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THE MULTIFRACTAL SPECTRUM OF CONTINUED FRACTIONS WITH NONDECREASING PARTIAL QUOTIENTS

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Abstract

Let $[a_1(x), a_2(x), \dots, a_n(x), \dots]$ be the continued fraction expansion of $x \in [0,1)$ and $q_n(x)$ be the denominator of its nth convergent. The irrationality exponent and Khintchine exponent of x are respectively defined by

$$\overline{v}(x) = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} \quad \text{and} \quad \gamma(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \log a_i(x).$$

We study the multifractal spectrum of the irrationality exponent and the Khintchine exponent for continued fractions with nondecreasing partial quotients. For any $v > 2$, we completely determine the Hausdorff dimensions of the sets $\{x \in [0,1) : a_1(x) \le a_2(x) \le \cdots, \overline{v}(x) = v\}$ and

$$\Big\{x \in [0,1) : a_1(x) \le a_2(x) \le \cdots, \lim_{n\to\infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{\psi(n)} = 1\Big\},$$

where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\psi(n) \to \infty$ as $n \to \infty$.

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1. Introduction

Diophantine approximation is a branch of number theory that can be described as a quantitative analysis of the density of the rational numbers in the real numbers. The first result is due to Dirichlet and is a simple consequence of the pigeonhole principle.

THEOREM 1.1 (Dirichlet, 1842). *For any $x \in [0,1)$ and $t > 1$, there exists $(q,p) \in \mathbb{N}^2$ such that*

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$$\left|x - \frac{p}{q}\right| < \frac{1}{qt} \quad \textit{and} \quad 1 \le q \le t.$$

Denote

$$J(v) = \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^v} \text{ for infinitely many } (q,p) \in \mathbb{N}^2\right\}.$$

Dirichlet's theorem implies that the set $J(v)$ equals $[0,1)$ for any $v \le 2$. Khintchine [16] proved that the set $J(v)$ is of Lebesgue measure zero for any $v > 2$. Jarník [13] and Besicovitch [1] independently showed that the Hausdorff dimension of these null sets $J(v)$ is $2/v$. Since the map $v \to J(v)$ is nonincreasing, it is natural to define

$$\overline{v}(x) = \sup\{v \in \mathbb{R} : x \in J(v)\}. \tag{1.1}$$

We call $\overline{v}(x)$ the *irrationality exponent* of an irrational number $x \in [0,1)$. The irrationality exponent $\overline{v}(x)$ reflects how well an irrational number x can be approximated by rational numbers: the higher the exponent, the better the approximation.

The theory of continued fractions is closely related to Diophantine approximation. It is well known that continued fraction expansions can be induced by the Gauss map $T : [0,1) \to [0,1)$ defined by

$$T(0) := 0, \quad T(x) := 1/x \pmod 1 \quad \text{for } x \in (0,1).$$

Each irrational number $x \in [0,1)$ admits a unique *continued fraction expansion*

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}} = [a_1(x), a_2(x), \ldots, a_n(x), \ldots], \tag{1.2}$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$ $(n \ge 2)$ are called the *partial quotients* of the continued fraction expansion of x. For each $n \ge 1$, let the fraction

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots + \cfrac{1}{a_n(x)}}}} = [a_1(x), a_2(x), \ldots, a_n(x)]$$

be the nth convergent of the continued fraction expansion of x. Via continued fractions, the irrationality exponent defined in (1.1) can be represented by

$$\overline{v}(x) = 2 + \limsup_{n \to \infty} \frac{\log a_{n+1}(x)}{\log q_n(x)}. \tag{1.3}$$

From the fundamental work of Khintchine [16] (see Bugeaud [3, Ch. 1]), $\overline{v}(x) = 2$ for Lebesgue almost all irrational numbers. The *Khintchine exponent* of x with continued fraction expansion (1.2) is defined (if the limits exist) by

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$$\gamma(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log a_i(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log a_1(T^{i-1}(x)).$$

The Gauss map T is ergodic (see, for example, [11]) with respect to the Gauss measure $dx/((x+1)\log 2)$. By Birkhoff's ergodic theorem, for Lebesgue almost all $x \in [0, 1)$,

$$\gamma(x) = \int_0^1 \frac{\log a_1(x)}{(x+1)\log 2}\, dx = \log(2.6584\cdots).$$

For more details about continued fractions, we refer to [11, 17].

