The asymptotic periods of integral and meromorphic functions

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1. In a former paper published in these *Proceedings*[1] it was shown that an integral function of order less than 1 cannot have any asymptotic periods, and it was suggested that a function of order 1 can have at most a set $k\omega$ $(k = \pm 1, \pm 2, \ldots)$. This was subsequently found to be the case.[2] Meromorphic functions for which κ, the exponent of convergence of the poles, is less than ρ, the order, behave in many ways like integral functions, so we should expect that (i) if $0 \leqq \kappa < \rho < 1$ there should be no asymptotic periods, (ii) if $0 \leqq \kappa < \rho = 1$ either none or else a single sequence $k\omega$ $(k = \pm 1, \pm 2, \ldots)$. It will be shown that this is so.

(i) was originally proved by a modification of the method used for integral functions. A quite different method was subsequently found, and this is superior in that it also establishes (ii). As the first proof has points of interest a sketch of it is included.

2. We first establish the following result:

Theorem I. *Let $f(z)$ be an integral function, or a meromorphic function for which $\kappa < \rho$, which has an asymptotic period of argument α. Then there is a number $\sigma < \rho$ such that*

$$(1) \qquad \frac{1}{2R}\int_{-R}^{R} \log^{+} |f(xe^{i\alpha})|\, dx < R^{\sigma} \qquad (R \geqq R_0).$$

This is proved by the one-dimensional analogue of the "jig-saw puzzle" argument used in **T**, 96-99. The details may be omitted. Now suppose that $\rho < 1$ and that $f(z)$ has an asymptotic period, which may be supposed real without loss of generality. Write $f(z) = F(z)/G(z)$, where $F(z)$ is an integral function of order ρ and $G(z)$ is a canonical product of order κ, ($G(z) \equiv 1$ if $f(z)$ is an integral

[1] 3 (1933), 241-258.

[2] J. M. Whittaker, *Interpolatory Function Theory* (Cambridge Tract No. 35, 1935), 86. This work will be referred to as **T**.

function). Then $F(z)\, F(-z) = H(z^2)$, where $H(z)$ is an integral function of order $\frac{1}{2}\rho$. It follows readily from (1), that, for some number $\lambda < \frac{1}{2}\rho$,

$$(2) \qquad \frac{1}{R}\int_0^R \log^+ |H(x)|\, dx < R^\lambda \qquad (R \geqq R_1).$$

This, however, is impossible. For choose any number μ such that $\lambda < \mu < \frac{1}{2}\rho$. Then a result of Besicovitch[1] and Pennycuick[2] states that

$$H(x) > x^\mu$$

in a set of upper density greater than $1 - \rho$, and this contradicts (2). $f(z)$ has therefore no asymptotic periods.

3. The second proof depends on Theorem A (**T**, 2). With the notation and terminology there described, express $f(z)$ in normal form

$$(3) \qquad f(z) = g(z) + \Sigma \left\{ \frac{P(z)}{D(z)} + Q(z) \right\},$$

with associated numbers σ, τ, τ_1, κ. The following result will be proved:

THEOREM 2. *A meromorphic function for which the inequalities* $\kappa < \tau = \rho$, $\kappa < 1$ *are satisfied has no asymptotic periods.*

If possible, let β be an asymptotic period. Some of the nebulae of $f(z+\beta)$ may intersect those of $f(z)$, and so it may be necessary to bracket together some terms in the expansion

$$(4) \qquad f(z+\beta) - f(z) = g(z+\beta) - g(z) + \Sigma \left\{ \frac{P(z+\beta)}{D(z+\beta)} + Q(z+\beta) \right\} - \Sigma \left\{ \frac{P(z)}{D(z)} + Q(z) \right\}$$

to get it in normal form. If there is no such intersection the result is evident, since the numbers τ_1 associated with (3) and (4) will clearly be the same, and the order of $f(z+\beta) - f(z)$ will be not less than

$$\max(\tau_1, \kappa) = \max(\tau, \kappa) = \rho.$$

If there are intersections, assume for the moment that no $f(z+\beta)$-nebula cuts (or touches) more than one $f(z)$-nebula. Let N_1, a typical nebula of $f(z)$, cut M_2, one of $f(z+\beta)$; and let N_2, the

[1] *Math. Ann.*, **97** (1927), 677-695.

[2] *Journal London Math. Soc.*, **10** (1935), 210-212.

