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ON COHEN-MACAULAY AND GORENSTEIN SIMPLICIAL AFFINE SEMIGROUPS

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We give arithmetic characterizations which allow us to determine algorithmically when the semigroup ring associated to a simplicial affine semigroup is Cohen-Macaulay and/or Gorenstein. These characterizations are then used to provide information about presentations of this kind of semigroup and, in particular, to obtain bounds for the cardinality of their minimal presentations. Finally, we show that these bounds are reached for semigroups with maximal codimension.

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Introduction

Given a semigroup S, one can consider the semigroup ring $K[S] = \bigoplus_{s \in S} K y_s$. Some properties of $K[S]$ can be characterized in terms of S, such as being a Cohen-Macaulay ring or a Gorenstein ring. If the semigroup S is affine (a finitely generated subsemigroup of $\mathbb{N}^k$ for some positive integer k), some work has been developed in this line, see for example [**4**, **2**, **11**, **5**]. In [**11**] a characterization of Cohen-Macaulayness (and Gorensteiness) of semigroup rings is given in terms of a property of the semigroup and a property of the extended homology of a certain simplicial complex related to the semigroup S.

In numerical semigroups, every element can be written in an unique way as a sum of an element in the Apéry set associated to a generator of the semigroup and a multiple of the aforementioned generator (see [**1**]). In this paper, we show that a similar condition characterizes Cohen-Macaulayness of $K[S]$, provided that S is simplicial. Instead of taking the Apéry set of a given generator (which is infinite in general) we consider the intersection of the Apéry sets of the extremal rays of the given semigroup. This characterization allows us to give an algorithmic method to decide if $K[S]$ is Cohen-Macaulay when S is a simplicial affine semigroup.

Another feature is that a numerical semigroup is symmetric (Gorenstein) if and only if the Apéry set associated to an element of the semigroup has a maximum (with respect to the ordering $s \leq s'$ if and only if $s' - s \in S$). In this paper, we show that a simplicial affine semigroup is Gorenstein if and only if it is Cohen-Macaulay and the intersection of the Apéry sets of the extremal rays has a maximum. Thus, we have an algorithmic way to check if $K[S]$ is Gorenstein when S is a simplicial affine semigroup.

The above-mentioned properties of the intersection of the Apéry sets of the extremal rays of a given simplicial affine semigroup, together with the fact that this set is finite, is used to generalize the algorithm given in [7] to compute a minimal system of generators for the congruence associated to a Cohen-Macaulay semigroup. This construction is then used to give bounds for the number of elements of such a system of generators for the Cohen-Macaulay and Gorenstein case. These bounds are reached and characterize the so-called Cohen-Macaulay and Gorenstein simplicial affine semigroups with maximal codimension, generalizing in this sense the results obtained for numerical semigroups in [8], [9].

1. How to know if a simplicial affine semigroup is Cohen-Macaulay

Let S be the semigroup of $\mathbb{N}^r$ generated by $A = \{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$, with $r > 1$ and $m \geq 1$. The semigroup S is *simplicial* if $r = \dim(S)$ and $\mathrm{L}_{\mathbb{Q}^+}(S) = \mathrm{L}_{\mathbb{Q}^+}(\{n_1, \ldots, n_r\})$, where $\mathrm{L}_{\mathbb{Q}^+}(B)$ is the set $\{\sum_{i=1}^n q_i b_i | q_i \in \mathbb{Q}^+, b_i \in B\}$. Assume that A is a minimal system of generators for S (i.e. $n_k \notin \langle A \backslash \{n_k\} \rangle$, for all $k \in \{1, \ldots, r+m\}$). The natural number m is called the *codimension* of S.

We say that S is Cohen-Macaulay (Gorenstein) if $K[S]$ is Cohen-Macaulay (Gorenstein), where $K[S]$ is the S-graded ring $\bigoplus_{s \in S} K y_s$ (see [3]).

It is possible to translate Cohen-Macaulayness of $K[S]$ in terms of properties of S. The same holds for Gorensteiness. This problem has been studied by S. Goto, N. Suzuki and K. Watanabe in [2]; by N. V. Trung and L. T. Hoa in [11]; and by Y. Kamoi in [5]. We are going to use two results given by these authors for the special case of simplicial affine semigroups which connect Cohen-Macaulayness and Gorensteiness of $K[S]$ with properties of S.

We define the i-th face of $\mathrm{L}_{\mathbb{Q}^+}(S)$ as $F_i = \mathrm{L}_{\mathbb{Q}^+}(\{n_1, \ldots, n_r\} \backslash \{n_i\})$. We define also the set $S_i = S - (S \cap F_i)$, that is, the set of elements in $\mathfrak{G}(S)$ (the group generated by S) that are the difference of an element in S and an element in the i-th face of the cone which is also in S. We define the set $G_{[1,r]} = \mathfrak{G}(S) \backslash \bigcup_{i=1}^r S_i$. The following theorem gives different conditions to check whether a simplicial affine semigroup is Cohen-Macaulay.

Theorem 1.1. *Under the above hypothesis, the following conditions are equivalent:*

(i) *$K[S]$ is Cohen-Macaulay;*

(ii) *for any $\alpha, \beta \in S$ with $\alpha + n_i = \beta + n_j (1 \leq i \neq j \leq r)$, $\alpha - n_j = \beta - n_i \in S$;*

(iii) *for any $\alpha \in \mathbb{N}^k$, if $\alpha - n_i \in S$ and $\alpha - n_j \in S$, then $\alpha - (n_i + n_j) \in S (1 \leq i \neq j \leq r)$;*

(iv) $S = \bigcap_{i=1}^r S_i$.

Proof. The equivalence between (i), (ii) and (iv) is given in [11]. The equivalence between (ii) and (iii) is trivial. □

A version for the Gorenstein case can be found in [11, Corollary 4.4].

Theorem 1.2. *Let S be a simplicial affine semigroup. Then the following conditions are equivalent:*

(i) *$K[S]$ is Gorenstein;*

(ii) *there exists $g \in \mathfrak{G}(S)$ such that $g - S = G_{\{1,r\}}$.*

These results translate properties in the semigroup ring $K[S]$ to properties of the semigroup S. In this paper we give a method to check if the semigroup S fulfils these conditions. In order to do that, we still need some background.

Definition 1.3. Given $n \in S - \{0\}$ we define the set

$$\mathrm{S}(n) = \{s \in S : s - n \notin S\}.$$

These sets are called the Apéry-sets (see [1]). In the case of numerical semigroups, these sets are finite, but in general they are infinite. For simplicial affine semigroups, the set $\bigcap_{i=1}^{r} \mathrm{S}(n_i)$ is finite. Note that, since A is a minimal system of generators for S, $\{0, n_{r+1}, \ldots, n_{r+m}\} \subseteq \bigcap_{i=1}^{r} \mathrm{S}(n_i)$. This set is going to play a very important role in the present paper.

Let us see that $\bigcap_{i=1}^{r} \mathrm{S}(n_i)$ is finite, and let us give a bound for its cardinality.

Since S is simplicial, then for each $i \in \{1, \ldots, m\}$, there exists the following natural number:

$$c_{r+i} = \min\{k \in \mathbb{N} - \{0\} : kn_{r+i} \in \langle n_1, n_2, \ldots, n_r \rangle\}.$$

Note also that these numbers can be easily computed. We can see n_i as a vector in $\mathbb{Q}^r$, for all $i \in \{1, \ldots, r+m\}$. Given $\{n_{i_1}, \ldots, n_{i_r}\} \subseteq \{n_1, \ldots, n_{r+m}\}$, let us denote the determinant of $\{n_{i_1}, \ldots, n_{i_r}\}$ by $\det(n_{i_1}, \ldots, n_{i_r})$. Since $\{n_1, \ldots, n_r\}$ is a basis of $\mathbb{Q}^r$, then for each $i \in \{1, \ldots, m\}$, we can find $\lambda_{i_1}, \ldots, \lambda_{i_r}$ such that

$$n_{r+i} = \sum_{j=1}^{r} \lambda_{i_j} n_j,$$

where the λ_{i_j}'s are exactly

$$\lambda_{i_j} = \frac{\det(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r)}{\det(n_1, \ldots, n_r)}.$$

Using this information, it is easy to show that

$$c_{r+i} = \frac{\det(n_1, \ldots, n_r)}{\gcd\{\det(n_1, \ldots, n_{j-1}, n_{r+i}, n_{j+1}, \ldots, n_r) : j \in \{1, \ldots, r\}\}},$$

where gcd denotes greatest common divisor.

