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THE BOUNDED APPROXIMATION PROPERTY FOR THE WEIGHTED SPACES OF HOLOMORPHIC MAPPINGS ON BANACH SPACES

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Abstract. In this paper, we study the bounded approximation property for the weighted space $\mathcal{HV}(U)$ of holomorphic mappings defined on a balanced open subset U of a Banach space E and its predual $\mathcal{GV}(U)$, where $\mathcal{V}$ is a countable family of weights. After obtaining an $\mathcal{S}$-absolute decomposition for the space $\mathcal{GV}(U)$, we show that E has the bounded approximation property if and only if $\mathcal{GV}(U)$ has. In case $\mathcal{V}$ consists of a single weight v, an analogous characterization for the metric approximation property for a Banach space E has been obtained in terms of the metric approximation property for the space $\mathcal{G}_v(U)$.

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1. Introduction and terminology. Weighted spaces of holomorphic functions defined on a balanced open subset of a finite-dimensional space have been studied extensively in the literature by Bierstedt [**3–5**], Summers [**6**], Rubel and Shields [**28**], etc. The study of infinite-dimensional analogues of such spaces was initiated in [**29**] and further carried out in [**1, 15, 16, 20**]. The main aim of this paper is to study the bounded approximation property (BAP) for these spaces, of which the finite-dimensional case was considered in [**4**] for the weighted spaces of holomorphic mappings containing polynomials; indeed, in this case, Cesàro mean operators C_n are the finite rank operators converging uniformly on compact sets to the identity operator. For weighted spaces of holomorphic mappings defined on an open balanced subset of a Banach space, the techniques involving $\mathcal{S}$-absolute decompositions come to our rescue to establish results for the BAP.

To begin with, let us denote by $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{C}$ the set of natural numbers, $\mathbb{N} \cup \{0\}$ and the complex plane, respectively. The letters E and F are used for the complex Banach spaces and the symbols E' and E^* denote, respectively, the algebraic dual and topological dual of E. We denote by U a non-empty open subset of E and by U_E, the open unit ball of E. The symbol B_E^λ denotes the set consisting of the elements with norm $\leq \lambda$. For $\lambda = 1$, B_E^1 is the closed unit ball B_E of E. The symbols X and Y are used for locally convex spaces and X_b^* for the strong dual of X.

For each $m \in \mathbb{N}$, $\mathcal{L}(^m E; F)$ denotes the Banach space of all continuous m-linear mappings from E to F endowed with the sup norm. A mapping $P : E \to F$ is said to be a *continuous m-homogeneous polynomial* if there exists a continuous m-linear map $A \in \mathcal{L}(^m E; F)$ such that $P(x) = A(x, \ldots, x)$, $x \in E$. The space of all m-homogeneous continuous polynomials from E to F is denoted by $\mathcal{P}(^m E; F)$ which is a Banach space endowed with the norm $\|P\| = \sup_{\|x\|\leq 1} \|P(x)\|$. *A continuous polynomial* P is a mapping from E into F which can be represented as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}(^mE; F)$ for $m = 0, 1, \ldots, k$. The vector space of continuous polynomials from E into F is denoted by $\mathcal{P}(E; F)$. A polynomial $P \in \mathcal{P}(^mE, F)$ is said to be of *finite type* if it is of the form

$$P(x) = \sum_{j=1}^{k} \phi_j^m(x) y_j, \ x \in E,$$

where $\phi_j \in E^*$ and $y_j \in F$, $1 \leq j \leq k$. We denote by $\mathcal{P}_f(^mE, F)$, the space of finite type polynomials from E into F. A continuous polynomial P from E into F is said to be of finite type if it has a representation as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}_f(^mE; F)$ for $m = 0, 1, \ldots, k$. The vector space of continuous polynomials of finite type from E into F is denoted by $\mathcal{P}_f(E; F)$. $\mathcal{P}_w(^mE, F)$ denotes the space of m-homogeneous polynomials which are weakly continuous on bounded subsets of E. The predual $\mathcal{Q}(^mE)$ of $\mathcal{P}(^mE)$, $m \in \mathbb{N}$, constructed by Ryan [**30**], is defined as

$$\mathcal{Q}(^mE) = \{\phi \in \mathcal{P}(^mE)' : \phi|B_m \text{ is } \tau_0 - \text{continuous}\},$$

where B_m is the unit ball of $P(^mE)$. The space $\mathcal{Q}(^mE)$ is endowed with the topology τ_m of uniform convergence on B_m.

A mapping $f : U \to F$ is said to be *holomorphic*, if for each $\xi \in U$, there exists a ball $B(\xi, r)$ with centre at ξ and radius $r > 0$, contained in U and a sequence $\{P^j f(\xi)\}_{j=0}^{\infty}$ of polynomials with $P^j f(\xi) \in \mathcal{P}(^jE; F)$, $j \in \mathbb{N}_0$ such that

$$f(x) = \sum_{j=0}^{\infty} P^j f(\xi)(x - \xi), \tag{1}$$

where the series converges uniformly for each $x \in B(\xi, r)$. The series in (1) is called the Taylor series of f at ξ. The space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U; F)$. In case U is an open subset of a finite-dimensional Banach space E, $(\mathcal{H}(U; F), \tau_0)$ is a Fréchet space, where τ_0 denotes the topology of uniform convergence on compact subsets of U. For $F = \mathbb{C}$, we write $\mathcal{H}(U)$ for $\mathcal{H}(U; \mathbb{C})$. A subset A of U is called *U-bounded* if A is bounded and $dist(A, \partial U) > 0$, where ∂U denotes the boundary of U. A mapping f in $\mathcal{H}(U; F)$ is of *bounded type* if it maps U-bounded sets to bounded sets. The space of holomorphic mappings of bounded type is denoted by $\mathcal{H}_b(U; F)$.