Much attention has been paid to the multifractal analysis of the level sets of the irrationality exponent and Khintchine exponent. For any $v > 2$, a result of Good [10, Theorem 9] implies that the set $E_{\overline{v}(x)\geq v}(v) = \{x \in [0, 1) : \overline{v}(x) \geq v\}$ is of Hausdorff dimension $2/v$. The main result of Bugeaud [2, Theorem 1] shows that the set $E_{\overline{v}(x)=v}(v) = \{x \in [0, 1) : \overline{v}(x) = v\}$ is also of Hausdorff dimension $2/v$. Sun and Wu [22] considered the set

$$E(v) = \Big\{x \in [0, 1) : \ 2 + \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = v\Big\}$$

and proved that $E(v)$ has Hausdorff dimension $1/v$. Replacing the lim sup by lim inf in (1.3), one can define the corresponding irrationality exponent by

$$\underline{v}(x) = 2 + \liminf_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)}.$$

Tan and Zhou [23] calculated the Hausdorff dimension of the intersection of level sets defined by $\overline{v}(x)$ and $\underline{v}(x)$, and also showed that the set $E_{\underline{v}(x)\geq v}(v) = \{x \in [0, 1) : \underline{v}(x) \geq v\}$ is of Hausdorff dimension $1/v$ for any $v > 2$. Based on these dimensional results for the sets $E(v)$ and $E_{\underline{v}(x)\geq v}(v)$, it follows easily that the set $E_{\underline{v}(x)=v}(v) = \{x \in [0, 1) : \underline{v}(x) = v\}$ is of Hausdorff dimension $1/v$ for any $v > 2$. For the multifractal analysis of level sets of the Khintchine exponent $\gamma(x)$, Fan *et al.* [6, Theorem 1.2] presented a complete characterisation for the Hausdorff dimension of the sets

$$E_{\gamma(x)=\xi}(\xi) = \{x \in [0, 1) : \gamma(x) = \xi\} \quad (0 \leq \xi \leq \infty).$$

More precisely, they proved that the Hausdorff dimension of the set $E_{\gamma(x)=\xi}(\xi)$, as a function of $\xi \in [0, \infty)$, is neither concave nor convex, and that the set $E_{\gamma(x)=\xi}(\infty)$ is of Hausdorff dimension $1/2$. This shows that there exist uncountably many points with infinite Khintchine exponent. Fan *et al.* [6, 7] gave a more refined classification for the set $E_{\gamma(x)=\xi}(\infty)$ by considering the multifractal spectrum of the level sets of the fast Khintchine exponent defined by

$$K(\psi) = \Big\{x \in [0, 1) : \lim_{n\to\infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{\psi(n)} = 1\Big\},$$

where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$.

Various related exponents have been investigated. For example, Pollicott and Weiss [19] studied the Lyapunov exponent of the Gauss map, Kesseböhmer and Stratmann

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[15] the Minkowski's question mark function, Nicolay and Simons [18] the Hölder exponent, Jaffard and Martin [12] the Brjuno function, Fang *et al.* [8] the convergence exponent and Song *et al.* [21] the irrationality exponent and the convergence exponent.

Multifractal analysis of sets characterised by two (or more) different Diophantine characteristics could potentially show that they are independent or, conversely, help to detect profound links between these characteristics. This paper is mainly concerned with the multifractal spectrum of the irrationality exponent and the Khintchine exponent defined by a nondecreasing sequence of partial quotients. That is, we investigate the Hausdorff dimension of the intersection of the sets $E_{\overline{v}(x)=v}(v)$, $K(\psi)$ and Λ, where

$$\Lambda = \{x \in [0,1) : \ a_n(x) \le a_{n+1}(x) \text{ for all } n \ge 1\}.$$

By a result of Ramharter [20], the set Λ is of Hausdorff dimension $1/2$ (see also Jordan and Rams [14] for general results in the setting of infinite iterated function systems).

Throughout this paper, we use the notation $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension (see [5]). We are now in a position to state our main results.

THEOREM 1.2. *For any $v > 2$,*

$$\begin{cases} \dim_{\mathrm{H}}(E(v) \cap \Lambda) = \dim_{\mathrm{H}}(E_{\overline{v}(x)=v}(v) \cap \Lambda) = \dim_{\mathrm{H}}(E_{\overline{v}(x)\ge v}(v) \cap \Lambda) = 1/v, \\ \dim_{\mathrm{H}}(E_{\underline{v}(x)=v}(v) \cap \Lambda) = \dim_{\mathrm{H}}(E_{\underline{v}(x)\ge v}(v) \cap \Lambda) = 1/v. \end{cases}$$

We are also interested in the Hausdorff dimension of the intersection of Λ with the sets $E_{\overline{v}(x)\le v}(v) = \{x \in [0,1) : \overline{v}(x) \le v\}$ and $E_{\underline{v}(x)\le v}(v) = \{x \in [0,1) : \underline{v}(x) \le v\}$.