We define the set

$$\Gamma = \left\{ \sum_{i=1}^{m} \gamma_{r+i} n_{r+i} : \gamma_{r+i} < c_{r+i} \textit{ for all } i \in \{1, \ldots, m\} \right\}.$$

If $n \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$, then n trivially belongs to Γ, and therefore $\# \bigcap_{i=1}^{r} \mathrm{S}(n_i) \leq \#\Gamma$. Note also that this property allows us to compute the elements of $\bigcap_{i=1}^{r} \mathrm{S}(n_i)$, because we only have to check among a finite number of elements of S whether they belong to $\bigcap_{i=1}^{r} \mathrm{S}(n_i)$ or not, and this is easy to check.

Now, we show that every element in a Cohen-Macaulay simplicial affine semigroup can be written in a "unique" way. This property will allow us to give a method to characterize Cohen-Macaulay simplicial affine semigroups.

Lemma 1.4. *Let $S = \langle n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m} \rangle \subset \mathbb{N}^r$ be an affine simplicial semigroup. Then every element $s \in S$ can be written as*

$$s = \sum_{i=1}^{r} a_i n_i + x,$$

where $x \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$ and $a_i \in \mathbb{N}$ for all $i \in \{1, \ldots, r\}$.

Proof. Take $s \in S$. If $s \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$, then we are done. Otherwise, there exists $i \in \{1, \ldots, r\}$ such that $s \notin \mathrm{S}(n_i)$. This means that $s_1 = s - n_i \in S$. If $s_1 \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$, then we have that $s = n_i + s_1$; otherwise there exists $j \in \{1, \ldots, r\}$ such that $s_1 \notin \mathrm{S}(n_j)$. Take $s_2 = s_1 - n_j$. Once we have s_k, we can construct s_{k+1} in a similar way. Note that this process must stop because we cannot have an infinite descending chain of elements in $\mathbb{N}^r$ ($s_{i+1} < s_i$, with $<$ the usual partial ordering in $\mathbb{N}^r$). Hence, there must be $l \in \mathbb{N}$ such that $s_l \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$. Passing all the n_i's to the other side we have what we want. □

Note that the last result holds also for affine semigroups not being simplicial. If the given semigroup is Cohen-Macaulay, then the expression appearing in the previous lemma is unique, for every element in S. Furthermore, the reverse is also true.

Theorem 1.5. *Under the above hypothesis, the following statements are equivalent:*

(i) *$K[S]$ is Cohen-Macaulay;*

(ii) *for every $s \in S$ such that*

$$s = \sum_{i=1}^{r} a_i n_i + x = \sum_{i=1}^{r} b_i n_i + y$$

with $x, y \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$, and $a_i, b_i \in \mathbb{N}$ for all $i \in \{1, \ldots, r\}$, we have that $a_i = b_i$ for all $i \in \{1, \ldots, r\}$ and $x = y$.

Proof. Assume that $K[S]$ is Cohen-Macaulay and that the second statement is false. Take α the least element in S (with respect to an ordering in $\mathbb{N}^r$ compatible with the addition) verifying

$$\alpha = \sum_{i=1}^{r} a_i n_i + x = \sum_{i=1}^{r} b_i n_i + y$$

with $a_i \neq b_i$ for some i or $x \neq y$. Note that not all a_i's and b_i's can be zero, because this would mean that $y = x$ and $a_i = b_i$ for all i. Hence, there must be an i such that $a_i \neq 0$. Note also that not all the b_j's can be zero, because this would lead to the fact that $y \notin \bigcap_{i=1}^{r} S(n_i)$. Thus, there exists j such that $b_j \neq 0$. (By the minimality of α, we have that $a_j = b_i = 0$.) This means that $\alpha - n_i \in S$ and $\alpha - n_j \in S$ which, by 1.1, leads to $\alpha - (n_i + n_j) \in S$. By the previous lemma, we have that there exist $c_1, \ldots, c_r \in \mathbb{N}$ and $z \in \bigcap_{i=1}^{r} S(n_i)$ such that $\alpha - (n_i + n_j) = \sum_{i=1}^{r} c_i n_i + z$. Therefore, we have the equality

$$\alpha' = \alpha - n_i = a_1 n_1 + \cdots + (a_i - 1)n_i + \cdots + a_r n_r + x = c_1 n_1 + \cdots + (c_j + 1)n_j + \cdots + c_r n_r + z,$$

with $a_j = 0 \neq (c_j + 1)$, which is a contradiction with the fact that α was the minimum verifying this.

Assume now that the second statement is true and let us show that $K[S]$ is Cohen-Macaulay. Take $\alpha \in S$ such that $\alpha - n_i \in S$ and $\alpha - n_j \in S$. By 1.1, it is enough to prove that $\alpha - (n_i + n_j) \in S$. Using the previous lemma, we have that

$$\alpha - n_i = \sum_{k=1}^{r} a_k n_k + x$$

$$\alpha - n_j = \sum_{k=1}^{r} b_k n_k + y$$

and therefore

$$a_i n_1 + \cdots + (a_i + 1)n_i + \cdots + a_r n_r + x = b_1 n_1 + \cdots + (b_j + 1)n_j + \cdots + b_r n_r + y.$$

Using the hypothesis we get $a_i + 1 = b_i$, which means that $b_i \neq 0$. Hence,

$$\alpha - (n_j + n_i) = b_1 n_1 + \cdots + (b_i - 1)n_i + \cdots + b_r n_r + y \in S. \qquad \square$$

Corollary 1.6. *Under the above hypothesis, the following statements are equivalent:*

(i) *$K[S]$ is Cohen-Macaulay,*

(ii) *for all $x, y \in \bigcap_{i=1}^{r} S(n_i)$, if $x \neq y$, then $x - y \notin \mathfrak{G}(\{n_1, \ldots, n_r\})$.*

Since we can compute the equations of $\mathfrak{G}(\{n_1, \ldots, n_r\})$ and the set $\bigcap_{i=1}^r \mathrm{S}(n_i)$, this corollary gives an algorithmic way to check if the semigroup ring of a given affine simplicial semigroup is Cohen-Macaulay.

Corollary 1.7. *If S is a Cohen-Macaulay simplicial affine semigroup, then*

(i) $\mathfrak{G}(S) = \{\sum_{i=1}^r z_r n_r + x \mid z_i \in \mathbb{Z}, x \in \bigcap_{i=1}^r \mathrm{S}(n_i)\}$;

(ii) *every element in $\mathfrak{G}(S)$ is equal to an unique expression of the form $z_1 n_1 + \cdots + z_r n_r + x$ with $z_i \in \mathbb{Z}$ and $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$;*

(iii) *the element $z_1 n_1 + \cdots + z_r n_r + x$ with $z_i \in \mathbb{Z}$ and $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ is in S if and only if $z_i \geq 0$ for all i.*

Proof. Let $A = \{\sum_{i=1}^r z_r n_r + x \mid z_i \in \mathbb{Z}, x \in \bigcap_{i=1}^r \mathrm{S}(n_i)\}$. For every $i \in \{1, \ldots, m\}$, we have that $n_{r+i} \in \mathrm{L}_{\mathbb{Q}^+}(\{n_1, \ldots, n_r\})$, and therefore there exists $c_{r+i} \in \mathbb{N}$ such that $c_{r+i} n_{r+i} \in \langle n_1, \ldots, n_r \rangle$.

Take $g \in \mathfrak{G}(S)$; then there must be $s, s' \in S$ such that $g = s - s'$. By 1.4, we have that $s = a_1 n_1 + \cdots + a_r n_r + x$ and $s' = a'_1 n_1 + \cdots + a'_r n_r + x'$, with $a_i, a'_i \in \mathbb{N}$ and $x, x' \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. To show $\mathfrak{G}(S) = A$, it is enough to show that $x - x' \in A$, for every $x, x' \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. Since $x - x' \in \mathfrak{G}(S)$, there must be $z_1, \ldots, z_{r+m}$ such that $x - x' = z_1 n_1 + \cdots + z_r n_r + z_{r+1} n_{r+1} + \cdots + z_{r+m} n_{r+m}$. For every $i \in \{1, \ldots, m\}$, take $q_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ such that $z_{r+i} = q_i c_{r+i} + b_i$. Thus, $x - x'$ can be written as $x - x' = z'_1 n_1 + \cdots + z'_r n_r + b_1 n_{r+1} + \cdots + b_m n_{r+m}$, with $z'_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}$. Note that $b_1 n_{r+1} + \cdots + b_m n_{r+m} \in S$, and by 1.4, we have that there exist $d_1, \ldots, d_r \in \mathbb{N}$ and $y \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ such that $b_1 n_{r+1} + \cdots + b_m n_{r+m} = d_1 n_1 + \cdots + d_r n_r + y$. Putting all together we get $x - x' = z''_1 n_1 + \cdots + z''_r n_r + y$, with $z''_i \in \mathbb{Z}$.