Let $\mathcal{V} = \{v_n\}$ be a countable family of positive continuous functions on U(referred to as weights) such that for each $x \in U$, there is $n \in \mathbb{N}$ such that $v_n(x) > 0$. In case $\mathcal{V}$ is a singleton set $\{v\}$, v is strictly positive on U. A weight v defined on an open balanced subset U of E is said to be *radial* if $v(tx) = v(x)$ for all $x \in U$ and $t \in \mathbb{C}$ with $|t| = 1$, and on E it is said to be *rapidly decreasing* if $\sup_{x \in E} v(x)\|x\|^m < \infty$ for each $m \in \mathbb{N}_0$. The weighted spaces of holomorphic functions is defined as

$$\mathcal{HV}(U; F) = \{f \in \mathcal{H}(U; F) : p_{v_n}(f) = \sup_{x \in U} v_n(x)\|f(x)\| < \infty \text{ for each } n \in \mathbb{N}\}$$

and

$$\mathcal{HV}_0(U;F) = \{f \in \mathcal{H}(U;F) : \text{for given } \epsilon > 0 \text{ and } n \in \mathbb{N}, \text{ there exists a } U\text{-bounded set } A \text{ such that } \sup_{x \in U \setminus A} v_n(x)\|f(x)\| < \epsilon\}.$$

For $F = \mathbb{C}$, we write $\mathcal{HV}(U)$ and $\mathcal{HV}_0(U)$ instead of $\mathcal{HV}(U,F)$ and $\mathcal{HV}_0(U,F)$. The spaces $\mathcal{HV}(U)$ and $\mathcal{HV}_0(U)$ are endowed with the topology τ_V generated by the family of semi-norms $\{p_{v_n} : n \in \mathbb{N}\}$.

A family $\mathcal{V}$ of weights satisfies *condition I* if for each U-bounded set A, there exists $n \in \mathbb{N}$ such that $\inf_{x \in A} v_n(x) > 0$ and, $\mathcal{V}$ satisfies *condition II* if for each $n \in \mathbb{N}$, there exist $R > 1$ and $m \in \mathbb{N}$ such that

$$p_{v_n}(P^j f(0)) \leq \frac{1}{R^j} p_{v_m}(f)$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{HV}(U)$.

We refer to **[15]** for the following result and notations.

Proposition 1.1. *Let $\mathcal{V}$ be a family of weights satisfying condition I and the space $\mathcal{HV}(U)$ contains all the polynomials. Then the topology τ_V restricted to $\mathcal{P}(^m E)$ coincides with the norm topology for each $m \in \mathbb{N}$.*

For $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, a sequence of strictly positive numbers, we write

$$D_\alpha = \{f \in \mathcal{HV}(U) : p_{v_n}(f) \leq \alpha_n \text{ for each n}\}.$$

Clearly, D_α is a τ_V-bounded set. Also, $\{D_\alpha\}_\alpha$ is a fundamental system of absolutely convex τ_V-bounded sets which are τ_0-compact. Further, it can be easily seen that $\{V_{n,\epsilon} : n \in \mathbb{N}, \epsilon > 0\}$, where $V_{n,\epsilon} = \{f \in \mathcal{HV}(U) : p_{v_n}(f) \leq \epsilon\}$, is a fundamental 0-neighbourhood system consisting of absolutely convex τ_0-closed sets. Then the space

$$\mathcal{GV}(U) = \{\phi \in \mathcal{HV}(U)' : \phi | D_\alpha \text{ is } \tau_0 \text{ -continuous}\}$$

is a complete barrelled DF-space, cf. **[3]** whose strong dual is topologically isomorphic to $\mathcal{HV}(U)$. For $U = U_E$, the following linearization result for $\mathcal{HV}(U,F)$ is given in **[16]**, p. 216 and for an arbitrary open set U, the proof follows analogously.

Theorem 1.2 (Linearization theorem). *For an open subset U of a Banach space E and a family $\mathcal{V}$ of weights on U satisfying condition I, there exists a complete barrelled (DF)-space $\mathcal{GV}(U)$ and a mapping $\Delta \in \mathcal{HV}(U, \mathcal{GV}(U))$ with the following property: for each Banach space F and each mapping $f \in \mathcal{HV}(U,F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{GV}(U), F)$ such that $T_f \circ \Delta = f$. Also, the correspondence $f \to T_f$ is a topological isomorphism.*

A sequence of subspaces $\{X_n\}_{n=1}^{\infty}$ of a locally convex space X is called a *Schauder decomposition* of X if for each $x \in X$, there exists a unique sequence $\{x_n\}$ of vectors $x_n \in X_n$ for all n, such that

$$x = \sum_{n=1}^{\infty} x_n = \lim_{m \to \infty} u_m(x),$$

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where the projection maps $\{u_m\}_{m=1}^{\infty}$ defined by $u_m(x) = \sum_{j=1}^{m} x_j$, $m \geq 1$ are continuous. If each X_n is one dimensional and $X_n = span\{e_n\}$, then $\{e_n\}_n$ is a Schauder basis of X.

Let us denote by $\mathcal{S}$, the subspace $\{(\beta_n)\}_{n=1}^{\infty} : \beta_n \in \mathbb{C}$ and $\limsup_{n\to\infty} |\beta_n|^{\frac{1}{n}} \leq 1\}$ of the vector space of all scalar sequences. Corresponding to $\mathcal{S}$, a Schauder decomposition $\{X_n\}_n$ is said to be $\mathcal{S}$-*absolute* if

(i) for each $\beta = (\beta_j) \in \mathcal{S}$ and $x = \sum_{j=1}^{\infty} x_j \in X$, $x_j \in X_j$, $\beta \cdot x = \sum_{j=1}^{\infty} \beta_j x_j \in X$; and
(ii) if p is a continuous semi-norm on X and $\beta \in \mathcal{S}$, then $p_\beta(x) = \sum_{j=1}^{\infty} |\beta_j| p_\beta(x_j)$ defines a continuous semi-norm on X.