$f(z)$-nebula corresponding to M_2, cut M_3, and so on. The process will come to an end after a finite number of stages, say when N_p cuts M_{p+1}. For otherwise every circle of radius R would contain at least $R/|\beta|$ nebulae of $f(z)$, and so at least $R/|\beta|$ poles, and this is false since the number of poles is $O(R^{\kappa+\epsilon})$, where $\kappa+\epsilon<1$. If the most distant nebula in the chain is distant d from the origin it follows from this argument that $p<d^{\kappa+\epsilon}$, and thus the diameter of the chain is $O(d^{\kappa+\epsilon})$. Now, denoting by $P_1(z)$ the $P(z)$ associated with N_1 so that $P_1(z+\beta)$ is the $P(z)$ associated with M_1, etc.,

$$\frac{P_1(z)}{D_1(z)}=\left\{\frac{P_1(z)}{D_1(z)}-\frac{P_2(z+\beta)}{D_2(z+\beta)}\right\}+\left\{\frac{P_2(z+\beta)}{D_2(z+\beta)}-\frac{P_3(z+2\beta)}{D_3(z+2\beta)}\right\}$$
$$+\ldots\ldots+\left\{\frac{P_p\{z+(p-1)\beta\}}{D_p\{z+(p-1)\beta\}}-\frac{P_{p+1}(z+p\beta)}{D_{p+1}(z+p\beta)}\right\}$$
$$+\frac{P_{p+1}(z+p\beta)}{D_{p+1}(z+p\beta)},$$

and so

$$\left|\frac{P_{p+1}(z+p\beta)}{D_{p+1}(z+p\beta)}\right|\geqq\left|\frac{P_1(z)}{D_1(z)}\right|-\left|\frac{E_1(z)}{D_1(z)\,D_2(z+\beta)}\right|-\ldots\ldots$$
$$-\left|\frac{E_p\{z+(p-1)\beta\}}{D_p\{z+(p-1)\beta\}\,D_{p+1}(z+p\beta)}\right|,$$

where

$$E_i(z)=P_i(z)\,D_{i+1}(z+\beta)-D_i(z)\,P_{i+1}(z+\beta)\qquad(i=1,2,\ldots\ldots,p).$$

Write

$$\max_{|z|=R}|P_i(z)|=\delta_i(R),\qquad\max_{|z|=R}|E_i(z)|=\eta_i(R),$$

and take z_1 to be a point of modulus $2d$ such that

$$P_1(z_1)=\delta_1(2d).$$

Then

$$\left|\frac{P_{p+1}(z_1+p\beta)}{D_{p+1}(z_1+p\beta)}\right|\geqq\frac{\delta_1(2d)}{|D_1(z_1)|}-\eta_1(3d)-\ldots\ldots-\eta_p(3d),$$

since it is clear that

$$|D_1(z)|,\ |D_2(z+\beta)|,\ \ldots\ldots,\ |D_{p+1}(z+p\beta)|\geqq1\quad\text{on }|z|=2d.$$

Again, if $b_0,\ldots\ldots,b_k$ are the poles in N_1, of multiplicities $\lambda_0,\ldots\ldots,\lambda_k$,

$$|D_1(z_1)|=|z_1-b_0|^{\lambda_0}\,|z_1-b_1|^{\lambda_1}\ldots\ldots|z_1-b_k|^{\lambda_k}\leqq(3d)^{\mu}<(3d)^{\kappa+\epsilon},$$

where $\mu=\lambda_0+\lambda_1+\ldots\ldots+\lambda_k$, so that

$$\delta_{p+1}(3d)\geqq\frac{\delta_1(2d)}{(3d)^{\kappa+\epsilon}}-\eta_1(3d)-\ldots\ldots-\eta_p(3d).\tag{5}$$

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We may suppose that $P_1(z)$ is one of a sequence for which

$$\frac{\log^+ \log^+ \delta_1(r)}{\log r} \to \tau_1. \tag{6}$$

This being so, if we write

$$H(r) = \max[\eta_1(r), \ldots, \eta_p(r)],$$

it follows from (5), (6) that at least one of the relations

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ H(r)}{\log r} = \tau_1, \quad \frac{\log^+ \log^+ \delta_{p+1}(r)}{\log r} \to \tau_1$$

is satisfied, and in either case $f(z+\beta) - f(z)$ is of order at least $\tau_1 = \rho$, since its expansion in normal form contains terms like

$$\frac{E_1(z)}{D_1(z) D_2(z+\beta)}, \quad \frac{E_2(z)}{D_2(z) D_3(z+\beta)}, \ldots,$$

and also terms like

$$\frac{P_{p+1}(z)}{D_{p+1}(z)}.$$

The assumption that β is an asymptotic period has therefore been shown to be false. Multiple intersections of nebulae can be dealt with similarly.

4. Now suppose that $\rho \leqq 1$. If $\tau = \rho$ we have shown that there are no asymptotic periods. If, on the other hand, $\tau < \rho$, it follows that

$$f(z) = g(z) + \text{ a function of order less than } \rho,$$

and the asymptotic periods of $f(z)$ are therefore the same as those of the integral function $g(z)$. The latter is of order ρ and so has no asymptotic periods if $\rho < 1$ and at most a single sequence $k\omega$ if $\rho = 1$.

5. I take this opportunity of correcting a slight error in a former paper "On the asymptotic periods of meromorphic functions."[1] On p. 37, line 14, there is mention of the line joining a point u to the origin. It should be made clear that this line must be one on which the set E is everywhere dense. If there is no such line it is easy to see that we get one of cases (α), (β), (δ). If there is, we get cases (γ), (ϵ), or (ζ) according as the projections on the perpendicular line correspond to cases (α), (β), or (γ).

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[1] *Quart. J. of Math.* (Oxford), 5 (1934), 34-42.