ON GROUPS WITH ALL COMPOSITION FACTORS ISOMORPHIC

R. Bercov

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By the celebrated theorem of Jordan [3] and Hölder [2], there is associated with each finite group G a family of distinct simple groups H_i such that every composition series of G has n_i factor groups isomorphic to H_i and no others. We denote the collection of pairs (H_i, n_i) by $CF(G)$. Conversely, given k pairs (H_i, n_i), we may construct by an easy direct product procedure a group G with $CF(G) = \{(H_i, n_i) | i = 1, \ldots, k\}$. The composition factors, of course, do not in general determine the group. The purpose of this note is to give a necessary and sufficient condition in the case $k = 1$ where $H = H_1$ is non-abelian that there should be only one isomorphism class of groups G with $CF(G) = \{(H, n)\}$. We require a very weak form of the well known conjecture of Schreier [4] that the outer automorphism group of a finite simple group is solvable.

THEOREM. Let H be a non-abelian finite simple group such that no subgroup of the outer automorphism group of H is isomorphic to H. Let m be the smallest degree of a faithful permutation representation of H. Let H^n be the direct product of n copies of H. Then there exists G not isomorphic to H^n with $CF(G) = CF(H^n)$ if and only if $n > m$.

<u>Proof</u>. If $n > m$ holds, H has a faithful permutation representation $\overline{H}$ on $n-1$ letters (not necessarily moving all letters). It is easily seen that the wreath product G of $\overline{H}$ with itself [1, p. 81] satisfies $CF(G) = CF(H^n)$ and is of larger exponent than H^n.

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We now choose n smallest such that G not isomorphic to H^n exists with $CF(G) = CF(H^n)$. Let M be a maximal normal subgroup of G. By the minimality of n, we have $M = M_1 x \ldots x M_{n-1}$, where M_i is isomorphic to H for $i = 1, \ldots, n-1$. For $y \in G$, we denote by $a(y)$ the automorphism of M which maps $z \in M$ to $y^{-1} z y$. Since an automorphism of a direct product of finitely many non-abelian simple groups permutes the factors [1, p. 135], $a(y)$ induces a permutation on the M_i which we denote by $\bar{y}$. Let $\bar{G} = \{\bar{y} | y \in G\}$.

We claim that no M_i is fixed by all $\bar{y} \in \bar{G}$. Suppose the contrary. Let $L = M_i$ be fixed by $\bar{G}$ and let C be the product of the M_j, $j \neq i$. Then L is normal in G and for $y \in G$ there is an automorphism y^* of L which maps $z \in L$ to $y^{-1} z y$. Let K, the centralizer of L in G, be the kernel of the map $y \to y^*$. Since $M = L \times C$ we have C contained in K. Since L is isomorphic to H, it has trivial center; thus $K \cap L = 1$. If K were larger than C, KL would be larger than the maximal normal subgroup M. We would then have $G = K \times L$, K isomorphic to M by the minimality of n, and G isomorphic to H^n contrary to hypothesis. Thus $K = C$, and $G^* = \{y^* | y \in G\}$ has normal subgroup $L^* = \{y^* | y \in L\}$ with factor group isomorphic to H. Since L^* is the inner automorphism group of L, the outer automorphism group of L, hence of H, has a subgroup isomorphic to H, contrary to hypothesis.

$\bar{G}$ is therefore non-trivial of degree $n-1$. Since the M_j are normal in M, the kernel of the map from G to $\bar{G}$ contains M. Since $\bar{G}$ is non-trivial, it follows from the maximality of M that $\bar{G}$ is isomorphic to G/M; hence to H. Thus H has a faithful permutation representation of degree $n - 1$. We conclude that $n - 1 \geq m$ and $n > m$.

We now illustrate our result by applying the theorem in its simplest case.

COROLLARY. There exists a group G with n composition factors, each isomorphic to A_k, the alternating group on

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$k \geq 5$ letters, such that G is not the direct product of n subgroups isomorphic to A_k if and only if $n > k$.

Proof. The Schreier conjecture is valid for A_k [5, p. 314]. A_k has a natural faithful representation of degree k but none of smaller degree since the symmetric group on $k - 1$ letters has fewer elements than the alternating group on k letters.

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University of Alberta, Edmonton

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