THEOREM 1.3. *For any $v > 2$,*

$$\dim_{\mathrm{H}}(E_{\overline{v}(x)\le v}(v) \cap \Lambda) = \dim_{\mathrm{H}}(E_{\underline{v}(x)\le v}(v) \cap \Lambda) = \tfrac{1}{2}.$$

Let ψ and $\tilde{\psi}$ be positive functions defined on $\mathbb{N}$. We say ψ and $\tilde{\psi}$ are equivalent if $\psi(n)/\tilde{\psi}(n) \to 1$ as $n \to \infty$. Fan *et al.* [7, Lemma 3.1] proved that $K(\psi) \ne \emptyset$ if and only if ψ is equivalent to a nondecreasing function. This also applies to the subset $K(\psi) \cap \Lambda$. In the following we always assume that ψ is nondecreasing.

THEOREM 1.4. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\psi(n) \to \infty$ as $n \to \infty$.*

(i) *If $\psi(n)/(n \log n) \to \alpha \ \ (0 \le \alpha < \infty)$ as $n \to \infty$, then*

$$\dim_{\mathrm{H}}(K(\psi) \cap \Lambda) = \begin{cases} 0, & 0 \le \alpha < 1, \\ (\alpha - 1)/2\alpha, & 1 \le \alpha < \infty. \end{cases}$$

(ii) *If $\psi(n)/(n \log n) \to \infty$ as $n \to \infty$ and the sequence $\{\psi(n) - \psi(n-1)\}_{n \ge 1}$ is nondecreasing, then*

$$\dim_{\mathrm{H}}(K(\psi) \cap \Lambda) = \frac{1}{1 + \limsup_{n\to\infty} \psi(n+1)/\psi(n)}.$$

From the proof of Theorem 1.4, we can calculate the Hausdorff dimension of the intersection of the level sets of the Khintchine exponent $\gamma(x)$ and Λ.

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COROLLARY 1.5. *For any* $0 \le \xi \le \infty$,

$$\dim_{\mathrm{H}}(E_{\gamma(x)=\xi}(\xi) \cap \Lambda) = \begin{cases} 0, & 0 \le \xi < \infty, \\ 1/2, & \xi = \infty. \end{cases}$$

The Lyapunov exponent of a dynamical system is a quantity that characterises the rate of separation of infinitesimally close trajectories. In the dynamical system of continued fractions, the Lyapunov exponent of orbits of the Gauss map T is defined whenever the limits exist by

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x))| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j(x))|$$

(see Devaney [4]). The Hausdorff dimension of the level sets

$$E_{\lambda(x)=\xi}(\xi) = \{x \in [0,1) : \lambda(x) = \xi\} \quad (0 \le \xi \le \infty)$$

has been completely characterised in Fan *et al.* [6, Theorem 1.3]. Similarly, we can define the so-called fast Lyapunov exponent of the Gauss map T by

$$\lambda^{\psi}(x) := \lim_{n\to\infty} \frac{1}{\psi(n)} \log |(T^n)'(x))| = \lim_{n\to\infty} \frac{1}{\psi(n)} \sum_{j=0}^{n-1} \log |T'(T^j(x))|,$$

where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$. Let

$$L(\psi) = \{x \in [0,1) : \lambda^{\psi}(x) = 1\}.$$

From [6, Lemma 2.7],

$$\lambda(x) = \lim_{n\to\infty} \frac{2 \log q_n(x)}{n}. \tag{1.4}$$

The following result follows directly from the proof of Corollary 1.5.

COROLLARY 1.6. *For any* $0 \le \xi \le \infty$,

$$\dim_{\mathrm{H}}(E_{\lambda(x)=\xi}(\xi) \cap \Lambda) = \begin{cases} 0, & 0 \le \xi < \infty, \\ 1/2, & \xi = \infty. \end{cases}$$

Under the condition $\psi(n)/n \to \infty$ as $n \to \infty$, we deduce from (1.4) and (2.2) (see below) that $K(\psi) = L(2\psi)$. Then from Theorem 1.4, the Hausdorff dimension of the intersection of the sets $L(\psi)$ and Λ is also determined.

COROLLARY 1.7. *Let* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be a function satisfying* $\psi(n)/n \to \infty$ *as* $n \to \infty$.