The second statement is a direct consequence of 1.5 and the third statement is a consequence of 1.4 and 1.5. □

2. On relations of Cohen-Macaulay simplicial affine semigroups

In this section we give, using the fact that $\bigcap_{i=1}^r \mathrm{S}(n_i) \subseteq \Gamma$, a bound for the number of elements in a minimal system of generators for the congruence associated to a Cohen-Macaulay simplicial affine semigroup. After that, we sharpen this bound using some extra results.

Let φ be the map defined in the following way:

$$\varphi : \mathbb{N}^{r+m} \to S$$

$$\varphi(a_1, \ldots, a_{r+m}) = \sum_{i=1}^{r+m} a_i n_i.$$

Let us denote the kernel congruence of φ by σ. Then S is isomorphic to $\mathbb{N}^{r+m}/\sigma$.

We say that ρ is a *minimal system of generators for* σ if ρ generates σ and its cardinal is minimal among the cardinal of the sets generating σ. It can be shown that $\#\rho \geq r+m-r=m$ (see [3]).

Definition 2.1. Let $n \in S-\{0\}$. We define the graph G_n as the graph whose vertices are

$$V(G_n)=\{n_i : n-n_i \in S, i \in \{1,\ldots,r+m\}\},$$

and whose edges are

$$E(G_n)=\{\overline{n_i n_j} : n-(n_i+n_j) \in S, i,j \in \{1,\ldots,r+m\}, i \neq j\}.$$

A minimal system of generators, ρ, for σ can be constructed in the following way (which follows from a straightforward generalization of the results given in [7] and is presented in [10]). For any $n \in S$, define ρ_n as follows:

1. If G_n is connected, then $\rho_n=\emptyset$;
2. If G_n is not connected and $G_n^1,\ldots,G_n^t$ are the connected components of G_n, then choose a vertex $n_{j_i} \in V(G_n^i)$ and an element $\alpha_1^n=(a_1^i,\ldots,a_{r+m}^i) \in \mathbb{N}^{r+m}$ such that $\varphi(\alpha_i^n)=n$ and $a_{j_i}^i \neq 0$; define
$$\rho_n=\{(\alpha_2^n,\alpha_1^n),\ldots,(\alpha_t^n,\alpha_1^n)\};$$
3. Take $\rho=\bigcup_{n\in S}\rho_n$.

Lemma 2.2. *If S is Cohen-Macaulay, then the elements in $\{n_1,\ldots,n_r\}\cap V(G_n)$ are all in the same connected component of G_n.*

Proof. This result is a consequence of 1.1. □

This result is the main idea used in this paper to generalize the results achieved for subsemigroups of $\mathbb{N}$ in [7].

Lemma 2.3. *Under the above hypothesis, if G_n is not connected, then there exist $j \in \{1,\ldots,m\}$ and $s \in \bigcap_{i=1}^r S(n_i)$ such that*

$$n=s+n_{r+j}.$$

Proof. Assume that for all $n_{r+k} \in V(G_n)\backslash\{n_1,\ldots,n_r\}$, there exists $i \in \{1,\ldots,r\}$ such that $n-n_{r+k} \notin S(n_i)$. This would mean that $n-(n_{r+k}+n_i) \in S$, and therefore n_{r+k} would be in the same connected component as $\{n_1,\ldots,n_r\}\cap V(G_n)$. As a consequence of this fact, G_n would be connected, a contradiction. Thus, there exists $j \in \{1,\ldots,m\}$ such that $n-n_{r+j} \in \bigcap_{i=1}^r S(n_i)$. This concludes the proof. □

Thus, a bound for $\#\rho$ could be $m^2\prod_{i=1}^{m} c_{r+i}$ (there are at most $m\prod_{i=1}^{m} c_{r+i}$ non-connected G_n and each of them has at most m connected components).

In order to find a better bound for $\#\rho$, we need some technical lemmas.

Lemma 2.4. *Let S be a Cohen-Macaulay simplicial affine semigroup and $n \in S$ be such that G_n is not connected. Then there exist a number $k > r$ and an element $s \in \bigcap_{j=1}^{k-1} \mathrm{S}(n_j)$ such that:*

(i) $n = n_k + s$;

(ii) $n \notin \bigcap_{j=1}^{k-1} \mathrm{S}(n_j)$;

(iii) *for all $s' \in S$, $s' \neq s$ and $s \notin \mathrm{S}(s')$, $s' + n_k \in \bigcap_{j=1}^{k-1} \mathrm{S}(n_j)$.*

Proof. Let n_i be such that $i = \min\{j : n_j \in \mathrm{V}(G_n)\}$. Since G_n has more than one connected component, we can choose another vertex of G_n which is not in the same connected component of n_i. Let C_i be the connected component of G_n which contains n_i. Choose n_k so that $k = \min\{j : n_j \in \mathrm{V}(G_n)\backslash C_i\}$. Since S is Cohen-Macaulay, then if $\{n_1, \ldots, n_r\} \cap \mathrm{V}(G_n) \neq \emptyset$, we have that $i \leq r$ and therefore $k > r$, because n_k is in a connected component different from the one of $\{n_1, \ldots, n_r\} \cap \mathrm{V}(G_n)$; if $\{n_1, \ldots, n_r\} \cap \mathrm{V}(G_n) = \emptyset$, then $i > r$ and so is k.

Since $n_k \in G_n$, then $s = n - n_k \in S$. Moreover, for any $j < k$, $\overline{n_j n_k}$ is not an edge of G_n, so $s - n_j = n - (n_j + n_k) \notin S$ and therefore $s \in \bigcap_{j=1}^{k-1} \mathrm{S}(n_j)$. This proves (i) and we also have (ii), since $n - n_i \in S$.

Now, suppose that $s' \in S$ is such that $s \neq s'$ and $s \notin \mathrm{S}(s')$. Then $0 \neq s - s' \in S$ and so there is a generator n_t such that $s - s' - n_t \in S$. This generator, n_t, is a vertex of G_n, since $n - n_t = (s - s' - n_t) + s' + n_k \in S$. Moreover, it is clear, from the above expression, that $n - (n_t + n_k) \in S$. Therefore, n_t and n_k are in the same connected component of G_n. If there exists $j \leq k - 1$ such that $s' + n_k \notin \mathrm{S}(n_j)$ (i.e. $s' + n_k - n_j \in S$), then we have that $n - n_j = (s - s' - n_t) + n_t + (s' + n_k - n_j) \in S$. Thus, n_j is a vertex of G_n which is in the same connected component that n_t (notice that $n - (n_j + n_t) \in S$). We conclude that n_j and n_k are in the same connected component; but $j < k$, which is a contradiction. □

Note that the first statement of the previous lemma sharpens the result given in 2.3.

Lemma 2.5. *For any $r + 1 \leq k \leq r + m$, let D_k be the set*

$$D_k = \{S \in \bigcap_{j=1}^{k-1} \mathrm{S}(n_j) : s + n_k \notin \bigcap_{j=1}^{k-1} \mathrm{S}(n_j) \text{ and for all } s' \in S, s' \neq s$$

$$\text{with } s \notin \mathrm{S}(s'), s' + n_k \in \bigcap_{j=1}^{k-1} \mathrm{S}(n_j)\}$$

and let $\dot{\bigcup}_{k=r+1}^{r+m} D_k$ *be the disjoint union of the sets* D_k. *Then there is an injective map* $i : \rho \to \dot{\bigcup}_{k=r+1}^{r+m} D_k$.

Proof. For every $n \in S$, let $G_n^1, \ldots, G_n^{t_n}$ be the connected components of G_n. We can assume that G_n^1 contains the vertex of G_n with the lowest index and $t_n > 2$. Recall that ρ_n is of the form

$$\rho_n = \{(\alpha_2^n, \alpha_1^n), \ldots, (\alpha_{t_n}^n, \alpha_1^n)\},$$

where each α_i^n is associated to G_n^i. Let us define, for each n and each (α_i^n, α_1^n), $i \geq 2$, the element $i(\alpha_i^n, \alpha_1^n)$. Take $n_{j_i} \in G_n^i$ such that $j_i = \min\{k : n_k \in V(G_n^i)\}$. Similar to the proof of the previous lemma, it is not difficult to show that $n - n_{j_i} \in D_{j_i}$. We define $i(\alpha_i^n, \alpha_1^n) = n - n_{j_i}$.