A particular case of a result given in [**13**], p. 196, is:

PROPOSITION 1.3. *If the sequence of spaces $\{X_n\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for a locally convex space X, then $\{(X_n)_b{}^*\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for X_b^*.*

Let $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X into Y. An operator T in $\mathcal{L}(X, Y)$ is said to be of *finite rank* if the range of T is finite dimensional. The class of all finite rank operators from X into Y is denoted by $\mathcal{F}(X, Y)$. A locally convex space X is said to have the *approximation property* if for every compact set K of X, p a continuous semi-norm on X and $\epsilon > 0$, there exists a finite rank operator $T = T_{\epsilon,K}$ such that $\sup_{x\in K} p(T(x) - x) < \epsilon$. If $\{T_{\epsilon,K} : \epsilon > 0$ and K varies over compact subsets of $X\}$ is an equicontinuous subset of $\mathcal{F}(X, X)$, X is said to have the BAP. When X is normed and above T can be chosen with $\|T\| \leq 1$, we say X has the metric approximation property.

One can easily prove:

PROPOSITION 1.4. *Let E be a Banach space with the λ-BAP. Then each complemented subspace of E with the projection map P has the $\lambda\|P\|$-BAP.*

We need the following results for our work.

PROPOSITION 1.5 ([**11**]). *If $\{X_n\}_{n=1}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of the locally convex space X, then X has the BAP if and only if each X_n has the BAP.*

THEOREM 1.6 ([**7**]). *Let E be a Banach space. Then E^* has the BAP if and only if $\mathcal{P}_w({}^nE)$ has the BAP for each $n \in \mathbb{N}$.*

We refer to [**12, 13, 23, 26**] for the theory of infinite-dimensional holomorphy and $\mathcal{S}$-absolute Schauder decompositions, and [**12, 17, 21**] for the theory of approximation properties.

In this paper, we show that the sequence of spaces $\{Q({}^nE)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for the space $\mathcal{GV}(U)$; and consequently, E has the BAP if and only if $\mathcal{GV}(U)$ has the BAP. The last section is devoted to the study of the metric approximation property for the space E and $\mathcal{G}_v(E)$.

2. The bounded approximation property for the space $\mathcal{HV}(U)$ and its predual $\mathcal{GV}(U)$. Throughout this section, we assume that U is a balanced open subset of a Banach space E and $\mathcal{V}$ is a family of radial weights defined on U. For studying BAP for the spaces $\mathcal{GV}(U)$ and $\mathcal{HV}(U)$, we first show that the sequence of spaces $\{Q({}^nE)\}_{n=0}^{\infty}$ and $\{\mathcal{P}({}^nE)\}_{n=0}^{\infty}$ forms an $\mathcal{S}$-absolute decomposition for the spaces $\mathcal{GV}(U)$ and $\mathcal{HV}(U)$, respectively.

Let us begin with the following lemmas which are useful for establishing the main result.

LEMMA 2.1. *Let $\mathcal{V}$ be a family of weights satisfying condition II. Then for $\beta = (\beta_n) \in \mathcal{S}$ and a given sequence $\alpha = (\alpha_n)$ of strictly positive numbers, there exists a sequence $\alpha' = (\alpha'_n)$ with $\alpha'_n > 0$ for each n such that*

$$\sum_{j=0}^{\infty} |\beta_j| p_{v_n}(P^j f(0)) \leq \alpha'_n \tag{2}$$

for each $n \in \mathbb{N}$ and $f \in D_\alpha$. In particular,

$$p_{v_n}(n^2 \sum_{j=n}^{\infty} P^j f(0)) \leq \sum_{j=n}^{\infty} j^2 p_{v_n}(P^j f(0) \leq \alpha'_n \tag{3}$$

for each $n \in \mathbb{N}$ and $f \in D_\alpha$.

Proof. Fix $\beta \in \mathcal{S}$. Then, $\beta.f = \sum_{j=0}^{\infty} \beta_j P^j f(0) \in \mathcal{H}(U)$ for each $f \in D_\alpha$, cf. **[12]**, p. 119. Let $f \in D_\alpha$ and $n \in \mathbb{N}$. Then by using our hypothesis on $\mathcal{V}$, there exist $R > 1$ and $m \in \mathbb{N}$ such that

$$p_{v_n}(P^j f(0)) \leq \frac{1}{R^j} p_{v_m}(f)$$

for each $j = 0, 1, 2 \ldots$. Therefore,

$$p_{v_n}(\beta.f) \leq \sum_{j=0}^{\infty} |\beta_j| p_{v_n}(P^j f(0)) \leq \sum_{j=0}^{\infty} \frac{|\beta_j|}{R^j} p_{v_m}(f).$$

Since $\frac{1+R}{2} > 1$, there exists $j_0 \in \mathbb{N}$ such that $|\beta_j| \leq (\frac{(1+R)}{2})^j$, for each $j \geq j_0$. Choose $C = \max\{|\beta_0|, |\beta_1|, \cdots |\beta_{j_0-1}|, 1\}$. Then, $|\beta_j| \leq C(\frac{(1+R)}{2})^j$ for each j. Hence,

$$p_{v_n}(\beta.f) \leq C\alpha_m \sum_{j=0}^{\infty} (\frac{(1+R)}{2R})^j = \alpha_n{}'$$

for each $n \in \mathbb{N}$ and $f \in D_\alpha$. Therefore, the set $\{\beta.f : f \in D_\alpha\} \subset D_{\alpha'}$. Also, (3) follows for $\beta = (j^2)$. □

REMARK 2.2. Note that $\{n^2\beta_n\}_{n=1}^{\infty} \in \mathcal{S}$ for each $\beta = \{\beta_n\}_{n=1}^{\infty} \in \mathcal{S}$. Hence by Lemma 2.1, there exists a sequence $\gamma^{\alpha,\beta} = \{\gamma_n^{\alpha,\beta}\}$ of strictly positive numbers such that

$$p_{v_n}(n^2 \sum_{j=n}^{\infty} \beta_j P^j f(0)) \leq \sum_{j=n}^{\infty} j^2 |\beta_j| p_{v_n}(P^j f(0) \leq \gamma_n^{\alpha,\beta} \tag{4}$$

for each $n \in \mathbb{N}$ and $f \in D_\alpha$. When $\beta_j = 1$ for each j, we get the inequality (3).