(i) *If* $\psi(n)/(n \log n) \to \alpha$ $(0 \le \alpha < \infty)$ *as* $n \to \infty$, *then*

$$\dim_{\mathrm{H}}(L(\psi) \cap \Lambda) = \begin{cases} 0, & 0 \le \alpha < 2, \\ (\alpha - 2)/2\alpha, & 2 \le \alpha < \infty. \end{cases}$$

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(ii) *If $\psi(n)/(n\log n) \to \infty$ as $n \to \infty$ and the sequence $\{\psi(n) - \psi(n-1)\}_{n\geq 1}$ is nondecreasing, then*

$$\dim_{\mathrm{H}}(L(\psi) \cap \Lambda) = \frac{1}{1 + \limsup\limits_{n\to\infty} \psi(n+1)/\psi(n)}.$$

We use $\mathbb{N}$ to denote the set of all positive integers, $|\cdot|$ denotes the length of a subinterval of $[0, 1)$, $\exp(x)$ the natural exponential function, $\lfloor x \rfloor$ the largest integer not exceeding x and $\mathcal{H}^s$ the s-dimensional Hausdorff measure of a set.

The paper is organised as follows. In Section 2 we present some elementary properties and useful lemmas concerning the dimensional results in continued fractions. Section 3 is devoted to the proofs of the main results.

2. Preliminaries

2.1. Elementary properties of continued fractions. For $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we call

$$I_n(a_1, \ldots, a_n) := \{x \in [0, 1) : \ a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

a *basic interval* of order n for the continued fraction. All points in $I_n(a_1, \ldots, a_n)$ have the same $p_n(x)$ and $q_n(x)$. Thus, for $x \in I_n(a_1, \ldots, a_n)$, we write

$$p_n(a_1, \ldots, a_n) = p_n = p_n(x) \quad \text{and} \quad q_n(a_1, \ldots, a_n) = q_n = q_n(x).$$

It is well known (see [17, page 4]) that p_n and q_n satisfy the recursive formula:

$$\begin{cases} p_{-1} = 1, & p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 1); \\ q_{-1} = 0, & q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1). \end{cases} \tag{2.1}$$

By the second formula of (2.1),

$$\prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n} (a_k + 1) \leq 2^n \prod_{k=1}^{n} a_k. \tag{2.2}$$

PROPOSITION 2.1 [11, page 18]. *For any $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, $I_n(a_1, a_2, \ldots, a_n)$ is the interval with the endpoints*

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

As a result, the length of $I_n(a_1, a_2, \ldots, a_n)$ is

$$|I_n(a_1, a_2, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}.$$

Combining (2.2) and Proposition 2.1, we deduce that

$$2^{-2n-1}\Big(\prod_{k=1}^{n} a_k\Big)^{-2} \leq |I_n(a_1, a_2, \ldots, a_n)| \leq \Big(\prod_{k=1}^{n} a_k\Big)^{-2}. \tag{2.3}$$

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2.2. Some useful lemmas. The first lemma below gives a lower bound of the Hausdorff dimension of some sets of points whose partial quotients are nondecreasing.

LEMMA 2.2 [8, Lemma 3.4]. *Let* $\{s_n\}_{n\geq 1}$ *be a sequence of positive integers tending to infinity with* $s_n \geq 2$ *for any* $n \geq 1$. *Set*

$$\mathbb{F}(\{s_n\}_{n\geq 1}) = \{x \in [0,1) : ns_n \leq a_n(x) < (n+1)s_n \text{ for all } n \geq 1\}.$$

Then

$$\dim_{\mathrm{H}} \mathbb{F}(\{s_n\}_{n\geq 1}) = \frac{1}{2 + \limsup_{n\to\infty} (2\log((n+1)!\,) + \log s_{n+1})/\log(s_1 s_2 \cdots s_n)}.$$

Combining [8, Theorem 2.4] and [9, Lemma 3.1] immediately yields the Hausdorff dimension of some lim inf level sets whose partial quotients are nondecreasing.

LEMMA 2.3. *For any* $0 \leq \alpha < \infty$,

$$\dim_{\mathrm{H}} \left\{x \in \Lambda : \liminf_{n\to\infty} \frac{\log a_n(x)}{\log n} \leq \alpha\right\} = \begin{cases} 0, & 0 \leq \alpha < 1, \\ (\alpha - 1)/2\alpha, & 1 \leq \alpha < \infty. \end{cases}$$

3. Proofs of main results

This section is devoted to the proofs of the main results. Our method is inspired by those of Fan *et al.* [7] and Fang *et al.* [8].