Assume now that $i(a, b) = i(a', b')$. There must be a natural number k such that $i(a, b) = i(a', b') \in D_k$, and therefore $\varphi(a) - n_k = i(a, b) = i(a', b') = \varphi(a') - n_k$. Thus, $\varphi(a) = \varphi(a')$. Hence, (a, b) and (a', b') are both in $\rho_{\varphi(a)}$. This implies that $b = b' = \alpha_1^{\varphi(a)}$. If $a \neq a'$, then their corresponding connected components are different and therefore $i(a, b)$ and $i(a', b')$ belong to different D_k's, which is a contradiction. □

Putting all together, we get the following theorem.

Theorem 2.6. *The cardinality of a minimal system of generators for* σ *is less than or equal to*

$$\frac{(2d - m)(m - 1)}{2} + 1,$$

where $d = \# \bigcap_{i=1}^{r} \mathrm{S}(n_i)$.

Proof. Using the previous lemma, we only have to compute the number of elements in D_k for $r + 1 \leq k \leq r + m$. Let c be the least natural number such that $cn_{r+m} \in \langle n_1, \ldots, n_{r+m-1} \rangle$ (note that $c \leq c_{r+m}$). Then, it can easily be shown that

$$\bigcap_{i=1}^{r+m-1} \mathrm{S}(n_i) = \{0, n_{r+m}, \ldots, (c - 1)n_{r+m}\}.$$

Hence, the only element in D_{r+m} is $(c - 1)n_{r+m}$, and therefore $\# D_{r+m} = 1$.

For any $r + 1 \leq k \leq r + m - 1$,

$$D_k \subseteq \bigcap_{i=1}^{k-1} \mathrm{S}(n_i) \backslash \{0\} \subseteq \bigcap_{i=1}^{r} \mathrm{S}(n_i) \backslash \{0, n_{r+1}, \ldots, n_{k-1}\}.$$

Since $\{0, n_{r+1}, \ldots, n_{k-1}\} \subseteq \bigcap_{i=1}^r S(n_i)$, $\#D_{r+i} \le d - i$ for all $i \in \{1, \ldots, m-1\}$, and therefore, by 2.5,

$$\#\rho \le 1 + \sum_{i=1}^{m-1}(d-i) = \frac{(2d-m)(m-1)}{2} + 1. \qquad \square$$

Note that this bound is reached if $m = 1$. In this case, the bound in 2.6 is 1, and therefore, on the one hand, 2.6 ensures $\#\rho \le 1$ and, on the other hand, the results given in [3] ensure that $\#\rho \ge 1 = m$. Thus $\#\rho = 1$. Note that these semigroups are a complete intersection and therefore they are all Cohen-Macaulay.

Another consequence of this result and 1.6 is the following. Let us consider the quotient group $\mathbb{Z}^r/\mathfrak{G}(\{n_1, \ldots, n_r\})$. Each equivalence class has a representative in the "box" whose vertices are $\sum_{i=1}^r \varepsilon_i n_i$, with $\varepsilon_i \in \{0, 1\}$ for all i. Note also that, since for all $x, y \in \bigcap_{i=1}^r S(n_i)$, with $x \neq y$, we have that $x - y \notin \mathfrak{G}(\{n_1, \ldots, n_r\})$, the representatives of $[x]$ and $[y]$ in the mentioned box must be different. Since there are $\det(n_1, \ldots, n_r)$ elements in the box, we have that $\#\bigcap_{i=1}^r S(n_i) \le \det(n_1, \ldots, n_r)$. This gives the following bound for the cardinality of a minimal relation for S:

$$\frac{(2\det(n_1, \ldots, n_r) - m)(m-1)}{2} + 1.$$

There are semigroups for which $\#\bigcap_{i=1}^r S(n_i) = \det(n_1, \ldots, n_r)$, as the following example shows:

Example 2.7. Take $r = m = 2$ and $S = \langle (2,0), (0,4), (1,2), (1,1)\rangle$. It is not hard to show that

$$\bigcap_{i=1}^r S(n_i) = \{(0,0), (1,2), (1,1), (2,2), (3,3), (2,3), (3,4), (4,5)\},$$

and therefore $\#\bigcap_{i=1}^r S(n_i) = \det((2,0),(0,4))$. Note that, by 1.6, $K[S]$ is Cohen-Macaulay.

In the next section we are going to prove that in the case $m = d - 1$, the bound is also reached.

3. Cohen-Macaulay affine simplicial semigroups with maximal codimension

Let $S = \langle n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\rangle \subset \mathbb{N}^r$ be a simplicial affine semigroup with $\bigcap_{i=1}^r S(n_i) = \{0, x_1, \ldots, x_{d-1}\}$. Note that m must be less or equal than $d-1$. We say that S has *maximum codimension* when $m = d - 1$. For this kind of semigroup it is very easy to compute a minimal relation as the next theorem shows:

Theorem 3.1. *Let S be a Cohen-Macaulay simplicial affine semigroup with maximal codimension. Then $\#\rho = d(d-1)/2$.*

Proof. If $m = d - 1$, then the bound given by 2.6 is $\#\rho \leq \frac{d(d-1)}{2}$. Recall that there are as many elements in ρ_n as connected components of G_n minus one. Thus, what we have to do is to count the number of connected components of each G_n. Observe that:

1. Since $m = d - 1$, then

$$\bigcap_{i=1}^{r} \mathrm{S}(n_i) = \{0, n_{r+1}, \dots, n_{r+m}\}.$$

2. If G_n is not connected, then, by 2.3, there exist $s \in \bigcap_{i=1}^{r} \mathrm{S}(n_i)$ (note that $s \neq 0$) and $j \in \{1, \dots, m\}$ such that $n = s + n_{r+j}$. Hence, $n = n_{r+i} + n_{r+j}$, for some $1 \leq i, j \leq m$.
3. If $n = n_{r+i} + n_{r+j}$, $1 \leq i, j \leq m$, then $n = n_{r+i} + n_{r+j} \notin \{0, n_{r+1}, \dots, n_{r+m}\} = \bigcap_{k=1}^{r} \mathrm{S}(n_k)$. Thus, there exists $k \in \{1, \dots, r\}$ such that $n \notin \mathrm{S}(n_k)$ and therefore $\{n_1, \dots, n_r\} \cap \mathrm{V}(G_n) \neq \emptyset$.
4. If $n = n_{r+i} + n_{r+j}$, $1 \leq i, j \leq m$, then
   (a) If $i \neq j$ then $\{n_{r+i}, n_{r+j}\}$ is a connected component of G_n. This is because $n_{r+i}, n_{r+j} \in \mathrm{V}(G_n)$, and if $n - (n_{r+i} + n_k) \in S$ with $k \neq r + j$ (which would mean that there is an edge between n_{r+i} and n_k), then $n_{r+j} - n_k \in S$, which is not possible.
   (b) If $i = j$ then, reasoning as in the previous case, $\{n_{r+i}\}$ is a connected component of G_n.

Taking all of this into account, we have that there are as many elements in ρ_n as possible expressions of the form $n_{r+i} + n_{r+j} = n$, $1 \leq i, j \leq m$. Thus, there are at least $d(d-1)/2$ elements in ρ. □

Let us see that the reverse is also true.

Theorem 3.2. *Let ρ be a minimal relation for S. If $\#\rho = d(d-1)/2$, then S has maximal codimension.*

Proof. By 2.6, $\#\rho \leq (2d - m)(m - 1)/2 + 1$. Thus $d(d-1)/2 \leq (2d - m)(m - 1)/2 + 1$. Making some computations, this leads to $m^2 - (2d + 1)m + d^2 + d - 2 \leq 0$, which implies that $m \in [d - 1, d + 2]$, and since $m \leq d - 1$ we have that $m = d - 1$. □

The next theorem shows how to construct a Cohen-Macaulay simplicial affine semigroup with maximal codimension from an arbitrary Cohen-Macaulay simplicial affine semigroup.

Theorem 3.3. *Let* $\overline{S} = \langle n_1, \dots, n_r, n_1 + x_1, \dots, n_1 + x_{d-1}\rangle$. *If* S *is Cohen-Macaulay, then the following statements hold.*

(i) *The set* $\{n_1, \dots, n_r, n_1 + x_1, \dots, n_1 + x_{d-1}\}$ *is a minimal set of generators of* $\overline{S}$.