LEMMA 2.3. *Let $\mathcal{V}$ be a countable family of weights satisfying condition I and $\mathcal{HV}(U)$ contains all the polynomials. Then the following are true*

(*a*) $\sup_{x \in U} v(x)\|x\|^m < \infty$ *for each $v \in \mathcal{V}$ and $m \in \mathbb{N}$.*

(*b*) *The subspace topology on $\mathcal{Q}(^nE)$, $n \in \mathbb{N}$ induced by $\mathcal{GV}(U)$coincides with its topology τ_n of uniform convergence on B_n.*

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Proof.

(a) For each $x \in U$, choose $\phi_x \in E^*$ with $\|\phi_x\| = 1$ and $\phi_x(x) = \|x\|$. Then for $m \in \mathbb{N}$, the set $\{\phi_x^m : x \in U\}$ is a norm-bounded subset of $\mathcal{P}(^mE)$, and hence τ_V-bounded by Proposition 1.1. Therefore, for $n \in \mathbb{N}$, we have

$$\sup_{x\in U} v_n(x)\|x\|^m \leq \sup_{x\in U}\sup_{y\in U} v_n(y)|\phi_x^m(y)| = \sup_{x\in U} p_{v_n}(\phi_x^m) < \infty.$$

(b) Fix $n \in \mathbb{N}$ arbitrarily and denote by τ_s, the subspace topology induced by $\mathcal{GV}(U)$ on $\mathcal{Q}(^nE)$. Let us note that $B_n \subset D_{t^n}$ for some sequence $t^n = \{t_m^n\}_{m\geq 1}$ by (a). Consequently, $\tau_n \leq \tau_s$.

On the other hand, for each $x \in U$, there exists $m \in \mathbb{N}$ such that $v_m(x) > 0$. Choose $r > 0$ such that $x + rB_E \subset U$. Then for $P \in D_\alpha \cap \mathcal{P}(^nE)$, we have

$$\|P\| = \frac{1}{r^n}\|P\|_{rB_E} \leq \frac{1}{r^n}\|P\|_{x+rB_E} \leq \frac{1}{r^n v_m(x)} p_{v_m}(P) \leq \frac{\alpha_m}{r^n v_m(x)} = C$$

by Lemma 1.13 in [**12**], p. 9. Hence, $D_\alpha \cap \mathcal{P}(^nE) \subset CB_n$. Thus, $\tau_n \geq \tau_s$. □

REMARK 2.4. Let us note that the condition (a) in the above lemma is equivalent to $\mathcal{P}(^nE) \subset \mathcal{HV}(E)$ (or $\mathcal{P}(^nE) \subset \mathcal{HV}(U)$) if and only if each $v \in \mathcal{V}$ is rapidly decreasing (or each $v \in \mathcal{V}$ is bounded and U is bounded).

We now prove

THEOREM 2.5. *Let $\mathcal{V}$ be a family of weights satisfying the conditions I and II, and $\mathcal{HV}(U)$ contains all the polynomials. Then the sequence of spaces $\{\mathcal{Q}(^nE)\}\}_{n=0}^\infty$ is an $\mathcal{S}$-absolute decomposition for $\mathcal{GV}(U)$.*

Proof. For $\phi \in \mathcal{GV}(U)$ and $n \in \mathbb{N}$, define $\phi_n : \mathcal{HV}(U) \to \mathbb{C}$ by

$$\phi_n(f) = \phi(P^nf(0)), \ \ f \in \mathcal{HV}(U).$$

As $\phi|D_\alpha$ is τ_0-continuous for each $\alpha = \{\alpha_n\}_{n=1}^\infty$ and $B_n \subset D_{t^n}$ for $t^n = \{t_m^n\}_{m\geq 0}$, $t_n^m = \sup_{x\in U} v_m(x)\|x\|^n < \infty$ for each m, $\phi_n|B_n$ is τ_0-continuous. Thus, $\phi_n \in \mathcal{Q}(^nE)$. Further, for $\phi \in \mathcal{GV}(U)$ and a sequence $\alpha = \{\alpha_n\}_{n=1}^\infty$ of strictly positive numbers,

$$\|\phi - \sum_{j=0}^{n-1}\phi_j\|_{D_\alpha} = \sup_{f\in D_\alpha}|\phi(\sum_{j\geq n} P^jf(0))| \leq \frac{1}{n^2}\|\phi\|_{D_{\alpha'}} \to 0 \text{ as } n \to \infty$$

since by (3), $\{n^2\sum_{j\geq n} P^jf(0) : f \in D_\alpha\} \subset D_{\alpha'}$. Thus, $\phi = \sum_{j=0}^\infty \phi_j$ in $\mathcal{GV}(U)$.

In order to prove the continuity of the projection maps $R_n : \mathcal{GV}(U) \to \mathcal{GV}(U)$, $R_n(\phi) = \phi_n$, $\phi \in \mathcal{GV}(U)$, $n \in \mathbb{N}$, consider a net (ϕ^η) in $\mathcal{GV}(U)$ converging to 0. Then by (3) and Lemma 2.3(b), we have

$$\|R_n(\phi^\eta)\|_{B_n} \leq \sup_{P\in D_{t^n}}|\phi_n^\eta(P)| = \sup_{P\in D_{t^n}}|\phi^\eta(P)| \leq \frac{1}{n^2}\|\phi^\eta\|_{D_{\alpha'}} \to 0.$$