PROOF OF THEOREM 1.2. For any $v > 2$, it is clear that

$$E(v) \subseteq E_{\overline{v}(x)=v}(v) \subseteq E_{\overline{v}(x)\geq v}(v) \quad \text{and} \quad E(v) \subseteq E_{\underline{v}(x)=v}(v) \subseteq E_{\underline{v}(x)\geq v}(v) \subseteq E_{\overline{v}(x)\geq v}(v).$$

The next lemma follows from the monotonicity of Hausdorff dimension [5, page 32].

LEMMA 3.1. *For any* $v > 2$,

$$\begin{cases} \dim_{\mathrm{H}}(E(v) \cap \Lambda) \leq \dim_{\mathrm{H}}(E_{\overline{v}(x)=v}(v) \cap \Lambda) \leq \dim_{\mathrm{H}}(E_{\overline{v}(x)\geq v}(v) \cap \Lambda), \\ \dim_{\mathrm{H}}(E(v) \cap \Lambda) \leq \dim_{\mathrm{H}}(E_{\underline{v}(x)=v}(v) \cap \Lambda) \leq \dim_{\mathrm{H}}(E_{\underline{v}(x)\geq v}(v) \cap \Lambda) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \dim_{\mathrm{H}}(E_{\overline{v}(x)\geq v}(v) \cap \Lambda). \end{cases}$$ □

In view of Lemma 3.1, we divide the proof of Theorem 1.2 into two steps: the upper bound of $\dim_{\mathrm{H}}(E_{\overline{v}(x)\geq v}(v) \cap \Lambda)$ and the lower bound of $\dim_{\mathrm{H}}(E(v) \cap \Lambda)$.

The upper bound of $\dim_{\mathrm{H}}(E_{\overline{v}(x)\geq v}(v) \cap \Lambda)$. Our method is to choose a suitable positive real number s such that $\mathcal{H}^s(E_{\overline{v}(x)\geq v}(v) \cap \Lambda) < \infty$. Let us remark that countable sets are of Hausdorff dimension zero, and the difference of the sets Λ and

$$\Lambda_\infty = \{x \in [0,1) : \ a_n(x) \leq a_{n+1}(x) \text{ for all } n \geq 1 \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}$$

is a countable set. Thus we only need to consider the Hausdorff dimension of the set $E_{\overline{v}(x)\geq v}(v) \cap \Lambda_\infty$. For $0 < \varepsilon < v - 2$ and $M \geq 1$, let

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$$J_n(\sigma_1,\dots,\sigma_n) = \bigcup_{\ell \ge (\sigma_1\cdots\sigma_n)^{\nu-2-\varepsilon}} I_{n+1}(\sigma_1,\dots,\sigma_n,\ell)$$

and $C_n = \{(\sigma_1,\dots,\sigma_n) \in \mathbb{N}^n : \sigma_1\cdots\sigma_n \ge M^n\}$. Then by (2.2),

$$\begin{aligned}
&E_{\overline{\nu}(x)\ge\nu}(\nu) \cap \Lambda_\infty \\
&\subseteq \Big\{x \in \Lambda_\infty : \limsup_{n\to\infty} \frac{\log a_{n+1}(x)}{\log a_1(x) + \cdots + \log a_n(x)} \ge \nu - 2\Big\} \\
&\subseteq \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\{x \in [0,1) : a_{n+1}(x) \ge (a_1(x)\cdots a_n(x))^{\nu-2-\varepsilon} \text{ and } a_1(x)\cdots a_n(x) \ge M^n\} \\
&= \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{(\sigma_1,\dots,\sigma_n)\in C_n} J_n(\sigma_1,\dots,\sigma_n). \qquad (3.1)
\end{aligned}$$

It follows from (2.3) that

$$\Big|\bigcup_{\ell \ge (\sigma_1\cdots\sigma_n)^{\nu-2-\varepsilon}} I_{n+1}(\sigma_1,\dots,\sigma_n,\ell)\Big| \le \sum_{\ell \ge (\sigma_1\cdots\sigma_n)^{\nu-2-\varepsilon}} \frac{1}{\ell^2\cdot(\sigma_1\cdots\sigma_n)^2} \le \frac{1}{(\sigma_1\cdots\sigma_n)^{\nu-\varepsilon}}. \qquad (3.2)$$

We are now in a position to obtain the upper bound of $\dim_{\mathrm{H}}(E_{\overline{\nu}(x)\ge\nu}(\nu)\cap\Lambda)$. Let s, M be two real numbers satisfying

$$s = \frac{1+2\varepsilon}{\nu-\varepsilon} \quad \text{and} \quad \frac{1}{M^\varepsilon}\cdot\sum_{j=1}^{\infty}\frac{1}{j^{1+\varepsilon}} < \frac{1}{2}. \qquad (3.3)$$