(ii) *The semigroup* $\overline{S}$ *is simplicial.*

(iii) *The semigroup* $\overline{S}$ *verifies*

$$\bigcap_{i=1}^{r} \overline{S}(n_i) = \{0, n_1 + x_1, \dots, n_1 + x_{d-1}\}.$$

(iv) *The semigroup* $\overline{S}$ *is Cohen-Macaulay with maximal codimension.*

Proof. (i) Assume that

$$n_1 + x_{d-1} = a_1 n_1 + \cdots + a_r n_r + b_1(n_1 + x_1) + \cdots + b_{d-1}(n_1 + x_{d-1});$$

then

$$n_1 + x_{d-1} = (a_1 + b_1 + \cdots + b_{d-1})n_1 + a_2 n_2 + \cdots + a_r n_r + b_1 x_1 + \cdots + b_{d-1} x_{d-1}.$$

There are two possible cases:

(a) $a_1 + b_1 + \cdots + b_{d-1} = 0$. Then $n_1 + x_{d-1} = a_2 n_2 + \cdots + a_r n_r$. Since $n_1 + x_{d-1} \neq 0$, there must exist i such that $a_i \neq 0$. This means that $(n_1 + x_{d-1}) - n_1 \in S$ and $(n_1 + x_{d-1}) - n_i \in S$ and, by 1.1, $(n_1 + x_{d-1}) - (n_1 + n_i) = x_{d-1} - n_i \in S$, which is a contradiction with the fact $x_{d-1} \in \bigcap_{i=1}^{r} S(n_i)$.

(b) $a_1 + b_1 + \cdots + b_{d-1} \neq 0$. Since $x_{d-1} \in \bigcap_{i=1}^{r} S(n_i)$, $a_1 + b_1 + \cdots + b_{d-1} = 1$ and $a_2 = \cdots = a_r = 0$. Two subcases must be considered:

  (i) $a_1 = 1$ and $b_i = 0$ for all i. This leads to $n_1 + x_{d-1} = n_1$, a contradiction.

  (ii) $a_1 = 0$, $b_i = 1$ for some i and $b_j = 0$ for $j \neq i$. This leads to $n_1 + x_{d-1} = n_1 + x_i$, which is again a contradiction.

This proves that $n_1 + x_{d-1}$ cannot be written as a linear combination with natural coefficients of the rest of the generators of $\overline{S}$. For the generators of the form $n_1 + x_i$ the proof is analogous, and it is clear that n_i, for $i \in \{1, \dots, r\}$, cannot be written as a linear combination of the rest of the generators.

(ii) Trivial.

(iii) Since $\{n_1, \dots, n_r, n_1 + x_1, \dots, n_1 + x_{d-1}\}$ is a minimal system of generators, $\{0, n_1 + x_1, \dots, n_1 + x_{d-1}\} \subseteq \bigcap_{i=1}^{r} \overline{S}(n_i)$. In order to show the other inclusion, let us prove that every element $s \in \overline{S} \backslash \langle n_1, \dots, n_r\rangle$ can be written as

$$s = a_1 n_1 + \cdots + a_r n_r + (n_1 + x_i),$$

for some i. Take $s \in \overline{S} \setminus \langle n_1, \ldots, n_r \rangle$, then

$$s = b_1 n_1 + \cdots + b_r n_r + c_1(n_1 + x_1) + \cdots + c_{d-1}(n_1 + x_{d-1})$$

with $c_k \neq 0$ for some k. Since $c_1 x_1 + \cdots + c_{d-1} x_{d-1} \in S$, then there exist $e_1, \ldots, e_r \in \mathbb{N}$ and j such that $c_1 x_1 + \cdots + c_{d-1} x_{d-1} = e_1 n_1 + \cdots + e_r n_r + x_j$. Therefore,

$$s = (b_1 + c_1 + \cdots + c_k - 1 + \cdots + c_{d-1} + e_1)n_1 + (b_2 + e_2)n_2 + \cdots + (b_r + e_r)n_r + (x_j + n_1).$$

(iv) This is trivial using 1.6 and $\bigcap_{i=1}^r \overline{S}(n_i) = \{0, n_1 + x_1, \ldots, n_1 + x_{d-1}\}$. □

We can repeat this process as many times as we want, each time adding a different generator in $\{n_1, \ldots, n_r\}$. Then we have the following result:

Corollary 3.4. *Let $b \in \langle n_1, \ldots, n_r \rangle$, $b \neq 0$, and $\tilde{S} = \langle n_1, \ldots, n_r, b + x_1, \ldots, b + x_{d-1} \rangle$. If S is Cohen-Macaulay, then $\tilde{S}$ is a simplicial Cohen-Macaulay affine semigroup with maximal codimension.*

Next, we show how to construct some other Cohen-Macaulay simplicial affine semigroups with maximal codimension.

Let $n_i = k_i e_i$ where $k_i \in \mathbb{N}$ and $e_i \in \mathbb{N}^r$ has all its coordinates equal to zero but the i-th coordinate, which is equal to one. Let $<$ be a total degree ordering in $\mathbb{N}^r$ compatible with the addition. Let $\rho = \{(n_i, 0) : i \in \{1, \ldots, r\}\}$ and let $\sigma = \langle \rho \rangle$ be the congruence generated by ρ. We can define the map

$$\mu : \mathbb{N}^r/\sigma \to \mathbb{N}^r$$

$$\mu([n]) = \min_{<}[n]$$

It is easy to prove the following lemma (see [**6**]).

Lemma 3.5. (i) *The set ρ is a canonical ("good") system of generators for σ.*

(ii) *For every $n \in \mathbb{N}^r$ there exist $a_1, \ldots, a_r \in \mathbb{N}$ such that $n = \mu([n]) + \sum_{i=1}^r a_i n_i$.*

(iii) $\operatorname{Im}(\mu)$ *is the set of elements included in the box whose vertices are the points $\sum_{i=1}^r \varepsilon_i n_i$ with $\varepsilon_i \in \{0, 1\}$, removing the faces not containing the zero element.*

(iv) *The element $(x, y) \in \sigma$ if and only if $x - y \in \mathfrak{G}(\{n_1, \ldots, n_r\})$.*

The next definition and lemma give a useful way to check if a given set is $\bigcap_{i=1}^r S(n_i)$.

Definition 3.6. Let H be a subgroup of $\mathbb{Z}^r$ and let $\{x_1, \ldots, x_p\} \subset \mathbb{Z}^r$. The set $\{x_1, \ldots, x_p\}$ is a *complete system modulo H* if the following two conditions hold:

(i) For all $i, j \in \{1, \ldots, p\}$, if $x_i - x_j \in H$ then $i = j$;

(ii) For all $i, j \in \{1, \ldots, p\}$ there exists $k \in \{1, \ldots, p\}$ such that $x_i + x_j - x_k \in H$.

Lemma 3.7. *Let $\{n_1, \ldots, n_r\} \subset \mathbb{N}^r$, $H = \mathfrak{G}(\{n_1, \ldots, n_r\})$, and let $\{x_0 = 0, x_1, x_2, \ldots, x_{d-1}\}$ be a complete system modulo H. Assume that $S = \langle n_1, \ldots, n_r, x_1, \ldots, x_{d-1}\rangle \subseteq \mathbb{N}^r$ is a simplicial affine semigroup as usual. Then, the following conditions are equivalent:*

(i) $\bigcap_{i=1}^r S(n_i) = \{0 = x_0, x_1, \ldots, x_{d-1}\}$;

(ii) *For all $i, j \in \{0, 1, \ldots, d-1\}$ there exist $k \in \{0, 1, \ldots, d-1\}$ and $(a_1, \ldots, a_r) \in \mathbb{N}^r$ such that $x_i + x_j = \sum_{i=1}^r a_i n_i + x_k$.*

Proof. Let us assume that $\bigcap_{i=1}^r S(n_i) = \{0 = x_0, x_1, \ldots, x_{d-1}\}$. Since $x_i + x_j \in S$, then by 1.4 there exists $(a_1, \ldots, a_r) \in \mathbb{N}^r$ such that $x_i + x_j = \sum_{i=1}^r a_i n_i + x_k$.

Assume now that the second statement holds. Let us show that $\{0 = x_0, x_1, \ldots, x_{d-1}\} \subseteq \bigcap_{i=1}^r S(n_i)$. If this were not true, then there would exist i, j such that $x_i - n_j \in S$. Hence, $x_i - n_j = \sum_{i=1}^r a_i n_i + \sum_{i=1}^{d-1} b_i x_i$. Applying several times the hypothesis we get $\sum_{i=1}^{d-1} b_i x_i = \sum_{i=1}^r c_i n_i + x_k$, and therefore $x_i - n_j = \sum_{i=1}^r (a_i + c_i) n_i + x_k$ which leads to $x_i - x_k \in H$. By hypothesis this means that $x_i = x_k$, and this is a contradiction, because $-n_j$ cannot be equal to $\sum_{i=1}^r (a_i + c_i) n_i$.

