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Zero-cycles on projective varieties and the norm principle

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Abstract

Using the Gille–Merkurjev norm principle we compute in a uniform way the image of the degree map for quadrics (Springer's theorem), for twisted forms of maximal orthogonal Grassmannians (a theorem of Bayer-Fluckiger and Lenstra), and for E_6- (a theorem of Rost) and E_7-varieties.

1. Introduction

Let G be a simple algebraic group over a field k and X a projective G-homogeneous variety. Consider the degree map

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}.$$

The goal of the present paper is to provide a method to compute the image of this map (see [PSZ08] for a computation of its kernel).

This problem has a long history starting probably with the Springer theorem, which says that an anisotropic quadratic form remains anisotropic over odd degree field extensions. This statement is equivalent to the fact that the image of the degree map when X is an anisotropic quadric equals $2\mathbb{Z}$.

To stress the difficulty of the above problem note that a computation of the degree map for the varieties of Borel subgroups of groups of type E_8 implies Serre's conjecture II for fields of cohomological dimension $\leqslant 2$ (see [Gil01]). Moreover, the above problem generalizes Serre–Tit's question, whether the map

$$H^1(k, G_0) \to \prod H^1(K_i, G_0)$$

has trivial kernel when K_i are finite field extensions of k such that $\gcd[K_i : k] = 1$ and G_0 is a split group.

The image of the degree map is known in the following cases: X is a quadric (Springer's theorem), X is a twisted form of a maximal orthogonal Grassmannian (a theorem of Bayer-Fluckiger and Lenstra [BL90]), and X is the variety of Borel subgroups of a group of type F_4, E_6 (a theorem of Rost, where cohomological invariants of Albert algebras are used), and E_7 (Gille's theorem [Gil97], where the norm principle is used). Note also that there are numerous papers of M. Florence, R. Parimala, B. Totaro, and many others concerning closely related problems.

In the present paper we apply the Gille–Merkurjev norm principle [BM02, Gil97, Mer96] and give a uniform proof of the above results. We also compute the image of the degree map for the

varieties of parabolic subgroups of type 7 of groups G of type E_7 and prove that anisotropic groups of type E_7 remain anisotropic over odd degree field extensions. Note that this property is used in [PS07] to relate the Rost invariant of G and its isotropy.

2. Norm principle and strategy of the proof

2.1 Let k be a perfect field with char $k \neq 2, 3$, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group, G a connected reductive algebraic group over k, $G' = [G, G]$ the commutator subgroup, Δ its Dynkin diagram, and Δ_0 its Tits index (see [Tit66]).

2.2 Special cocharacters

Let G_1 be a reductive algebraic group over k and

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1 \qquad (1)$$

an exact sequence. The cocharacter group T_* can be canonically identified with the group $\mathbb{Z}$. In particular, we may speak about odd cocharacters or, more generally, about cocharacters coprime to a prime number p. A cocharacter $\varphi \in T_*$ is called *f-special* if there is a k-homomorphism $g : \mathbb{G}_m \to G$ such that $f \circ g = \varphi$.

2.3 Set $X(\varphi)$

Denote $Z' = G/G'$, C the center of the simply connected cover of G', Z the center of G, and μ the center of G'.

We can represent the homomorphism f as a composition $G \to Z' \to T$. In particular, there is the induced homomorphism $\alpha : Z'^{\Gamma}_* \to T_*$ between the cocharacter groups. The exact sequence

$$1 \to \mu \to Z \to Z' \to 1$$

induces a homomorphism $\beta : Z'^{\Gamma}_* \to \mu(-1)^{\Gamma}$, and the canonical epimorphism $C \to \mu$ induces a map $\gamma : C^{\Gamma}_* \to \mu(-1)^{\Gamma}$, where $\mu(-1)$ is the Tate twist, i.e. $\mu(-1) = \mathrm{Hom}(\mu_n, \mu)$ for any n with $\mu^n = 1$. For a cocharacter $\varphi \in T_*$ we define a subset $X(\varphi) \subset C^{\Gamma}_*$ as the set $\gamma^{-1}(\beta(\alpha^{-1}(\{\varphi\})))$.