Then we deduce from (3.1), (3.2) and (3.3) that

$$\begin{aligned}
\mathcal{H}^s(E_{\overline{\nu}(x)\ge\nu}(\nu)\cap\Lambda_\infty) &\le \liminf_{n\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\dots,\sigma_n)\in C_n}|J_n(\sigma_1,\dots,\sigma_n)|^s \\
&\le \liminf_{n\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\dots,\sigma_n)\in C_n}\frac{1}{(\sigma_1\cdots\sigma_n)^{1+2\varepsilon}} \\
&\le \liminf_{n\to\infty}\sum_{n=N}^{\infty}\Big(\frac{1}{M^\varepsilon}\cdot\Big(\sum_{j=1}^{\infty}\frac{1}{j^{1+\varepsilon}}\Big)\Big)^n = 0.
\end{aligned}$$

This shows that

$$\dim_{\mathrm{H}}(E_{\overline{\nu}(x)\ge\nu}(\nu)\cap\Lambda) = \dim_{\mathrm{H}}(E_{\overline{\nu}(x)\ge\nu}(\nu)\cap\Lambda_\infty) \le \frac{1+2\varepsilon}{\nu-\varepsilon}$$

and letting $\varepsilon \to 0^+$ gives the desired upper bound.

The lower bound of $\dim_{\mathrm{H}}(E(\nu)\cap\Lambda)$. Recall that

$$E(\nu)\cap\Lambda = \Big\{x \in \Lambda : \lim_{n\to\infty}\frac{\log a_{n+1}(x)}{\log q_n(x)} = \nu - 2\Big\}.$$

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To bound $\dim_{\mathrm{H}}(E(v) \cap \Lambda)$ from below, we shall construct a Cantor subset of $E(v) \cap \Lambda$. Let $s_n = \exp((v-1)^n)$ and

$$\mathbb{F}(\{s_n\}_{n\geq 1}) = \{x \in [0,1) : \ ns_n \leq a_n(x) < (n+1)s_n \text{ for all } n \geq 1\}.$$

We claim that

$$\mathbb{F}(\{s_n\}_{n\geq 1}) \subseteq E(v) \cap \Lambda. \tag{3.4}$$

If $x \in \mathbb{F}(\{s_n\}_{n\geq 1})$, then it is easy to see that $a_n(x) \leq a_{n+1}(x)$ for any $n \geq 1$. Now it remains to show that

$$\lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = v - 2.$$

In fact, we deduce from (2.2) that

$$\begin{aligned} v - 2 &= \lim_{n\to\infty} \frac{\log(n+1) + (v-1)^{n+1}}{n\log 2 + \log(n+1)! + (v-1) + \cdots + (v-1)^n} \\ &\leq \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{n\log 2 + \log a_1(x) + \cdots + \log a_n(x)} \\ &\leq \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} \leq \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log a_1(x) + \cdots + \log a_n(x)} \\ &\leq \lim_{n\to\infty} \frac{\log(n+2) + (v-1)^{n+1}}{\log n! + (v-1) + \cdots + (v-1)^n} = v - 2. \end{aligned}$$

It follows from (3.4) and Lemma 2.2 that

$$\begin{aligned} \dim_{\mathrm{H}}(E(v) \cap \Lambda) \geq \dim_{\mathrm{H}} \mathbb{F}(\{s_n\}_{n\geq 1}) &= \frac{1}{2 + \limsup\limits_{n\to\infty} \frac{2\log(n+1)! + (v-1)^{n+1}}{(v-1)+\cdots+(v-1)^n}} \\ &= \frac{1}{2 + (v-2)} = \frac{1}{v}. \end{aligned}$$

PROOF OF THEOREM 1.3. For any $v > 2$, recall that

$$E_{\overline{v}(x)\leq v}(v) \cap \Lambda = \Big\{x \in \Lambda : \ 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} \leq v\Big\}.$$

It is clear that $E_{\overline{v}(x)\leq v}(v) \cap \Lambda \subseteq E_{\underline{v}(x)\leq v}(v) \cap \Lambda \subseteq \Lambda$ and so

$$\dim_{\mathrm{H}}(E_{\overline{v}(x)\leq v}(v) \cap \Lambda) \leq \dim_{\mathrm{H}}(E_{\underline{v}(x)\leq v}(v) \cap \Lambda) \leq \dim_{\mathrm{H}} \Lambda = \tfrac{1}{2}.$$