Now take $s \in \bigcap_{i=1}^r S(n_i)$. Then $s = \sum_{i=1}^{d-1} b_i x_i$. As before, we can apply several times the hypothesis, getting $s = \sum_{i=1}^r c_i n_i + x_k$ for some $(c_1, \ldots, c_r) \in \mathbb{N}^r$ and some $k \in \{0, 1, \ldots, d-1\}$. Since $s \in \bigcap_{i=1}^r S(n_i)$, we have that $c_i = 0$ for all i, and therefore $s = x_k$. This concludes the proof. $\square$

Proposition 3.8. *Let $\{0 = x_0, x_1, \ldots, x_{p-1}\} \subseteq \operatorname{Im}(\mu)$ be a complete system modulo $\mathfrak{G}(\{n_1, \ldots, n_r\})$. Then $S = \langle n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{p-1}\rangle$ is a Cohen-Macaulay simplicial affine semigroup with maximal codimension.*

Proof. It is enough to prove that $\bigcap_{i=1}^r S(n_i) = \{0, n_1 + x_1, \ldots, n_1 + x_{p-1}\}$ and that $\{n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{p-1}\}$ is a minimal system of generators for S.

In order to prove the first condition, let us use 3.7. Note that, since $\{0, x_1, \ldots, x_{p-1}\}$ is complete, then for all i, j there exists k such that $x_i + x_j - x_k \in \mathfrak{G}(\{n_1, \ldots, n_r\})$. Note also that $x_i + x_j - \mu([x_i + x_j]) \in \mathfrak{G}(\{n_1, \ldots, n_r\})$, and that $x_k \in \operatorname{Im}(\mu)$, which implies that $x_k = \mu([x_i + x_j])$. Using 3.5, we get that there exist $a_1, \ldots, a_r \in \mathbb{N}$ such that $x_i + x_j = x_k + \sum_{i=1}^r a_i n_i$. Thus, $(n_1 + x_i) + (n_1 + x_j) = 2n_1 + \sum_{i=1}^r a_i n_i + x_k = (n_1 + x_k) + (a_1 + 1)n_1 + \sum_{i=2}^r a_i n_i$.

Let us prove now that $\{n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{p-1}\}$ is a minimal system of generators for S. It is enough to prove that $n_1 + x_i$ cannot be expressed as $\sum_{j=1, j\neq i}^{p-1} b_j(n_1 + x_j)$. If there exists such an expression, then $\sum b_j$ must be greater than one, because $\{0, x_1, \ldots, x_{p-1}\}$ is a complete system modulo $\mathfrak{G}(\{n_1, \ldots, n_r\})$. Applying several times the same procedure used before, we get that there must be $c_1, \ldots, c_r$ such that $\sum b_j x_j = x_k + \sum c_i n_i$, for some $k \in \{0, \ldots, p-1\}$. Therefore, $x_i = x_k + y$, with $y \in \mathfrak{G}(\{n_1, \ldots, n_r\})$, and since $\{0, x_1, \ldots, x_{p-1}\}$ is a complete system modulo

$\mathfrak{G}(\{n_1, \ldots, n_r\})$, i must be equal to k and therefore $y = 0$, which leads to $c_i = 0$ for all i and $\sum b_j = 1$, which is a contradiction.

Finally, it is easy to check that $S = \langle n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{p-1} \rangle$ is Cohen-Macaulay using the definition of complete system modulo a group and 1.6. □

We can do the same for the rest of n_i's and get the same result as before. Using this fact and 3.4 we get the following corollary.

Corollary 3.9. *If $\{0, x_1, \ldots, x_{p-1}\} \subseteq \mathrm{Im}(\mu)$ is a complete system modulo $\mathfrak{G}(\{n_1, \ldots, n_r\})$, then $S = \langle n_1, \ldots, n_r, b + x_1, \ldots, b + x_{p-1} \rangle$ is a Cohen-Macaulay simplicial affine semigroup with maximal codimension for all $b \in \langle n_1, \ldots, n_r \rangle \backslash \{0\}$.*

Note also that, in particular, $\mathrm{Im}(\mu)$ is a complete system modulo $\mathfrak{G}(\{n_1, \ldots, n_r\})$.

4. How to know if a simplicial affine semigroup is Gorenstein

In this section, we are going to give a characterization for Gorenstein simplicial affine semigroups so that we are able to check algorithmically if a given simplicial affine semigroup is Gorenstein. We show that a simplicial affine semigroup, S, is Gorenstein if and only if it is Cohen-Macaulay and the set $\bigcap_{i=1}^r \mathrm{S}(n_i)$ has a maximum with respect to the ordering: $s \leq s'$, $s, s' \in S$ if and only if there exists $s'' \in S$ such that $s' = s + s''$. In the following, when we refer to an ordering concept, we always mean the ordering just described.

Once we get the aforementioned result, it will be easy to check if S is Gorenstein, because we can compute $\bigcap_{i=1}^r \mathrm{S}(n_i)$ and then see if it has a maximum and if S is Cohen-Macaulay. Note that this test is not as hard as it would seem to be at first glance, because '$s \leq s'$, $s, s' \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ then $s' = s + s''$ for some $s'' \in S$' forces s'' to be in $\bigcap_{i=1}^r \mathrm{S}(n_i)$.

Throughout this section, S denotes a Cohen-Macaulay simplicial affine semigroup.

Lemma 4.1. *The set $S \cap F_i$ is the set of elements of S having the i-th coordinate (with respect to the basis $\{n_1, \ldots, n_r\}$) equal to 0.*

Proof. Trivial. □

Lemma 4.2. *Consider $s \in S$ and $g \in \mathfrak{G}(S)$ such that $g - S = G_{[1,r]}$. Then $g + s \in S$ if and only if $s \notin \bigcup_{i=1}^r (S \cap F_i)$ (i.e. s is not in any face of the cone).*

Proof. Assume $g + s \in S$; then $-s = g - (g + s) \in G_{[1,r]}$, which means that $-s \notin \bigcup_{i=1}^r S_i$, and this implies that $s \notin \bigcup_{i=1}^r (S \cap F_i)$.

Assume now that $s \notin \bigcup_{i=1}^r (S \cap F_i)$. If $-s \in \bigcup_{i=1}^r S_i$, then $-s = s' - s_i$ for some $s' \in S$ and $s_i \in S \cap F_i$, for some $i \in \{1, \ldots, r\}$. Thus, $s_i = s + s'$ and therefore the i-th coordinate of s with respect to the basis $\{n_1, \ldots, n_r\}$ is zero. Using the previous result, we get that

$s \in S \cap F_i$, which is a contradiction. This shows that $-s \notin \bigcup_{i=1}^r S_i$, and therefore $-s \in G_{[1,r]} = g - S$. Hence, $-s = g - s'$ for some $s' \in S$, which means that $g + s = s' \in S$. □

Lemma 4.3. *Let $g \in \mathfrak{G}(S)$ be such that $g - S = G_{[1,r]}$. Then $g - x \in S$ for all $x \in G_{[1,r]}$.*

Proof. Take $x \in G_{[1,r]} = g - S$. Then there exists $s \in S$ such that $x = g - s$, and therefore $g - x = s \in S$. □

Lemma 4.4. *Let $g \in \mathfrak{G}(S)$ be such that $g - S = G_{[1,r]}$. Then $g + (n_1 + \cdots + n_r)$ is in the set of maximals of $\bigcap_{i=1}^r \mathrm{S}(n_i)$.*

Proof. First of all, note that $-(n_1 + \cdots + n_r) \in G_{[1,r]}$, otherwise there would be $s \in S$ and $s_i \in S \cap F_i$ such that $-(n_1 + \cdots + n_r) = s - s_i$, and therefore $s_i = n_1 + \cdots + n_r + s$. Hence s_i has all its coordinates bigger than zero, and this contradicts the fact that $s_i \in S \cap F_i$. By 4.3, we have $g + (n_1 + \cdots + n_r) \in S$.

Note also that, for every $i \in \{1, \ldots, r\}$, $(n_1 + \cdots + n_r) - n_i \in S \cap F_i$ and hence $n_i - (n_1 + \cdots + n_r) \in S_i$. This implies that $n_i - (n_1 + \cdots + n_r) \notin G_{[1,r]}$ for all $i \in \{1, \ldots, r\}$. Thus, $g - (n_i - (n_1 + \cdots + n_r)) = (g + (n_1 + \cdots + n_r)) - n_i \notin S$. This implies that $g + (n_1 + \cdots + n_r) \in \bigcap_{i=1}^r \mathrm{S}(n_i)$.

