AUTOMORPHISMS AND SYMPLECTIC LEAVES
OF CALOGERO–MOSER SPACES

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Abstract

We study the symplectic leaves of the subvariety of fixed points of an automorphism of a Calogero–Moser space induced by an element of finite order of the normalizer of the associated complex reflection group. We give a parametrization *à la* Harish-Chandra of its symplectic leaves (generalizing earlier works of Bellamy and Losev). This result is inspired by the mysterious relations between the geometry of Calogero–Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper, C. Bonnafé ['Calogero–Moser spaces vs unipotent representations', *Pure Appl. Math. Q.*, to appear, Preprint, 2021, arXiv:2112.13684].

1. Introduction

Let V be a finite-dimensional vector space and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ generated by reflections. To a class function k on W supported on the set of reflections, Etingof and Ginzburg [EtGi] associated a normal irreducible affine complex variety $\mathcal{Z}_k(V, W)$ called a (generalized) *Calogero–Moser space*. If τ is an element of finite order of the normalizer of W in $\mathbf{GL}_{\mathbb{C}}(V)$ stabilizing the class function k, it induces an automorphism of $\mathcal{Z}_k(V, W)$. The main theme of this paper is the study of the symplectic leaves of the variety $\mathcal{Z}_k(V, W)^\tau$ of its fixed points in $\mathcal{Z}_k(V, W)$ (endowed with its reduced closed subscheme structure).

Note that W acts trivially on $\mathcal{Z}_k(V, W)$ so, by replacing τ by $w\tau$ for some $w \in W$ if necessary, we may assume that the natural morphism $V^\tau \longrightarrow (V/W)^\tau$ is onto (this argument is due to Springer [Spr] and will be recalled in Section 4): this will be assumed throughout this paper and will simplify some statements.

The Poisson bracket on $\mathfrak{Z}_k(V, W)$ induces a Poisson bracket on $\mathfrak{Z}_k(V, W)^\tau$ and we are interested in parametrizing the symplectic leaves of this fixed-point subvariety. For this, we define a τ-*cuspidal* symplectic leaf (or a τ-*cuspidal* point) to be a zero-dimensional symplectic leaf of $\mathfrak{Z}_k(V, W)^\tau$, and we define a τ-*split* parabolic subgroup of W to be the stabilizer of some point in V^τ. We also denote by W_τ the quotient of the setwise stabilizer of V^τ in W by the pointwise stabilizer. For its action on V^τ, the group W_τ is a reflection group [LeSp]. Our result is as follows.

THEOREM A. *Assume that the natural morphism $V^\tau \longrightarrow (V/W)^\tau$ is onto. Then there is a natural bijection between the set of symplectic leaves of $\mathfrak{Z}_k(V, W)^\tau$ and the set of W_τ-orbits of pairs (P, p), where P is a τ-split parabolic subgroup and p is a τ-cuspidal point of $\mathfrak{Z}_{k_P}(V_P, P)^\tau$.*

Moreover, the dimension of the symplectic leaf associated with (P, p) through this bijection is equal to $2 \dim(V^P)^\tau$.

Here, k_P is the restriction of k to P. In Section 9 we give an explicit description of the bijection. If $\tau = \mathrm{Id}_V$, this result was proved by Bellamy [Bel1] and Losev [Los] and might be viewed as a *Harish-Chandra theory* of symplectic leaves. So Theorem A can be thought as a τ-*Harish-Chandra theory*, inspired by the Broué–Malle–Michel d-*Harish-Chandra theory* of unipotent representations of finite reductive groups [BMM] (see [Bon2] for a further discussion of this analogy and applications of Theorem A). The main point is to combine Springer/Lehrer–Springer theory (which describes the action of the setwise stabilizer of V^τ on V^τ) with work of Bellamy/Losev. We propose the following conjecture about the geometry of symplectic leaves of $\mathfrak{Z}_k(V, W)^\tau$.

CONJECTURE B. *Let (P, p) be as in Theorem A and let $\mathcal{S}$ denote the corresponding symplectic leaf of $\mathfrak{Z}_k(V, W)^\tau$. Then there exist a parameter l for the pair $((V^P)^\tau, \mathrm{N}_{W_\tau}(P_\tau)/P_\tau)$ and a $\mathbb{C}^\times$-equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}}^{\mathrm{nor}} \simeq \mathfrak{Z}_l((V^P)^\tau, \mathrm{N}_{W_\tau}(P_\tau)/P_\tau).$$

Here, $\overline{\mathcal{S}}^{\mathrm{nor}}$ denotes the normalization of the closure of $\mathcal{S}$.