Now it suffices to construct a subset of $E_{\overline{v}(x)\leq v}(v) \cap \Lambda$ and then show that the subset is of Hausdorff dimension $1/2$. Let $s_n = 2^n$ and let

$$\mathbb{F}(\{s_n\}_{n\geq 1}) = \{x \in [0,1) : \ ns_n \leq a_n(x) < (n+1)s_n \text{ for all } n \geq 1\}.$$

Then by (2.2), it is easy to prove that

$$\mathbb{F}(\{s_n\}_{n\geq 1}) \subseteq E_{\overline{v}(x)\leq v}(v) \cap \Lambda. \tag{3.5}$$

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Applying Lemma 2.2, we conclude from (3.5) that

$$\dim_{\mathrm{H}}(E_{\overline{v}(x)\le v}(v)\cap\Lambda)\ge \dim_{\mathrm{H}}\mathbb{F}(\{s_n\}_{n\ge1})=\frac{1}{2+\limsup_{n\to\infty}\frac{2\log(n+1)!+(n+1)\log 2}{\frac{1}{2}n(n+1)\log 2}}=\frac{1}{2}. \quad (3.6)$$

□

PROOFS OF THEOREM 1.4 AND COROLLARY 1.5. We shall divide the proof of Theorem 1.4 into two cases. Recall that

$$K(\psi)\cap\Lambda=\Big\{x\in\Lambda:\ \lim_{n\to\infty}\frac{\log a_1(x)+\cdots+\log a_n(x)}{\psi(n)}=1\Big\}.$$

Case 1: $\psi(n)/(n\log n)\to\alpha$ $(0\le\alpha<\infty)$ as $n\to\infty$. For the upper bound of $\dim_{\mathrm{H}}(K(\psi)\cap\Lambda)$, we shall construct a larger set containing $K(\psi)\cap\Lambda$ by using the general form of the Stolz–Cesàro theorem which states that if $\{b_n\}_{n\ge1}$ and $\{c_n\}_{n\ge1}$ are two sequences such that $\{c_n\}_{n\ge1}$ is monotone and unbounded, then

$$\liminf_{n\to\infty}\frac{b_{n+1}-b_n}{c_{n+1}-c_n}\le\liminf_{n\to\infty}\frac{b_n}{c_n}\le\limsup_{n\to\infty}\frac{b_n}{c_n}\le\limsup_{n\to\infty}\frac{b_{n+1}-b_n}{c_{n+1}-c_n}. \quad (3.7)$$

It follows from (3.7) that

$$\begin{aligned}K(\psi)\cap\Lambda&\subseteq\Big\{x\in\Lambda:\ \liminf_{n\to\infty}\frac{\log a_1(x)+\cdots+\log a_n(x)}{n\log n}=\alpha\Big\}\\&\subseteq\Big\{x\in\Lambda:\ \liminf_{n\to\infty}\frac{\log a_n(x)}{\log n}\le\alpha\Big\}.\end{aligned}$$

Thus we conclude from Lemma 2.3 that

$$\dim_{\mathrm{H}}(K(\psi)\cap\Lambda)\le\begin{cases}0, & 0\le\alpha<1,\\ (\alpha-1)/2\alpha, & \alpha\ge1.\end{cases}$$

To bound $\dim_{\mathrm{H}}(K(\psi)\cap\Lambda)$ from below, we shall construct a suitable Cantor subset of $K(\psi)\cap\Lambda$. By the upper bound estimate, we have $\dim_{\mathrm{H}}(K(\psi)\cap\Lambda)=0$ for $\alpha=1$. In what follows, we assume that $\alpha>1$. Let $s_n=2\lfloor n^{\alpha-1}\rfloor$ and let

$$\mathbb{F}(\{s_n\}_{n\ge1})=\{x\in[0,1):\ ns_n\le a_n(x)<(n+1)s_n \text{ for all } n\ge1\}.$$

Then we claim that

$$\mathbb{F}(\{s_n\}_{n\ge1})\subseteq K(\psi)\cap\Lambda. \quad (3.8)$$

On the one hand, since the sequence of positive integers $\{s_n\}_{n\ge1}$ is nondecreasing, the set $\mathbb{F}(\{s_n\}_{n\ge1})$ is a subset of Λ. On the other hand, for each $x\in\mathbb{F}(\{s_n\}_{n\ge1})$,

$$\begin{aligned}\alpha=\lim_{n\to\infty}\frac{\sum_{k=1}^n\log(2(k^\alpha-k))}{n\log n}&\le\lim_{n\to\infty}\frac{\log a_1(x)+\cdots+\log a_n(x)}{n\log n}\\&\le\lim_{n\to\infty}\frac{\sum_{k=1}^n\log(4k^\alpha)}{n\log n}=\alpha.\end{aligned}$$