Let us show that $g + (n_1 + \cdots + n_r)$ belongs to the set of maximals of $\bigcap_{i=1}^r \mathrm{S}(n_i)$. For this purpose, let us show that $g + (n_1 + \cdots + n_r) + s$, with $s \neq 0$, does not belong to $\bigcap_{i=1}^r \mathrm{S}(n_i)$. We can compute the coordinates of the element $(n_1 + \cdots + n_r) + s$ with respect to the basis $\{n_1, \ldots, n_r\}$. Let $(x_1, \ldots, x_r)$ be these coordinates. It is clear that $x_i \geq 1$ for all i, and since $s \neq 0$, then there must exist $j \in \{1, \ldots, r\}$ such that $x_j > 1$. This means that the element $(n_1 + \ldots, n_r) + s - n_j$ has all its coordinates greater than zero, and therefore it does not belong to $S \cap F_i$ for any i. Using 4.2 we get that $g + (n_1 + \cdots + n_r) + s - n_j \in S$. Hence $g + (n_1 + \cdots + n_r) + s \notin S(n_j)$. This concludes the proof. □

Lemma 4.5. *Let $u = x - (n_1 + \cdots + n_r)$, with $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. Then $u \in G_{[1,r]}$.*

Proof. Assume that $u \notin G_{[1,r]}$. This means that $u \in S_i$ for some i. Without loss of generality, let us suppose that $u \in S_1$. Then, $u = s - s_1$ for some $s \in S$, $s_1 \in S \cap F_1$. Thus, $u + s_1 \in S$, which implies that $u + s_1 + s' \in S$ for all $s' \in S$. Since $s_1 \in F_1$, then there exists $k \in \mathbb{N}$ such that $ks_1 \in \langle n_2, \ldots, n_r \rangle$. Hence, $u + s_1 + (k-1)s_1 = u + (a_2 n_2 + \cdots + a_r n_r) \in S$ with $a_i \in \mathbb{N}$. Using 1.4 we get that $u + (a_2 n_2 + \cdots + a_r n_r) = b_1 n_1 + \cdots + b_r n_r + y$ with $b_i \in \mathbb{N}$ for all i and $y \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. Since $u = x - (n_1 + \cdots + n_r)$, we get that $a_2 n_2 + \cdots + a_r n_r + x = (b_1 + 1)n_1 + (b_2 + 1)n_2 + \cdots + (b_r + 1)n_r + y$ and by 1.5 we get that $x = y$ and $b_1 + 1 = 0$, which is a contradiction. □

Theorem 4.6. *If S is Gorenstein, then there is exactly one maximal element in the set $\bigcap_{i=1}^r \mathrm{S}(n_i)$.*

Proof. Since S is Gorenstein, $g - S = G_{[1,r]}$ for some $g \in \mathfrak{G}(S)$. By 4.4, we know that

$g+(n_1+\cdots+n_r)=x$ is a maximal element of $\bigcap_{i=1}^r \mathrm{S}(n_i)$. Take y another maximal element of $\bigcap_{i=1}^r \mathrm{S}(n_i)$. By the previous lemma we have that $y-(n_1+\cdots+n_r)\in G_{[1,r]}$, and therefore $y-(n_1+\cdots+n_r)\in g-S$. This implies $y+s=x$, for some $s\in S$. Since y is maximal, s must be the zero element and $x=y$. □

The reverse is also true, as the next results show.

Lemma 4.7.

$$G_{[1,r]}\subseteq\{z_1n_1+\cdots+z_rn_r+z\,|\,z_i<0,\, z\in\bigcap_{i=1}^r \mathrm{S}(n_i)\}.$$

Proof. By 1.7, we know that

$$G_{[1,r]}\subseteq \mathfrak{G}(S)=\{z_1n_1+\cdots+z_rn_r+x\,|\,z_i\in\mathbb{Z},\, x\in\bigcap_{i=1}^r \mathrm{S}(n_i)\}.$$

Take $z_1,\ldots,z_r\in\mathbb{Z}$, $x\in\bigcap_{i=1}^r \mathrm{S}(n_i)$, such that $z_1n_1+\cdots+z_rn_r+x\in G_{[1,r]}$. If $z_i\geq 0$ for some i, then $\sum_{z_j\geq 0} z_jn_j+x-(\sum_{z_j<0}-z_jn_j)\in S-(S\cap F_i)=S_i$, a contradiction. □

Theorem 4.8. *If S is Cohen-Macaulay and the set $\bigcap_{i=1}^r \mathrm{S}(n_i)$ has exactly one maximal element, then S is Gorenstein.*

Proof. Let x be the maximal element of $\bigcap_{i=1}^r \mathrm{S}(n_i)$. Define $g=x-(n_1+\cdots+n_r)$. Let us show that $g-S=G_{[1,r]}$.

Take an element $h\in G_{[1,r]}$. By the previous lemma, h can be written as $h=z_1n_1+\cdots+z_rn_r+z$, with $z_i\in\mathbb{Z}$, $z_i<0$, $z\in\bigcap_{i=1}^r \mathrm{S}(n_i)$. Since x is maximal in $\bigcap_{i=1}^r \mathrm{S}(n_i)$, we have that there must be an element $s\in S$ such that $z+s=x$. Thus,

$$\begin{aligned}h&=x-(n_1+\cdots+n_r)-s+((z_1+1)+n_1+\cdots+(z_r+1)n_r)\\&=x-(n_1+\cdots+n_r)-(s+(-(z_1+1)n_1-\cdots-(z_r+1)n_r)),\end{aligned}$$

and since $z_i<0$, $-(z_i+1)\geq 0$, which implies that $h\in g-S$.

Now take $g-s\in g-S$. If $g-s\notin G_{[1,r]}$, then $g-s\in S_i$ for some i. Hence there exist $s'\in S$, $s_i\in S\cap F_i$ such that $g-s=s'-s_i$, and therefore $g=(s'+s)-s_i\in S_i$, a contradiction with 4.5. □

5. On relations of Gorenstein simplicial affine semigroups

If S is Cohen-Macaulay, we already have a bound for the number of elements of a minimal system of generators for the associated congruence of S. We are going to show that if, in addition, S is Gorenstein, then this bound can be improved.

In this section, S is a simplicial affine semigroup with minimal system of generators $\{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$ and $m > 1$ (if $m = 1$, S is a complete intersection semigroup and therefore it has the minimum possible number of elements in a system of generators for the congruence associated to S).

Lemma 5.1. *If S is Gorenstein and $x = \max \bigcap_{i=1}^r \mathrm{S}(n_i)$, then $G_{x+n_{r+i}}$ is a connected graph.*

Proof. Note that:

1. Clearly $n_{r+i} \in V(G_{x+n_{r+i}})$.
2. Since $n_{r+j} \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ and $x = \max \bigcap_{i=1}^r \mathrm{S}(n_i)$ we have that $x - n_{r+j} \in S$, and therefore $(x + n_{r+i}) - n_{r+j} \in S$. Thus, $n_{r+j} \in V(G_{x+n_{r+i}})$.
3. For every $j \in \{1, \ldots, m\}$, $j \neq i$, we have that $(x + n_{r+i}) - n_{r+j} - n_{r+i} = x - n_{r+j} \in S$.

This means that all the elements in $\{n_{r+1}, \ldots, n_{r+m}\}$ are connected. By 2.2, the same holds for the elements in $\{n_1, \ldots, n_r\} \cap \mathrm{V}(G_{x+n_{r+i}})$. Thus, the number of connected components is less or equal than two.

Let us suppose that the number of connected components is two. Take $y \in \bigcap_{i=1}^r \mathrm{S}(n_i) \backslash \{x\}$. Since x is greater than y, there exists $s \in S$ such that $y + s = x$. The element s must be in $\bigcap_{i=1}^r \mathrm{S}(n_i)$, and therefore there exists $d_{r+j} \in \mathbb{N}$, $j \in \{1, \ldots, m\}$, such that $s = d_{r+1}n_{r+1} + \cdots + d_{r+m}n_{r+m}$. If $y + n_{r+i} \notin \bigcap_{i=1}^r \mathrm{S}(n_i)$, then there must exist $c_k \in \mathbb{N}$, $k \in \{1, \ldots, r + m\}$, such that

$$y + n_{r+i} = c_1 n_1 + \cdots + c_r n_r + c_{r+1} n_{r+1} + \cdots + c_{r+m} n_{r+m},$$

with $c_1 + \cdots + c_r > 0$ (since $y + n_{r+i} \notin \bigcap_{i=1}^r \mathrm{S}(n_i)$). Hence,

$$x + n_{r+i} = y + s + n_{r+i} = c_1 n_1 + \cdots + c_r n_r + (c_{r+1} + d_{r+1}) n_{r+1} + \cdots + (c_{r+m} + d_{r+m}) n_{r+m}.$$

Since we have assumed that there are two connected components, $c_{r+1} + d_{r+1} = \cdots = c_{r+m} + d_{r+m} = 0$. This leads to $s = 0$ and therefore $x = y$, which is not possible. Therefore, $y + n_{r+i} \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ for all $y \in \bigcap_{i=1}^r \mathrm{S}(n_i) \backslash \{x\}$. Thus

$$\bigcap_{i=1}^r \mathrm{S}(n_i) = \{0, n_{r+i}, \ldots, k n_{r+i} = x\}.$$

But $n_{r+j} \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ for $j > 0$ and this would mean that $n_{r+j} = t n_{r+i}$ which is a contradiction with the fact that $\{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$ is a minimal system of generators for S. This implies that the number of connected components is one. □

Theorem 5.2. *If S is Gorenstein and ρ is a minimal set defining relations for the associated congruence of S, then*

$$\#\rho \leq \frac{(2d-m)(m-1)}{2} + 2 - m,$$

with $d = \# \bigcap_{i=1}^{r} S(n_i)$.