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Now, we show that $\beta \cdot \phi \in \mathcal{GV}(U)$ for $\phi = \sum_{n=0}^{\infty} \phi_n$ in $\mathcal{GV}(U)$ and $\beta = \{\beta_n\} \in \mathcal{S}$. For $l \geq k$, consider

$$\| \sum_{n=k}^{l} \beta_n \phi_n \|_{D_\alpha} \leq \sum_{n=k}^{l} \frac{1}{n^2} \sup_{f \in D_\alpha} |\phi(n^2 \beta_n P^n f(0))| \leq \|\phi\|_{D_{\gamma(\alpha,\beta)}} \sum_{n=k}^{l} \frac{1}{n^2} \to 0 \text{ as } k, l \to \infty$$

by (4). Thus, the series $\sum_{n=0}^{\infty} \beta_n \phi_n$ converges in $\mathcal{GV}(U)$ which is a dual Fréchet space. Further, proceeding on similar lines, we can establish

$$\sum_{n \geq 1} |\beta_n| \|\phi_n\|_{D_\alpha} \leq \|\phi\|_{D_{\gamma(\alpha,\beta)}} \sum_{n \geq 1} \frac{1}{n^2}.$$

Thus, the sequence of spaces $\{\mathcal{Q}(^n E)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for $\mathcal{GV}(U)$. □

Using Proposition 1.3, the following result which is given in [**15**], is an immediate consequence of Theorem 2.5.

PROPOSITION 2.6. *If $\mathcal{V}$ satisfies the hypothesis of Theorem 2.5, then $\{\mathcal{P}(^n E)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for $\mathcal{HV}(U)$.*

In our next result, we consider spaces E for which $\mathcal{P}(^m E) = \mathcal{P}_w(^m E)$ holds for each $m \in \mathbb{N}$; cf. [**13**] for examples.

THEOREM 2.7. *Let U be an open balanced subset of a Banach space E with $\mathcal{P}(^m E) = \mathcal{P}_w(^m E)$, for each $m \in \mathbb{N}$. Assume that $\mathcal{V}$ satisfies the conditions I and II, and $\mathcal{HV}(U)$ contains all the polynomials. Then the following are equivalent:*

(a) *E^* has the BAP.*
(b) *$\mathcal{P}(^m E)$ has the BAP for each $m \in \mathbb{N}$.*
(c) *$\mathcal{HV}(U)$ has the BAP.*

Proof. (a) ⇒ (b). Follows from Proposition 1.6 since $\mathcal{P}(^m E) = \mathcal{P}_w(^m E)$ for each m.
(b) ⇒ (a). Take $m = 1$ in (b).
(b) ⇔ (c). Use Propositions 1.5 and 2.6. □

Combining the result of Caliskan [**10**], we state the main result of this section as

THEOREM 2.8. *Let $\mathcal{V}$ be a family of weights defined on an open balanced subset U of a Banach space E. Assume that $\mathcal{V}$ satisfies the conditions I and II, and $\mathcal{HV}(U)$ contains all the polynomials. Then the following are equivalent:*

(a) *E has the BAP.*
(b) *$\mathcal{Q}(^m E)$ has the BAP for each $m \in \mathbb{N}$.*
(c) *$\mathcal{GV}(U)$ has the BAP.*

Proof. (a) ⇔ (b) cf. [**9**], Proposition 2.
(b) ⇔ (c). It is a direct consequence of Proposition 1.5 and Theorem 2.5. □

REMARK 2.9. For characterizing the BAP of $\mathcal{G}_v(U)$, the method of $\mathcal{S}$-absolute decompositions is not applicable; indeed, in this case, the sequence $\{\mathcal{Q}(^n E)\}_{n=0}^{\infty}$ does not form an $\mathcal{S}$-absolute decomposition for $\mathcal{G}_v(U)$, as exhibited in

EXAMPLE 2.10. For $E = \mathbb{C}$, consider the weight $v(z) = e^{-|z|}$, $z \in \mathbb{C}$. If $\{\mathcal{Q}(^n \mathbb{C})\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for $\mathcal{G}_v(\mathbb{C})$, then by Proposition 2.1, $\{\mathcal{P}(^n \mathbb{C})\}_{n=0}^{\infty}$ is an

$\mathcal{S}$-absolute decomposition for $\mathcal{H}_v(\mathbb{C})$; but for $f(z) = e^z \in \mathcal{H}_v(\mathbb{C})$, and $z = x$, a real number, we have

$$v(x)|f(x) - \sum_{m=0}^{n} P^m f(0)(x)| = |\frac{e^x - \sum_{m=0}^{n} \frac{x^m}{m!}}{e^x}| \to 1$$

as $x \to \infty$ for any given n, implying that $\{\mathcal{P}(^n\mathbb{C})\}_{n=0}^{\infty}$ is not even a Schauder decomposition for $\mathcal{H}_v(\mathbb{C})$.

REMARK 2.11. Let us note that the condition II satisfied by the family of weights plays a vital role in proving Theorems 2.5 and 2.8. It does not happen in case $\mathcal{V}$ is a singleton set, for

$$p_v(P^j f(0)) \leq \frac{1}{R^j} p_v(P^j f(0))$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_v(U)$, yields that $R \leq 1$.

However, for examples of family of weights satisfying Conditions I and II, we refer to **[15]**.

3. The metric approximation property. In this section, we consider the family $\mathcal{V}$ consisting of a single weight v satisfying condition I. As observed in Example 2.10 of the preceding section, the techniques involving $\mathcal{S}$-absolute Schauder decompositions will not be applicable for proving the results on the BAP for the space $\mathcal{G}_v(U)$. Let us note that for $\mathcal{V} = \{v\}$, the weighted space

$$\mathcal{H}_v(U; F) = \{f \in \mathcal{H}(U; F) : \|f\|_v = \sup_{x \in U} v(x)\|f(x)\| < \infty\}$$

is a Banach space equipped with $\|\cdot\|_v$-norm. For $F = \mathbb{C}$, we write $\mathcal{H}_v(U) = \mathcal{H}_v(U; \mathbb{C})$ and the closed unit ball of $\mathcal{H}_v(U)$ is denoted by B_v. Since B_v is τ_0-compact, cf. **[29]**, p. 349, the predual of $\mathcal{H}_v(U)$ is given by

$$\mathcal{G}_v(U) = \{\phi \in \mathcal{H}_v(U)' : \phi | B_v \text{ is } \tau_0\text{-continuous}\}$$

by the Ng Theorem, cf. **[27]**. The evaluation map $J_U^v : \mathcal{H}_v(U) \to \mathcal{G}_v(U)^*$ defined as $J_U^v(f)(\phi) = \phi(f)$, $\phi \in \mathcal{G}_v(U)$, $f \in \mathcal{H}_v(U)$ is an isometric isomorphism. For $\mathcal{V} = \{v\}$, the Linearization Theorem 1.2 takes the following form, cf. **[1, 18]** .