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Applying Lemma 2.2, we deduce from (3.7) that

$$\dim_{\mathrm{H}} \mathbb{F}(\{s_n\}_{n\geq 1}) = \frac{1}{2+\limsup_{n\to\infty} \frac{2\log((n+1)!)+\log s_{n+1}}{\log(s_1 s_2\cdots s_n)}} \geq \frac{1}{2+\limsup_{n\to\infty} \frac{2\log(n+2)+\log s_{n+2}-\log s_{n+1}}{\log s_{n+1}}}$$
$$= \frac{1}{2+\frac{2}{\alpha-1}} = \frac{\alpha-1}{2\alpha}. \qquad (3.9)$$

Combining this with (3.8) and (3.9) completes the proof.

Case 2: $\psi(n)/(n\log n) \to \infty$ as $n\to\infty$. Note that $\psi(n)$ is a nondecreasing function. For the upper bound of $\dim_{\mathrm{H}}(K(\psi)\cap\Lambda)$, we deduce from [7, Theorem 1.1] that

$$\dim_{\mathrm{H}}(K(\psi)\cap\Lambda) \leq \dim_{\mathrm{H}} K(\psi) = \frac{1}{1+\limsup_{n\to\infty}\psi(n+1)/\psi(n)}.$$

For the lower bound, the strategy is again to construct a suitable Cantor subset. Let $s_n = 2\lfloor \exp(\psi(n)-\psi(n-1))\rfloor$ and set $\psi(0)=0$ for convenience. Let

$$\mathbb{F}(\{s_n\}_{n\geq 1}) = \{x\in[0,1):\ ns_n \leq a_n(x) < (n+1)s_n \text{ for all } n\geq 1\}.$$

The sequence $\{\psi(n)-\psi(n-1)\}_{n\geq 1}$ is nondecreasing and it is easy to check that

$$\mathbb{F}(\{s_n\}_{n\geq 1}) \subseteq K(\psi)\cap\Lambda.$$

Before proceeding, we remark that

$$\lim_{n\to\infty}\frac{\psi(n)}{n\log n} = \infty \quad \text{implies} \quad \limsup_{n\to\infty}\frac{\log((n+1)!\,)}{\psi(n)} = 0.$$

Combining these observations, we deduce from Lemma 2.2 that

$$\dim_{\mathrm{H}}(K(\psi)\cap\Lambda) \geq \dim_{\mathrm{H}}\mathbb{F}(\{s_n\}_{n\geq 1}) = \frac{1}{2+\limsup_{n\to\infty}\frac{2\log((n+1)!)+\log s_{n+1}}{\log(s_1s_2\cdots s_n)}}$$
$$\geq \frac{1}{2+\limsup_{n\to\infty}\frac{2\log((n+1)!)}{\psi(n)}+\limsup_{n\to\infty}\frac{\psi(n+1)-\psi(n)}{\psi(n)}}$$
$$= \frac{1}{1+\limsup_{n\to\infty}\frac{\psi(n+1)}{\psi(n)}}. \qquad \square$$

Proof of Corollary 1.5. For the case $0\leq\xi<\infty$, we deduce from the definition of the set $E_{\gamma(x)=\xi}(\xi)$ and (3.7) that

$$E_{\gamma(x)=\xi}(\xi)\cap\Lambda \subseteq \Big\{x\in\Lambda:\ \lim_{n\to\infty}\frac{\log a_1(x)+\log a_2(x)+\cdots+\log a_n(x)}{n\log n} = 0\Big\}$$
$$\subseteq \Big\{x\in\Lambda:\ \liminf_{n\to\infty}\frac{\log a_n(x)}{\log n} \leq 0\Big\}.$$

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Then by Lemma 2.3,

$$\dim_{\mathrm{H}}(E_{\gamma(x)=\xi}(\xi) \cap \Lambda) = 0.$$

For the case $\xi = \infty$, clearly

$$\dim_{\mathrm{H}}(E_{\gamma(x)=\xi}(\infty) \cap \Lambda) \le \dim_{\mathrm{H}} \Lambda = \tfrac{1}{2}.$$

It is easy to prove that the set $\mathbb{F}(\{s_n\}_{n\ge1})$ constructed in (3.5) is also a subset of $E_{\gamma(x)=\xi}(\infty) \cap \Lambda$. Combining this with (3.6) gives

$$\dim_{\mathrm{H}}(E_{\gamma(x)=\xi}(\infty) \cap \Lambda) \ge \dim_{\mathrm{H}} \mathbb{F}(\{s_n\}_{n\ge1}) = \tfrac{1}{2}. \qquad \square$$

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