2.4 Set $\Omega(\varphi)$

From now on we assume that the Dynkin diagram Δ has no multiple edges. Following [Mer96, (5.8)] we identify C_* and the character group C^* and consider $X(\varphi)$ as a subset of C^*. Let $\bar{\omega}_i$ denote the ith fundamental weight of the simply connected cover of G' (enumeration of simple roots follows Bourbaki). Now define $\Omega(\varphi)$ as the set of all subsets $\Theta \subset \Delta$ such that the elements $\{\sum_{i \in I} \bar{\omega}_i |_C, \ I \subset \Delta \backslash \Theta$ a Γ-orbit$\}$ generate a subgroup of $C^{*\Gamma}$ that intersects $X(\varphi)$.

2.5 Type of a parabolic subgroup

It is well-known that there is a bijective correspondence between the conjugacy classes of parabolic subgroups of $G'_{\bar{k}}$ and the subsets of the set Δ of simple roots.

The *type* of a parabolic subgroup is the corresponding subset of Δ. Under this identification the Borel subgroup has type $\emptyset$. If P is a maximal parabolic subgroup of type $\Delta \backslash \{\alpha_i\}$, where α_i is the ith simple root, then for simplicity of notation we say that P is of type i.

2.6 Tits homomorphism

Let

$$\beta : C^{*\Gamma} \to \mathrm{Br}(k)$$

be the Tits homomorphism for the simply connected cover of G' defined in [Tit71]. In order to compute the sets $\Omega(\varphi)$ we need to know the restrictions of the fundamental weights $\bar{\omega}_i$ to C and their images under the Tits homomorphism.

Below we describe them for groups of type $^1\mathrm{D}_n$, $^1\mathrm{E}_6$, and E_7. We use graphical notation, where the algebra over a vertex i of the Dynkin diagram stands for the image $\beta(\bar{\omega}_i|_C)$. Apart from this, the restriction $\bar{\omega}_i|_C$ is trivial if and only if the respective algebra is k.

Type D_n. A simply connected group of inner type D_n has the form $\mathrm{Spin}(A, \sigma)$, where A is a central simple algebra of degree $2n$ with an orthogonal involution σ of the first kind with trivial discriminant.

For the character group C^* of the center of $\mathrm{Spin}(A, \sigma)$ we have

$$C^* = \{0, \chi, \chi^+, \chi^-\},$$

where χ (respectively, χ^+, χ^-) is the restriction of the fundamental weight $\bar{\omega}_1$ (respectively, $\bar{\omega}_{n-1}$, $\bar{\omega}_n$) to the center.

Let $C^\pm(A, \sigma)$ be the direct summands of the Clifford algebra $C_0(A, \sigma) = C^+(A, \sigma) \oplus C^-(A, \sigma)$.
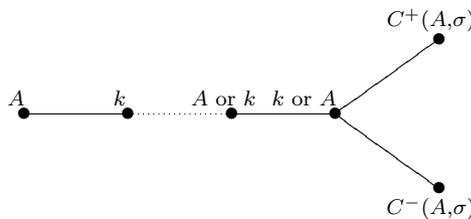

We associate the Tits algebras to the last two vertices $n-1$ and n in such a way that for $\varepsilon = +$ (respectively, $\varepsilon = -$) the algebra $C^\varepsilon(A, \sigma)$ splits over the field of rational functions of the projective homogeneous variety of maximal parabolic subgroups of type P_{n-1} (respectively, P_n). The latter are two irreducible components of the variety of $2n^2$-dimensional isotropic right ideals I of A with respect to σ.

Type E_6. The Tits algebra is a certain central simple algebra A of index 1, 3, 9, or 27 and of exponent 1 or 3.
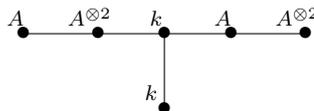
Type E_7. The Tits algebra is a certain central simple algebra A of index 1, 2, 4, or 8 and of exponent 1 or 2.
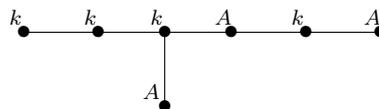
Under the above assumptions the following lemmas hold.

