1 and B_3 symmetric.

In order that A, B are amicable orthogonal designs YB_1 must be symmetric.

It is easy to show, however, that no such B_1 exists. Therefore, there are no amicable orthogonal designs of type $(1, 1, 5; 8)$.

The remaining results in this section will not be proved here. The proofs are longer and more involved but use the same type of reasoning as described in the above proofs. We summarize these results in the following lemma.

LEMMA 10. *There are no amicable orthogonal designs of types*

$$\begin{array}{ll} & (a, b; 1, 7), \quad a + b = 7, \quad a, b \neq 0. \\ (1, 3, 3; 8), & (7; 2, 2, 2, 2), \\ (2, 2, 3; 4, 4), & (7; 1, 1, 6), \end{array}$$

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4. Applications

In Section 2 we gave amicable orthogonal designs of type $(2, 2, 3; 2, 6)$ in order 8. By using these designs in Theorem 9 of Geramita and Wallis [1], we obtain an orthogonal design of type $(2, 3, 3, 3, 3, 6, 6, 6)$ in order 32 which gives $(3, 3, 3, 3, 20)$, $(3, 3, 6, 9, 11)$, and $(2, 3, 9, 9, 9)$ designs in order 32.

In [2] we constructed a $(1, 1, 1, 1, 1, 1, 1, 1, 8)$ and a $(1, 1, 1, 1, 1, 1, 5, 5)$ design in order 16 which give a $(1, 1, 1, 1, 2, 2, 2, 2, 16)$ and a $(1, 1, 1, 1, 2, 2, 10, 10)$ design in order 32. Hence we can construct designs of type $(1, 5, 5, 17)$, $(1, 5, 11, 11)$, and $(3, 9, 9, 9)$ in order 32. Hence we have

LEMMA 11. *In order* 32,

(i) all 5-*tuples*, $(a, b, c, d, 32-a-b-c-d)$, $0 \le a+b+c+d \le 32$, *are the types of orthogonal designs except possibly*

$(1, 3, 9, 9, 10)$,	$(1, 4, 5, 5, 17)$,
$(1, 3, 6, 11, 11)$,	$(1, 5, 6, 9, 11)$,
$(1, 5, 5, 5, 16)$,	$(1, 4, 5, 11, 11)$,
$(1, 5, 5, 10, 11)$,	$(3, 3, 4, 11, 11)$;

(ii) all 4-*tuples*, $(a, b, c, 32-a-b-c)$, $0 \le a+b+c \le 32$ *are the types of orthogonal designs in order* 32;

(iii) all 3-*tuples*, 2-*tuples*, *and* 1-*tuples are the types of orthogonal designs.*

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