Note that this conjecture is not known even in the case where $\tau = \mathrm{Id}_V$ (in which case $W_\tau = W$ and $P_\tau = P$). It has been proved by Maksimau and the author [BoMa] whenever $\mathfrak{Z}_k(V, W)$ is smooth and $\tau \in W \cdot \mathbb{C}^\times$.

The paper is organized as follows. We recall the set-up (reflection groups, Cherednik algebras, Calogero–Moser spaces, . . .) in Section 2, and Section 3 recalls useful results on Poisson structures and symplectic leaves. In Section 4 we recall the main results of Lehrer and Springer on the group W_τ and some of its consequences. In Section 5 we restate Theorem A and Conjecture B in more precise terms. The proof of Theorem A is given in Sections 6–9 (see the end of Section 5 for the description of the different steps). In Section 10 we give an overview of the known cases for Conjecture B. A short appendix summarizes easy results about completions of rings that are needed in Section 9 to conclude the proof of Theorem A.

2. Set-up

2.1. Complex numbers. Throughout this paper, we abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$ and all varieties will be algebraic, complex, quasi-projective and reduced. If $\mathfrak{X}$ is an irreducible variety, we denote by $\mathfrak{X}^{\text{nor}}$ its normalization. If $\mathfrak{X}$ is an affine variety, we denote by $\mathbb{C}[\mathfrak{X}]$ its coordinate ring: if, moreover, $\mathfrak{X}$ is irreducible, then $\mathfrak{X}^{\text{nor}}$ is also affine and $\mathbb{C}[\mathfrak{X}^{\text{nor}}]$ is the integral closure of $\mathbb{C}[\mathfrak{X}]$ in its fraction field (which is denoted by $\mathbb{C}(\mathfrak{X})$).

We fix in this paper a complex vector space V of finite dimension n. If X is a subset of V (or V^*), and if G is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by G_X^{set} (respectively, G_X^{pt}) the setwise (respectively, pointwise) stabilizer of X and we set $G[X] = G_X^{\text{set}}/G_X^{\text{pt}}$. Then $G[X]$ acts faithfully on X (and on the vector space spanned by X). If $X = \{v\}$ is a singleton, then $G_X^{\text{set}} = G_X^{\text{pt}}$ (and we denote both simply by G_X or G_v) and $G[X] = 1$. If H is a subgroup of G, we set $\overline{\mathrm{N}}_G(H) = \mathrm{N}_G(H)/H$.

If, moreover, G is finite, we identify $(V^G)^*$ and $(V^*)^G$, and we denote by V_G the unique G-stable subspace of V such that $V = V_G \oplus V^G$.

2.2. Reflections. Let W be a finite subgroup of $\mathrm{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}$$

and note that, for the moment, we *do not* assume that W is generated by $\mathrm{Ref}(W)$. We set $\varepsilon : W \to \mathbb{C}^{\times}, w \mapsto \det(w)$. We identify $\mathbb{C}[V]$ (respectively, $\mathbb{C}[V^*]$) with the symmetric algebra $\mathrm{S}(V^*)$ (respectively, $\mathrm{S}(V)$).

We denote by $\mathscr{A}$ the set of *reflecting hyperplanes* of W, namely

$$\mathscr{A} = \{V^s \mid s \in \mathrm{Ref}(W)\}.$$

If $H \in \mathscr{A}$, we denote by α_H an element of V^* such that $H = \mathrm{Ker}(\alpha_H)$ and by $\alpha_H^{\vee}$ an element of V such that $V = H \oplus \mathbb{C}\alpha_H^{\vee}$ and the line $\mathbb{C}\alpha_H^{\vee}$ is W_H^{pt}-stable. We set $e_H = |W_H^{\text{pt}}|$. Note that W_H^{pt} is cyclic of order e_H and that $\mathrm{Irr}(W_H^{\text{pt}}) = \{\mathrm{Res}_{W_H^{\text{pt}}}^{W} \varepsilon^j \mid 0 \leqslant j \leqslant e - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H^{\text{pt}}$ associated with the character $\mathrm{Res}_{W_H^{\text{pt}}}^{W} \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H^{\text{pt}}} \varepsilon(w)^j w \in \mathbb{C}W_H^{\text{pt}}.$$