LEMMA 2.1 [Mer96, Lemma 3.4]. *Let K/k be a finite field extension lying in the algebraic closure $\bar{k}$ and let $\varphi \in T_*$. If the cocharacter φ is f_K-special, then the cocharacter $[K : k]\varphi$ is f-special.*

459

Lemma 2.2 [Mer96, Theorem 5.6]. *For a cocharacter $\varphi \in T_*$ the following conditions are equivalent:*

(i) φ *is f-special;*

(ii) *there exists a parabolic subgroup of G defined over k whose type is contained in $\Omega(\varphi)$.*

Lemma 2.3 [Mer96, Proposition 5.8]. *Let $\beta : C^{*\Gamma} \to \mathrm{Br}(k)$ be the Tits homomorphism for the simply connected cover of G'. Assume that the Dynkin diagram Δ has no multiple edges. If a cocharacter $\varphi \in T_*$ is f-special, then $0 \in \beta(X(\varphi))$.*

Theorem 2.4. *Let X be an anisotropic smooth projective variety over k and p a prime number. In the above notation assume that the following conditions hold:*

(1) *for any field extension K/k and for any coprime to p cocharacter φ, if $0 \in \beta_K(X(\varphi)) \subset \mathrm{Br}(K)$ and G' has a parabolic subgroup defined over K whose type is contained in $\Omega(\varphi)$, then $X(K) \neq \emptyset$;*

(2) *for any field extension K/k and for any coprime to p cocharacter φ if $X(K) \neq \emptyset$, then there exists a parabolic subgroup of G' of type contained in $\Omega(\varphi)$ defined over K.*

Then $\deg(\mathrm{CH}_0(X)) \subset p\mathbb{Z}$.

Proof. Let K/k be a field extension. We show first the following.

Claim. $X(K) \neq \emptyset$ if and only if any coprime to p cocharacter φ is f_K-special.

Indeed, if $X(K) \neq \emptyset$, then by item 2 there is a parabolic subgroup of G' defined over K whose type is contained in $\Omega(\varphi)$. By Lemma 2.2 φ is f_K-special.

Conversely, if φ is f_K-special, then by Lemma 2.3 we have $0 \in \beta_K(X(\varphi))$, and by Lemma 2.2 there is a parabolic subgroup of G' defined over K of type contained in $\Omega(\varphi)$. Therefore, by item 1 we have $X(K) \neq \emptyset$.

Let now K/k be a finite field extension such that $X(K) \neq \emptyset$. To finish the proof of the theorem it suffices to show that $[K : k]$ is divisible by p. Assume the converse.

Since $X(K) \neq \emptyset$, by the claim any coprime to p cocharacter φ is f_K-special. By Lemma 2.1 the cocharacter $[K : k]\varphi$ is f-special. Therefore, by the claim $X(k) \neq \emptyset$. This is a contradiction. $\square$

3. Applications

Corollary 3.1 (Springer's theorem). *Let A be a central simple k-algebra of degree $2n \geqslant 4$ with an orthogonal involution σ of the first kind. Let X be the variety of isotropic with respect to σ right ideals of A of dimension $2n$. Assume X is anisotropic.*

Then $\deg(\mathrm{CH}_0(X)) \subset 2\mathbb{Z}$. In particular, if X is an anisotropic smooth even-dimensional projective quadric, then $\deg(\mathrm{CH}_0(X)) = 2\mathbb{Z}$.

Proof. (See [Mer96, Lemma 6.2].) There exists the following exact sequence of groups:

$$1 \to G_1 = \mathrm{Spin}(A, \sigma) \to G = \Gamma(A, \sigma) \xrightarrow{f} \mathbb{G}_m \to 1,$$

where $\Gamma(A, \sigma)$ is the Clifford group and f is the *spinor norm* homomorphism.

Let $p = 2$. It is easy to check that for any odd cocharacter φ the set $X(\varphi) = \{\chi\}$, where χ is the restriction of $\bar{\omega}_1$ to the center C.

Let K/k be a field extension. If $0 \in \beta_K(X(\varphi))$, then the algebra A_K is split (see §2.6). Thus, σ_K corresponds to a quadratic form, and X_K is a projective quadric. If additionally G'_K has a parabolic subgroup defined over K of type contained in $\Omega(\varphi)$, then it easy to see that this quadratic form is isotropic, and thus $X(K) \neq \emptyset$.