THEOREM 3.1 (Linearization theorem). *For an open subset U of a Banach space E and a weight v defined on U satisfying condition I, there exists a Banach space $\mathcal{G}_v(U)$ and a mapping $\Delta_v \in \mathcal{H}_v(U, \mathcal{G}_v(U))$ with the following property: for each Banach space F and each mapping $f \in \mathcal{H}_v(U, F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v(U), F)$ such that $T_f \circ \Delta_v = f$. Also, the correspondence Ψ between $\mathcal{H}_v(U, F)$ and $\mathcal{L}(\mathcal{G}_v(U), F)$ given by*

$$\Psi(f) = T_f$$

is an isometric isomorphism.

Writing

$$\mathcal{H}_v(U) \otimes F = \{f \in \mathcal{H}_v(U, F) : f \text{ has finite-dimensional range}\},$$

a direct application of the above theorem is the following result, also given in [**1, 20**]

Proposition 3.2. *Let v be a weight defined on an open subset U of a Banach space E. Then $f \in \mathcal{H}_v(U) \otimes F$ if and only if $T_f \in \mathcal{F}(\mathcal{G}_v(U), F)$.*

We now start this section with the following representation result for the members of $\mathcal{G}_v(U)$.

Theorem 3.3. *Let v be a weight on an open subset U of a Banach space E. Then each u in $\mathcal{G}_v(U)$ has a representation of the form*

$$u = \sum_{n\geq 1} \alpha_n v(x_n)\Delta_v(x_n) \tag{5}$$

for $(\alpha_n) \in l_1$ and $(x_n) \subset U$. Also,

$$\|u\| = \inf\{\|\alpha\|_1 : \alpha \textit{ varies over all represtations of } u \textit{ in } (5)\}. \tag{6}$$

Proof. Corresponding to the dual pair $\langle \mathcal{G}_v(U), \mathcal{H}_v(U)\rangle$, we have

$$\begin{aligned}(J_U^v(B_v))^\circ &= \{\phi \in \mathcal{G}_v(U) : |<\phi, J_U^v(f)>| \leq 1, \text{ for every } f \in B_v\}\\ &= \{\phi \in \mathcal{G}_v(U) : |\phi(f)| \leq 1, \text{ for every } f \in B_v\} = B_{\mathcal{G}_v(U)}.\end{aligned}$$

Also, $J_U^v(B_v) = \{(v\Delta_v)(x) : x \in U\}^\circ$. Consequently, the balanced closed convex hull $\overline{\Gamma}\{(v\Delta_v)(x) : x \in U\}$ of the set $\{(v\Delta_v)(x) : x \in U\}$ coincides with $B_{\mathcal{G}_v(U)}$ by Bipolar Theorem [**19**], p. 192. Thus, for any $u \in \mathcal{G}_v(U)$, there exist sequences $(\alpha_n) \in B_{l_1}$ and $\{x_n\} \subset U$ such that

$$u = \sum_{n\geq 1} \alpha_n \|u\| v(x_n)\Delta_v(x_n).$$

Write $(\beta_n) = (\alpha_n\|u\|)$. Clearly, $\|(\beta_n)\|_1 \leq \|u\|$. Also, for any representation of u in (5), $\|u\| \leq \|(\alpha_n)\|_1$ as $v(x_n)\|\Delta_v(x_n)\| \leq 1$. Thus, (6) follows. □

Proposition 3.4. *Let U be an open subset of a separable Banach space. Then, $\mathcal{G}_v(U)$ is separable for any bounded weight v on U.*

Proof. Since l_1 is separable, there exists a countable dense set Λ in l_1. Choose $\epsilon > 0$ arbitrarily. For α in l_1, there exists $\beta \in \Lambda$ such that $\|\alpha - \beta\|_1 < \frac{\epsilon}{3}$. Let $D = \{y_n : n \in \mathbb{N}\}$ be a countable dense subset of U. Define

$$M = \left\{\sum_{n=1}^{m} \beta_n v(y_n)\delta_{y_n} : y_n \in D,\ \beta \in \Lambda,\ m \in \mathbb{N}\right\}.$$

Then, M is clearly a countable subset of $\mathcal{G}_v(U)$. We now show that $\overline{M} = G_v(U)$. Let us consider $u \in \mathcal{G}_v(U)$. Then, $u = \sum_{n\geq 1} \alpha_n v(x_n)\delta_{x_n}$ for some $(\alpha_n) \in l_1$ and $(x_n) \subset U$. Set $v = \sum_{n\geq 1} \beta_n v(x_n)\delta_{x_n}$. Then, $\|u - v\| < \epsilon$. Let $\lambda = \sup\{v(x) : x \in U\}$. Choose $n_0 \in \mathbb{N}$