If Ω is a W-orbit of reflecting hyperplanes, we write e_{Ω} for the common value of all the e_H, where $H \in \Omega$. We denote by ∇ the set of pairs (Ω, j) where $\Omega \in \mathscr{A}/W$ and $0 \leqslant j \leqslant e_{\Omega} - 1$. The vector space of families of complex numbers indexed by ∇ is denoted by $\mathbb{C}^{\nabla}$: elements of $\mathbb{C}^{\nabla}$ are called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j) \in \nabla} \in \mathbb{C}^{\nabla}$, we define $k_{H,j}$, for all $H \in \Omega$ and $j \in \mathbb{Z}$, by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W-orbit of H and j_0 is the unique element of $\{0, 1, \ldots, e_H - 1\}$ such that $j \equiv j_0 \bmod e_H$.

2.3. Parabolic subgroups. We denote by $\mathrm{Parab}(W)$ the set of parabolic subgroups of W (that is, the set of subgroups of W that are stabilizers of some point of V)

and by $\mathrm{Parab}(W)/W$ the set of conjugacy classes of parabolic subgroups of W. If $P \in \mathrm{Parab}(W)$, we denote by $\mathscr{V}(P)$ the set of elements $v \in V$ such that $W_v = P$: it is a nonempty open subset of V^P. By definition, $W_{V^P}^{\mathrm{pt}} = P$ and $W_{V^P}^{\mathrm{set}} = \mathrm{N}_W(P)$, so that $W[V^P] = \overline{\mathrm{N}}_W(P)$. The family $(\mathscr{V}(P))_{P \in \mathrm{Parab}(W)}$ is a stratification of V (the order between strata corresponds to the reverse order of the inclusion of parabolic subgroups).

This stratification is stable under the action of the group W. If $\mathfrak{P} \in \mathrm{Parab}(W)/W$, we denote by $\mathscr{U}(\mathfrak{P})$ the image of $\mathscr{V}(P)$ in V/W, where P is any element of $\mathfrak{P}$. Then $(\mathscr{U}(\mathfrak{P}))_{\mathfrak{P} \in \mathrm{Parab}(W)/W}$ is a stratification of V/W (the order between strata corresponds to the reverse order of the inclusion, up to conjugacy, of parabolic subgroups). Replacing V by V^*, we similarly define $\mathscr{V}^*(P)$ and $\mathscr{U}^*(\mathfrak{P})$ for $P \in \mathrm{Parab}(W)$ and $\mathfrak{P} \in \mathrm{Parab}(W)/W$. By definition, $\overline{\mathrm{N}}_W(P)$ acts freely on $\mathscr{V}(P)$ or $\mathscr{V}^*(P)$. Moreover, for $P \in \mathfrak{P}$, the natural map $\mathscr{V}(P) \to \mathscr{U}(\mathfrak{P})$ induces an isomorphism of varieties

$$\mathscr{V}(P)/\overline{\mathrm{N}}_W(P) \xrightarrow{\sim} \mathscr{U}(\mathfrak{P}).$$

In particular, $\mathscr{U}(\mathfrak{P})$ is smooth.

2.4. Rational Cherednik algebra at $t = 0$. Let $k \in \mathbb{C}^{\nabla}$. We define the *rational Cherednik algebra* $\mathbf{H}_k$ (*at $t = 0$*) to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V \oplus V^*)$ with the group W) by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \displaystyle\sum_{H \in \mathscr{A}} \sum_{j=0}^{e_H - 1} e_H(k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^{\vee}, x \rangle}{\langle \alpha_H^{\vee}, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases} \tag{2-1}$$

for all $x, x' \in V^*$, $y, y' \in V$. Here $\langle \ , \ \rangle : V \times V^* \to \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_k$ and $\mathbb{C}[V^*] \to \mathbf{H}_k$. Recall [EtGi, Theorem 1.3] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k \tag{2-2}$$

induced by multiplication. (This the so-called *Poincaré–Birkhoff–Witt decomposition*, or *PBW decomposition*.)