Finally, if $X(K) \neq \emptyset$, then G' has a parabolic subgroup of type $\Delta \backslash \{\alpha_1\}$, where α_1 is the first simple root. However, $\Delta \backslash \{\alpha_1\} \in \Omega(\varphi)$.

Thus, we have checked all conditions of Theorem 2.4. □

COROLLARY 3.2 (Bayer-Fluckiger and Lenstra). *Let A be a central simple algebra of degree $2n \geqslant 4$ with an orthogonal involution σ of the first kind. Let Y be the variety of $2n^2$-dimensional isotropic right ideals of A and*

$$X = Y \times \mathrm{Spec}(k[t]/(t^2 - \mathrm{disc}(\sigma))).$$

Assume X is anisotropic. Then $\deg(\mathrm{CH}_0(X)) \subset 2\mathbb{Z}$.

Proof. (See [Mer96, 6.3].) Consider the following exact sequence of groups:

$$1 \to G_1 = \mathrm{O}^+(A, \sigma) \to G = \mathrm{GO}^+(A, \sigma) \xrightarrow{f} \mathbb{G}_m \to 1,$$

where f is the *multiplier map*.

Let $p = 2$. Denote as χ^+ (respectively, χ^-) the restriction of the fundamental weight $\bar{\omega}_{n-1}$ (respectively, $\bar{\omega}_n$) to the center. It is easy to check that for any odd cocharacter φ we have

$$X(\varphi) = \begin{cases} \emptyset, & \mathrm{disc}(\sigma) \neq 1, \\ \{\chi^+, \chi^-\}, & \mathrm{disc}(\sigma) = 1. \end{cases}$$

Finally, if $0 \in \beta_K(X(\varphi))$, then $\mathrm{disc}(\sigma_K) = 1$. Then the variety Y_K is the disjoint union of varieties of parabolic subgroups of G'_K of types $\Delta \backslash \{\alpha_{n-1}\}$ and $\Delta \backslash \{\alpha_n\}$. If, additionally, G' has a parabolic subgroup defined over K of type contained in $\Omega(\varphi)$, then the Tits index Δ_0 of G'_K contains at most one of the roots α_{n-1}, α_n (see §2.6). Therefore, in this case $X(K) \neq \emptyset$.

To finish the proof of the corollary it remains to note that condition (2) of Theorem 2.4 is obvious. □

Using similar arguments one can show the following well-known statement. Opposite to the traditional approach our proof does not use cohomological invariants of Albert algebras.

COROLLARY 3.3 (Rost). *Let G_1 be a simply connected algebraic group of type $^1\mathrm{E}_6$ over k, and X the variety of its parabolic subgroups of type 1 (respectively, 6). Assume that X is anisotropic. Then $\mathrm{CH}_0(X) \subset 3\mathbb{Z}$.*

Proof. If G_1 has a non-trivial Tits algebra, then the statement is obvious, since for a field extension K/k condition $X(K) \neq \emptyset$ implies that the Tits algebras of $(G_1)_K$ are split.

Assume that G_1 has trivial Tits algebras. Let J be an Albert algebra associated with G_1 (see [Gar01a]). A k-linear map

$$h : J \to J$$

is called a *similarity* if there exists $\alpha_h \in k^\times$ (the multiplier of h) such that

$$N(h(j)) = \alpha_h N(j)$$

461

for all $j \in J$, where N stands for the cubic norm on J. Then G_1 coincides with the similarities of this Jordan algebra with multiplier 1. Let G be the group of all similarities. Then G is a reductive group and there exists the following exact sequence of algebraic groups:

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1,$$

where the map f is defined on k-points as $h \mapsto \alpha_h$.

Let $p = 3$ and let $\varphi \in T_* = \mathbb{Z}$ be a cocharacter coprime to 3. We now check the conditions of Theorem 2.4.