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such that $\sum_{n\geq n_0}|\beta_n| < \frac{\epsilon}{3}$. Write $\epsilon' = \frac{\epsilon}{3\|\beta\|_1}(\frac{1}{\inf_{1\leq n\leq n_0} v(x_n)} + \lambda)$. Then by continuity of v and equicontinuity of B_v, there exists $\delta > 0$ such that

$$|v(x_n) - v(y)| < \epsilon', \quad \|f(x_n) - f(y)\| < \epsilon' \text{ for each } f \in B_v$$

for each y with $\|y - x_n\| < \delta$ and $1 \leq n \leq n_0$. Now for given n, $1 \leq n \leq n_0$, there exists $y_n \in D$ such that $\|x_n - y_n\| < \delta$. Write $u_0 = \sum_{n\leq n_0} \beta_n v(y_n)\delta_{y_n}$. Then, we have

$$\|u - u_0\| \leq \|u - v\| + \|v - v_0\| < \frac{2\epsilon}{3} + \|\sum_{n=1}^{n_0} \beta_n(v(x_n)\delta_{x_n} - v(y_n)\delta_{y_n})\|$$

$$\leq \frac{2\epsilon}{3} + \sum_{n=1}^{n_0} |\beta_n||v(x_n) - v(y_n)|\|\delta_{x_n}\| + \sum_{n=1}^{n_0} |\beta_n||v(y_n)|\|\delta_{x_n} - \delta_{y_n}\|$$

$$\leq \frac{2\epsilon}{3} + \epsilon' \sum_{n=1}^{n_0} |\beta_n| \frac{1}{\inf\limits_{1\leq n\leq n_0} v(x_n) + \lambda} \leq \epsilon.$$

□

The next theorem which is a consequence of Proposition 4.2 and Theorem 4.7 proved in [**18**], can be proved directly by using Theorems 3.1 and 3.3 as follows.

THEOREM 3.5. *Let v be a weight on an open subset of a Banach space E. Then the restriction of the map $\Psi : (\mathcal{H}_v(U, F), \tau_c) \to (\mathcal{L}(G_v(U), F), \tau_c)$ to $\|.\|_v$-bounded subsets of $\mathcal{H}_v(U, F)$ is a topological isomorphism.*

Proof. As $\Delta_v(K)$ is a compact subset of $\mathcal{G}_v(U)$ for any compact subset $K \subset U$, $\tau_c - \tau_c$ continuity of Ψ follows. For the converse, consider a bounded subset B of $\mathcal{H}_v(U, F)$ with $\|f\|_v \leq \lambda$ for all $f \in B$. Let L be a compact subset of $\mathcal{G}_v(U)$. Then,

$$L \subset \overline{\Gamma}\{u_m : m \in \mathbb{N}\}$$

for some null sequence (u_m) in $\mathcal{G}_v(U)$. For arbitrarily chosen $\epsilon > 0$, choose $m_0 \in \mathbb{N}$ such that $\|u_m\| < \epsilon$ for $m > m_0$. Then for $u \in L$ with $u = \sum_{m\geq 1} \alpha_m u_m$, $(\alpha_m) \in B_{l_1}$ and $f \in B$, we have

$$\|T_f(u)\| \leq \sum_{m\leq m_0} |\alpha_m|\|T_f(u_m)\| + \lambda\epsilon.$$

Now by Theorem 3.3,

$$u_m = \sum_{i\geq 1} \beta_i^m v(x_i^m)\delta_{x_i^m}$$

for $(\beta_i^m)_{i\geq 1} \subset l_1$ and $(x_i^m)_{i\geq 1} \subset U$, $1 \leq m \leq m_0$. Choose $j \in \mathbb{N}$ such that $\sum_{i>j} |\beta_i^m| < \epsilon$ for each $1 \leq m \leq m_0$. Write $K = \{x_i^m : 1 \leq m \leq m_0, \quad 1 \leq i \leq j\}$ and $C = \lambda \sum_{m\leq m_0} |\alpha_m| \sum_{i\leq j} |\beta_i^m|$. Then, K is a compact subset of U and

$$\sum_{m\leq m_0} |\alpha_m|\|T_f(u_m)\| \leq \lambda\epsilon + C \sup_{x\in K} \|f(x)\|.$$

Consequently,

$$\sup_{u\in L}\|T_f(u)\| \leq 2\lambda\epsilon + C\sup_{x\in K}\|f(x)\|.$$

Thus, $\Psi^{-1}|B$ is $\tau_c - \tau_c$ continuous. □

PROPOSITION 3.6. *Let v be a radial rapidly decreasing weight on a Banach space E. Then E is topologically isomorphic to a 1-complemented subspace of $\mathcal{G}_v(E)$.*

Proof. Since $\sup_{x\in U} v(x)\|x\| < \infty$, the identity map I from E to itself is a member of $\mathcal{H}_v(E, E)$. By Theorem 3.1, there exists $T \in L(\mathcal{G}_v(E), E)$ and $\Delta_v \in \mathcal{H}_v(E, \mathcal{G}_v(E))$ such that $T\circ\Delta_v = I$ and $\|T\| = \|I\|_v$. Write $S = P^1\Delta_v(0)$. Then, $S \in \mathcal{L}(E, \mathcal{G}_v(E))$ and by Cauchy's integral formula, for $t \in E$,

$$S(t) = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{\Delta_v(\zeta t)}{\zeta^2}d\zeta \;\Rightarrow\; T\circ S(t) = t \;\Rightarrow\; \|S(t)\| \geq \frac{1}{\|T\|}\|t\|.$$

Thus, S is an injective map and S^{-1} is continuous.
Define $P = S\circ T$. Then, P is a projection map from $\mathcal{G}_v(E)$ onto $P(\mathcal{G}_v(E)) = S(E)$. In order to show that $\|P\| \leq 1$, consider a $u \in B_{\mathcal{G}_v(E)}$. Then by Theorem 3.3,

$$u = \sum_{m=1}^{\infty}\alpha_m v(x_m)\Delta_v(x_m)$$

for some sequence $(\alpha_m) \in B_{l_1}$ and $(x_m) \subset E$. Consequently,

$$\|P(u)\| = \|\sum_{m=1}^{\infty}\alpha_m v(x_m)S(x_m)\| \leq \sum_{m=1}^{\infty}|\alpha_m|v(x_m)\|P^1\Delta_v(0)(x_m)\| \leq 1$$

as for $x \in E$, $\|P^1\Delta_v(0)(x)\| \leq \sup_{|\lambda|=1}\|\Delta_v(\lambda x)\| \leq \sup_{|\lambda|=1}\frac{1}{v(\lambda x)} = \frac{1}{v(x)}$.
Thus, $\|P\| = 1$ and E is topologically isomorphic to a 1-complemented subspace of $\mathcal{G}_v(E)$. □