REMARK 2.1. Let $(l_{\Omega})_{\Omega \in \mathscr{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\nabla}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_{\Omega}$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no restriction to generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω, or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_{\Omega}-1} = 0$ for all Ω (as in [BoRo]).

2.5. Calogero–Moser space. We denote by $\mathbf{Z}_k$ the centre of the algebra $\mathbf{H}_k$: it is well known [EtGi, Theorem 3.3 and Lemma 3.5] that $\mathbf{Z}_k$ is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [Gor, Proposition 3.6] (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$). If $W = \langle \mathrm{Ref}(W) \rangle$, then $\mathbf{Z}_k$ is a free $\mathbf{P}$-module of rank $|W|$ (see [EtGi, Proposition 4.15]). We denote by

$\mathfrak{X}_k$ the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathfrak{X}_k]$ is $\mathbf{Z}_k$: this is the *Calogero–Moser space* associated with the datum (V, W, k). It is irreducible and normal.

We set $\mathscr{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathscr{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a morphism of varieties

$$\Upsilon_k : \mathfrak{X}_k \longrightarrow \mathscr{P}$$

which is finite (and flat if $W = \langle \mathrm{Ref}(W) \rangle$).

2.6. Other structures on the Calogero–Moser space. The Calogero–Moser space $\mathfrak{X}_k$ is endowed with other structures (a $\mathbb{C}^\times$-action, a Poisson bracket, a filtration, an action of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ . . .) which are described below.

2.6.1. Grading, $\mathbb{C}^\times$-action. The algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees:

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a $\mathbb{Z}$-grading on $\mathbf{H}_k$, because the defining relations (2-1) are homogeneous. Since the centre of a graded algebra is always graded, the subalgebra $\mathbf{Z}_k$ is also $\mathbb{Z}$-graded. So the Calogero–Moser space $\mathfrak{X}_k$ inherits a regular $\mathbb{C}^\times$-action. Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $\mathbf{Z}_k$.

2.6.2. Poisson structure. Let $t \in \mathbb{C}$. One can define a deformation $\mathbf{H}_{t,k}$ of $\mathbf{H}_k$ as follows: $\mathbf{H}_{t,k}$ is the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = t\langle y, x \rangle + \displaystyle\sum_{H \in \mathcal{A}} \sum_{i=0}^{e_H - 1} e_H(k_{H,i} - k_{H,i+1}) \dfrac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,i}, \end{cases}$$

for all x, $x' \in V^*$, y, $y' \in V$. It is well known [EtGi] that the PBW decomposition still holds so that the family $(\mathbf{H}_{t,k})_{t \in \mathbb{C}}$ is a flat deformation of $\mathbf{H}_k = \mathbf{H}_{0,k}$. This allows us to define a Poisson bracket $\{ \, , \, \}$ on $\mathbf{Z}_k$ as follows: if $z_1, z_2 \in \mathbf{Z}_k$, we denote by z_1^t, z_2^t the corresponding element of $\mathbf{H}_{t,k}$ through the PBW decomposition and we define

$$\{z_1, z_2\} = \lim_{t \to 0} \frac{[z_1^t, z_2^t]}{t}.$$

Finally, note the following observation.

The Poisson bracket is $\mathbb{C}^\times$-equivariant.

2.6.3. Filtration. The tensor algebra $\mathrm{T}(V \oplus V^*)$ is naturally filtered by the subspaces $(\bigoplus_{j=0}^d (V \oplus V^*)^{\otimes j})$. This induces a filtration of $\mathrm{T}(V \oplus V^*) \rtimes W$ by putting W in degree 0

and so induces a filtration $(\mathcal{F}_j \mathbf{H}_k)_{j \geqslant 0}$ of the rational Cherednik algebra. By convention, we set $\mathcal{F}_{-1}\mathbf{H}_k = 0$. If M is any subspace of $\mathbf{H}_k$, we set $\mathcal{F}_j M = M \cap \mathcal{F}_j \mathbf{H}_k$, so that M also inherits a filtration, and we denote by $\mathrm{Rees}_{\mathcal{F}} M$ the *Rees module* of M (associated with the filtration $(\mathcal{F}_j X)_{j \geqslant 0}$), namely the $\mathbb{C}[\hbar]$-submodule of $\mathbb{C}[\hbar] \otimes M$ equal to

$$\mathrm{Rees}_{\mathcal{F}} M = \bigoplus_{j \geqslant 0} \hbar^j \mathcal{F}_j M.$$