First we compute $X(\varphi)$. In our situation $G' = [G, G] = G_1$, $Z' = T = \mathbb{G}_m$, $\mu = \mu_3$, $C = \mu_3$, $Z = \mathbb{G}_m$, and the group

$$C_* \simeq C^* = \mathbb{Z}/3 = \{0, \bar{\omega}_1|_C, \bar{\omega}_6|_C = -\bar{\omega}_1|_C\},$$

where $\bar{\omega}_i|_C$ denotes the restriction of the ith fundamental weight of G_1 to the center, $i = 1, 6$. Therefore, $X(\varphi) = \{\bar{\omega}_1|_C\}$ or $X(\varphi) = \{\bar{\omega}_6|_C\}$ (it depends on φ mod 3).

Let K/k be a field extension. Assume first that G'_K is isotropic and the type of a parabolic subgroup P of G' defined over K is contained in $\Omega(\varphi)$. If the parabolic subgroup of G' of type 1 is not defined, then by Tits's classification [Tit66, p. 58] the Tits index of G' equals $\Delta_0 = \Delta \setminus \{\alpha_2, \alpha_4\}$. However, the restrictions to the center of the second and the fourth fundamental weights are trivial (see [Tit90, p. 653] or §2.6). This contradicts the assumption that the type of P is contained in $\Omega(\varphi)$.

Finally, condition (2) of Theorem 2.4 is obvious. $\square$

Remark 3.4. If the Tits algebras of G_1 are trivial, then the image of the degree homomorphism $\mathrm{CH}_0(X) \to \mathbb{Z}$ equals $3\mathbb{Z}$.

COROLLARY 3.5. *Let G_1 be a simply connected algebraic group of type E_7 over k and let X be the variety of the maximal parabolic subgroups of G_1 of type 7. Assume that X is anisotropic. Then $\mathrm{CH}_0(X) \subset 2\mathbb{Z}$.*

Proof. Let (A, σ, π), where A is a central simple k-algebra with a symplectic involution σ and $\pi : A \to A$ is a linear map, be *a gift* associated with G_1 (see [Fer72, Gar01a, Gar01b]). An invertible element $h \in A$ is called a *similarity* if there exists $\alpha_h \in k^\times$ (the multiplier of h) such that

$$\sigma(h)h = \alpha_h \cdot 1$$

and

$$\pi(hah^{-1}) = \alpha_h h \pi(a) h^{-1}$$

for all $a \in A$. Then G_1 coincides with the similarities of this gift with multiplier 1. Let G be the group of all similarities. Then G is a connected reductive group and there exists the following exact sequence of algebraic groups:

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1,$$

where the map f is defined on k-points as $h \mapsto \alpha_h$.

Let $p = 2$ and let φ be an odd cocharacter.

First, we compute $X(\varphi)$. In our situation $G' = [G, G] = G_1$, $Z' = T = \mathbb{G}_m$, $\mu = \mu_2$, $C = \mu_2$, and $Z = \mathbb{G}_m$. Thus, the maps α and γ from § 2.3 are the identity maps. The map $\beta : Z'_* = \mathbb{Z} \to \mu_2(-1) = \mathbb{Z}/2$ from § 2.3 is the usual factor map. Therefore, $X(\varphi) = \{\chi\}$, where by χ we denote a unique non-trivial element of $C_* \simeq C^*$.

Let K/k be a field extension. Assume that $0 \in \beta_K(X(\varphi))$, G'_K is isotropic and the type of a parabolic subgroup of G' defined over K is contained in $\Omega(\varphi)$. The first assumption implies that the Tits algebra A of G_1 is split.

If the parabolic subgroup of G' of type 7 is not defined, then by Tits's classification [Tit66, p. 59] the Tits index of G' equals $\Delta_0 = \Delta \backslash \{\alpha_1\}$. However, the restriction to the center of the first fundamental weight is trivial (see [Tit90, p. 653]). Therefore, we have $X(K) \neq \emptyset$.

Finally, condition (2) of Theorem 2.4 is obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 3.6. If the Tits algebras of G_1 are trivial, then the image of the degree homomorphism $\mathrm{CH}_0(X) \to \mathbb{Z}$ equals $2\mathbb{Z}$.

COROLLARY 3.7. *A group G_1 as in the statement of the previous corollary does not split over an odd degree field extension.*

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