Finally, we prove

THEOREM 3.7. *Let v be a radial rapidly decreasing weight on a Banach space E satisfying $v(x) \leq v(y)$ whenever $\|x\| \geq \|y\|$, $x, y \in E$. Then the following assertions are equivalent:*

(*a*) *E has the MAP.*
(*b*) $\overline{\{P \in \mathcal{P}_f(E, F) : \|P\|_v \leq 1\}}^{\tau_c} = B_{\mathcal{H}_v(E,F)}$ *for any Banach space F.*
(*c*) $\overline{B_{\mathcal{H}_v(E)\bigotimes F}}^{\tau_c} = B_{\mathcal{H}_v(E,F)}$ *for any Banach space F.*
(*d*) $\Delta_v \in \overline{B_{\mathcal{H}_v(E)\bigotimes\mathcal{G}_v(E)}}^{\tau_c}$.
(*e*) *$\mathcal{G}_v(E)$ has the MAP.*

Proof. (*a*) ⇒ (*b*). Let $f \in \mathcal{H}_v(E, F)$ with $\|f\|_v \leq 1$ and $K \subset E$ be compact. Consider the Cesàro means of f defined as

$$C_m f(x) = \frac{1}{m+1}\sum_{k=0}^{m}(\sum_{j=0}^{k}P^j f(x)), \quad x \in E, \; m \in \mathbb{N}.$$

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Since $C_mf \to f$ in τ_0-topology, there exists $m \in \mathbb{N}$ such that $\sup_{x\in K} \|C_m(f)(x) - f(x)\| < \frac{\epsilon}{2}$. As $C_m(f) \in \mathcal{P}(E, F)$, there exists $\delta > 0$ such that

$$\|C_m(f)(x) - C_m(f)(y)\| < \frac{\epsilon}{2}, \tag{7}$$

whenever $x \in K$, $y \in E$ with $\|x - y\| < \delta$. By (a), there exists $T \in \mathcal{F}(E, E)$ with $\|T\| \leq 1$ such that

$$\sup_{x\in K} \|T(x) - x\| < \delta. \tag{8}$$

Using (3.3) and (3.4),

$$\sup_{x\in K} \|C_m(f) \circ T(x) - C_m(f)(x)\| < \frac{\epsilon}{2}.$$

Hence,

$$p_K(C_m(f) \circ T - f) < \epsilon.$$

Clearly, $C_m(f) \circ T \in \mathcal{P}_f(E, F)$. Also, $\|C_m(f)oT\|_v \leq 1$; indeed, $\|C_m(g)(u)\| \leq \sup_{|\lambda|=1} \|g(\lambda u)\|$, for any $u \in E$ and $g \in \mathcal{H}(E)$ cf. **[22]** yields

$$\begin{aligned}\|C_m(f) \circ T\|_v &= \sup_{x\in E} v(x)\|C_m(f)(T(x))\| \leq \sup_{x\in E} v(x) \sup_{|\lambda|=1} \|f \circ T(\lambda x)\| \\ &\leq \sup_{x\in E} \sup_{|\lambda|=1} v(T(\lambda x))\|f \circ T(\lambda x)\| \leq \|f\|_v\end{aligned}$$

as $\|T(\lambda x)\| \leq \|x\|$ for $|\lambda| = 1$ and v is norm decreasing.
$(b) \Rightarrow (c)$. As v is rapidly decreasing, $\mathcal{P}_f(E, F) \subset \mathcal{P}(E, F) \subset \mathcal{H}_v(E) \otimes F$ for any Banach space F. Thus, $\overline{B_{\mathcal{H}_v(E)\bigotimes F}}^{\tau_c} = B_{\mathcal{H}_v(E,F)}$ by (b).
$(c) \Rightarrow (d)$ follows as $\Delta_v \in B_{\mathcal{H}_v(E,G_v(E))}$.
$(d) \Rightarrow (e)$. Since $T_{\Delta_v} = I_{\mathcal{G}_v}$ and $\mathcal{H}_v(E) \bigotimes \mathcal{G}_v(E)$ can be identified with $\mathcal{F}(\mathcal{G}_v(E), \mathcal{G}_v(E))$ by Proposition 3.2, $I_{\mathcal{G}_v(E)} \in \overline{B_{\mathcal{F}(\mathcal{G}_v(E),\mathcal{G}_v(E))}}^{\tau_c}$ by Theorem 3.5 and (d). Thus, $\mathcal{G}_v(E)$ has the MAP.
$(e) \Rightarrow (a)$. Follows from Propositions 1.4 and 3.6. □

Since a reflexive Banach space E having AP is equivalent to having MAP, cf. **[21]**, p. 40, we have the following characterization.

THEOREM 3.8. *For a weight v satisfying the conditions of Theorem 3.7, a reflexive Banach space E has BAP if and only if $\mathcal{G}_v(E)$ has BAP.*

It is known **[9]** that for $v = 1$, a separable Banach space E has the BAP if and only if $\mathcal{G}_v(U_E)$ has the BAP. It would be interesting to know the solution of the following problem for the case when v is a radial rapidly decreasing weight, or the analogoue of Theorem 3.7 for the BAP.

PROBLEM 3.9. *Let E be a separable Banach space and v be a radial rapidly decreasing weight on E such that $v(x) \leq v(y)$, whenever $\|x\| \geq \|y\|$, $x, y \in E$. Then E has the BAP if and only $\mathcal{G}_v(E)$ has the BAP.*

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