Proof. Take $x = \max \bigcap_{i=1}^{r} S(n_i)$. Let us define D_k as in 2.5 and let us show that $x \notin D_k$, for every $r+1 \leq k < r+m$. The set D_k is contained in the set $\bigcap_{i=1}^{k-1} S(n_i)$. Let us show that $x \notin \bigcap_{i=1}^{k-1} S(n_i)$. Since $x = \max \bigcap_{i=1}^{r} S(n_i)$, $x - n_{r+j} \in S$, for all $j \in \{1, \dots, m\}$. This implies that $x \notin S(n_{r+j})$ for any $j \in \{1, \dots, m\}$.

Besides, $x \notin \{n_{r+1}, \dots, n_{r+m}\}$ (otherwise $\{n_1, \dots, n_r, n_{r+1}, \dots, n_{r+m}\}$ would not be a minimal system of generators for S, since $x = \max \bigcap_{i=1}^{r} S(n_i)$).

Thus, as in 2.6, $D_k \subseteq \bigcap_{i=1}^{r} S(n_i) \backslash \{0, n_{r+1}, \dots, n_{k-1}, x\}$ and

$$\#\rho \leq 1 + \sum_{i=1}^{m-1} (d - i - 1) = \frac{(2d-m)(m-1)}{2} + 2 - m. \qquad \square$$

6. Gorenstein simplicial affine semigroups with maximal codimension

Given a Gorenstein simplicial affine semigroup, we are going to show how to construct a Gorenstein semigroup with maximal codimension. The main difference between the construction exposed in this section and the one exposed in Section 3 is that a Gorenstein semigroup can never reach the bound $m = d - 1$.

Lemma 6.1. *If S is Gorenstein, then $m < d - 1$, with $d = \# \bigcap_{i=1}^{r} S(n_i)$. (Note that $m > 1$.)*

Proof. We already know that $m \leq d - 1$. Assume that $m = d - 1$. Then $\bigcap_{i=1}^{r} S(n_i) = \{0, n_{r+1}, \dots, n_{r+d-1}\}$. Since S is Gorenstein, there exists $i \in \{1, \dots, d-1\}$ such that $n_{r+i} = \max \bigcap_{i=1}^{r} S(n_i)$. This means that for every $j \in \{1, \dots, d-1\}$, $j \neq i$, there exists $s_j \in S$ such that $n_{r+j} + s_j = n_{r+i}$. Note that s_j must be in $\bigcap_{i=1}^{r} S(n_i)$. Hence there exists k such that $s_j = n_{r+k}$, and this is a contradiction with the fact that $\{n_1, \dots, n_r, n_{r+1}, \dots, n_{r+m}\}$ is a minimal system of generators. $\square$

Thus, if we want to get a Gorenstein simplicial affine semigroup with maximal codimension, it must fulfil the condition that $m = d - 2$.

The next theorem tells us the number of elements of a minimal system of generators ρ, for the congruence associated to S, in the case S is Gorenstein with maximal codimension.

Theorem 6.2. *Let S be a Gorenstein simplicial affine semigroup with maximal codimension ($m = d - 2$). Then $\#\rho = d(d-3)/2$.*

Proof. Let $S = \langle n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\rangle$. Since S is Gorenstein with maximal codimension, $\bigcap_{i=1}^r S(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}, x\}$, with $x = \max \bigcap_{i=1}^r S(n_i)$. In order to compute $\#\rho$, let us see for which $n \in S$, G_n is not connected. If G_n is not connected, then, by 2.3, $n = y + n_{r+j}$ with $y \in \bigcap_{i=1}^r S(n_i)$. Note also that, by 5.1, $y \neq x$. Hence, if G_n is not connected, then $n = n_{r+i} + n_{r+j}$, for some $i, j \in \{1, \ldots, m\}$. Let us show that the reverse is also true. Take $i, j \in \{1, \ldots, m\}$, and let us prove that $G_{n_{r+i}+n_{r+j}}$ is not connected. Two cases must be taken into account:

1. If $n_{r+i} + n_{r+j} \in \bigcap_{i=1}^r S(n_i)$, then, since $\{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$ is a minimal system of generators for S and $\bigcap_{i=1}^r S(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}, x\}$, we have that $n_{r+i} + n_{r+j} = x$. Note that $x \in \bigcap_{i=1}^r S(n_i)$ which means that $\{n_1, \ldots, n_r\} \cap V(G_x) = \emptyset$. Note also that, since $x = \max \bigcap_{i=1}^r S(n_i)$, for every $l \in \{1, \ldots, m\}$, there exists $k \in \{1, \ldots, m\}$ such that $n_{r+l} + n_{r+k} = x$. This implies that $V(G_x) = \{n_{r+1}, \ldots, n_{r+m}\}$ and that $E(G_x) = \{\overline{n_{r+l}n_{r+k}} : n_{r+l} + n_{r+k} = x\}$.

2. If $n_{r+i} + n_{r+j} \notin \bigcap_{i=1}^r S(n_i)$ then we proceed as in 3.1 and we get that $G_{n_{r+i}+n_{r+j}}$ is not connected and has a connected component with vertices in $\{n_1, \ldots, n_r\}$.

Thus, for every $n = n_{r+i} + n_{r+j}$, $\#\rho_n$ is equal to the number of the expressions of the form $n = n_{r+l} + n_{r+k}$, $l, k \in \{1, \ldots, m\}$, provided $n \neq x$ and it is equal to the number of such expressions minus one if $n = x$ (this is due to the fact that if $n \neq x$, there is an extra component whose vertices are contained in $\{n_1, \ldots, n_r\}$). Hence we must count the expressions of the form $n_{r+i} + n_{r+j}$, $1 \leq i, j \leq m$, and subtract one, getting $(d-1)(d-2)/2 - 1$. □

The reverse is also true, as the following result shows:

Theorem 6.3. *If S is Gorenstein and $\#\rho = d(d-3)/2$, then S has maximal codimension ($m = d - 2$).*

Proof. The proof is easy and similar to 3.2.

To conclude this section, we are going to construct a Gorenstein simplicial affine semigroup with maximal codimension from a given Gorenstein simplicial affine semigroup. Take S as at the beginning of the section and let $\bigcap_{i=1}^r S(n_i) = \{0, x_1, \ldots, x_{d-1}\}$. Assume that S is Gorenstein with $x_{d-1} = \max \bigcap_{i=1}^r S(n_i)$. We can define $\overline{S} = \langle n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{d-2}\rangle$. It can be shown, as we did in 3.3, that

1. The set $\{n_1, \ldots, n_r, n_1 + x_1, \ldots, n_1 + x_{d-2}\}$ is a minimal system of generators for $\overline{S}$;

2. The semigroup $\overline{S}$ is simplicial.

We do not introduce the element $n_1 + x_{d-1}$, because x_{d-1} can be obtained adding two other elements in $\bigcap_{i=1}^r S(n_i)$.

Theorem 6.4. *Under the above hypothesis,*

$$\bigcap_{i=1}^{r} \overline{S}(n_i) = \{0, n_1 + x_1, \ldots, n_1 + x_{d-2}, 2n_1 + x_{d-1}\}.$$

Proof. Straightforward using the fact that $\{0, n_1 + x_1, \ldots, n_1 + x_{d-2}, 2n_1 + x_{d-1}\}$ is a complete system modulo $\mathfrak{G}(S)$ and 3.7. $\square$

Corollary 6.5. *Under the above hypothesis, $\overline{S}$ is Gorenstein.*

Proof. Using 1.6 it is easy to show that $\overline{S}$ is Cohen-Macaulay. It is also clear that $2n_1 + x_{d-1} = \max_{\overline{S}} \bigcap_{i=1}^{r} \overline{S}(n_i)$, because $x_{d-1} = \max_S \bigcap_{i=1}^{r} S(n_i)$. $\square$

Finally, we can construct some other examples of Gorenstein simplicial affine semigroups with maximal codimension taking subsets of $\mathrm{Im}(\mu)$ (see Section 3) that are complete systems modulo $\mathfrak{G}(\{n_1, \ldots, n_r\})$ and have a maximum.

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