Recall that, if $\lambda \in \mathbb{C}$, then

$$\mathbb{C}[\hbar]/\langle \hbar - \lambda \rangle \otimes_{\mathbb{C}[\hbar]} \mathrm{Rees}_{\mathcal{F}} M \simeq \begin{cases} M & \text{if } \lambda \neq 0, \\ \mathrm{gr}_{\mathcal{F}}(M) & \text{if } \lambda = 0, \end{cases} \tag{2-3}$$

where $\mathrm{gr}_{\mathcal{F}}(M) = \bigoplus_{j \geqslant 0} \mathcal{F}_j M / \mathcal{F}_{j-1} M$ is the graded vector space associated with M and its filtration.

If A is a subalgebra of $\mathbf{H}_k$ and J is an ideal of A, then $\mathrm{Rees}_{\mathcal{F}}(A)$ is a subalgebra of $\mathbb{C}[\hbar] \otimes A$ (called the *Rees algebra* of A) and $\mathrm{Rees}_{\mathcal{F}}(J)$ is an ideal of $\mathrm{Rees}_{\mathcal{F}}(A)$. Recall [EtGi, Theorem 1.3] that

$$\mathrm{gr}_{\mathcal{F}} \mathbf{H}_k \simeq \mathbf{H}_0 = \mathbb{C}[V \times V^*] \rtimes W \quad \text{and} \quad \mathrm{gr}_{\mathcal{F}} \mathbf{Z}_k \simeq \mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W. \tag{2-4}$$

2.6.4. Action of the normalizer. The group $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ acts on the set ∇ and so on the space of parameters $\mathbb{C}^\nabla$. If $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$, then τ induces an isomorphism of algebras $\mathbf{H}_k \longrightarrow \mathbf{H}_{\tau(k)}$. So, if $\tau(k) = k$, then τ acts on the algebra $\mathbf{H}_k$, and so on its centre $\mathbf{Z}_k$ and on the Calogero–Moser space $\mathcal{Z}_k$, which preserves the $\mathbb{C}^\times$-action and the Poisson bracket. We set

$$\delta(\tau) = \max_{w \in W} \dim V^{w\tau}.$$

Of course, $\delta(\tau)$ depends only on the coset $W\tau$ and not on τ. We say that τ is *W-full* if $\delta(\tau) = \dim V^\tau$. Since W acts trivially on $\mathcal{Z}_k$, the study of the action of τ on $\mathcal{Z}_k$ is equivalent to the study of the action of $w\tau$. So, by replacing τ by $w\tau$ if necessary, we may assume that τ is W-full.

EXAMPLE 2.2. An element $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is called *W-regular* (or simply *regular* if W is clear from the context) if $V^\tau \cap V_{\mathrm{reg}} \neq \varnothing$. A W-regular element of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is W-full [Spr].

HYPOTHESIS AND NOTATION. From now on, and until the end of this paper, we assume that

$$W = \langle \mathrm{Ref}(W) \rangle,$$

we fix a parameter $k \in \mathbb{C}^\nabla$ and an element τ of *finite* order of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ such that $\tau(k) = k$. We also assume that τ is *W-full*.

If $\bigstar$ is one of the objects defined in the previous sections ($\mathbf{H}_k$, $\mathcal{Z}_k$, ∇, $\mathcal{A}$, …), we sometimes denote it by $\bigstar(W)$ or $\bigstar(V, W)$ if we need to emphasize the context.

3. Poisson structures and symplectic leaves

NOTATION. We fix in this section, and only in this section, a commutative noetherian Poisson $\mathbb{C}$-algebra R, whose Poisson bracket is denoted by $\{\ ,\ \}$.

3.1. Poisson ideals. An ideal I of R is called a *Poisson ideal* of R if $\{r, I\} \subset I$ for all $r \in R$. The following facts may be found in [Dix, Lemma 3.3.3].

PROPOSITION 3.1. *Let I be a Poisson ideal of R. Then the following assertions hold.*

(a)   *Every minimal prime ideal containing I is Poisson.*
(b)   *The radical of I is Poisson.*

3.2. Normalization. The next result is due to Kaledin [Kal].

THEOREM 3.2 (Kaledin). *Assume that R is a domain. Then there is a unique Poisson bracket on the normalization of R extending $\{\ ,\ \}$.*

3.3. Action of a finite group. We assume in this subsection that we are given a finite group G acting on the $\mathbb{C}$-algebra R in such a way that the Poisson bracket is G-equivariant (that is, $\{g(r), g(r')\} = g(\{r, r'\})$ for all $g \in G$ and $r, r' \in R$). Let I denote the ideal of R generated by the family $(g(r) - r)_{g \in G, r \in R}$. Then R/I is the biggest quotient algebra of R on which G acts trivially.

Since G is finite and $\mathbb{C}$ has characteristic 0, the natural map

$$R^G \longrightarrow (R/I)^G = R/I$$

is surjective and its kernel is I^G. Moreover, R^G is a Poisson subalgebra of R (because the Poisson bracket is G-equivariant). Note that I is not in general a Poisson ideal of R, but it is easily checked that

$$I^G \text{ is a Poisson ideal of } R^G.$$

Therefore, $R/I = R^G/I^G$ can be naturally endowed with a Poisson bracket. And, by Proposition 3.1(b), $R/\sqrt{I} = R^G/\sqrt{I^G}$ also inherits a Poisson bracket.

REMARK 3.3. If $R = \mathbb{C}[\mathcal{X}]$ is the coordinate ring of an affine variety $\mathcal{X}$, then R/I is the coordinate ring of the G-fixed points scheme of $\mathcal{X}$ (which is denoted by $\mathcal{X}^{(G)}$), while $R/\sqrt{I}$ is the coordinate ring of its reduced subscheme (which is denoted by $\mathcal{X}^G$). The above construction shows that the closed subvariety $\mathcal{X}^G$ of $\mathcal{X}$ inherits a Poisson structure from that on $\mathcal{X}$, even though it is not in general a Poisson subvariety of $\mathcal{X}$ (that is, the natural map $\mathcal{X}^G \hookrightarrow \mathcal{X}$ is not Poisson). However, $\mathcal{X}/G$ is also a Poisson variety and the natural map $\mathcal{X}^G \hookrightarrow \mathcal{X}/G$ is Poisson; that is, $\mathcal{X}^G$ is a closed Poisson subvariety of $\mathcal{X}/G$.

If, moreover, $\mathcal{X}$ is smooth, then $\mathcal{X}^{(G)} = \mathcal{X}^G$ is also smooth, and if the Poisson structure on $\mathcal{X}$ makes it into a symplectic variety, then $\mathcal{X}^G$ is also symplectic for the induced Poisson structure.

EXAMPLE 3.4. Let E be a $\mathbb{C}$-vector space endowed with a symplectic form ω and assume here that $R = \mathbb{C}[E]$ and that $G \subset \mathbf{Sp}(E, \omega)$. Then the restriction of ω to E^G is nondegenerate, so this endows E^G with the structure of a Poisson (even more, symplectic) variety. On the other hand, via the above Remark 3.3, the variety E^G also inherits from E a structure of Poisson variety. It is easily checked that both structures coincide.

3.4. Symplectic leaves. Assume in this subsection that $R = \mathbb{C}[\mathfrak{X}]$ is the coordinate ring of an affine variety $\mathfrak{X}$. Brown and Gordon [BrGo] defined a stratification of $\mathfrak{X}$ by *symplectic leaves*, which are in general not algebraic subvarieties of $\mathfrak{X}$. We denote by $\mathcal{S}ymp(\mathfrak{X})$ the set of symplectic leaves of $\mathfrak{X}$.

When $\mathfrak{X}$ has finitely many symplectic leaves, then the symplectic leaves are algebraic [BrGo, Proposition 3.7] and the stratification of $\mathfrak{X}$ into symplectic leaves is given as follows. Let $(\mathcal{S}_j)_{j \geqslant 0}$ be the sequence of closed subvarieties of $\mathfrak{X}$ defined by

$$\begin{cases} \mathcal{S}_0 = \mathfrak{X}, \\ \text{if } j \geqslant 0, \text{ then } \mathcal{S}_{j+1} \text{ is the reduced singular locus of } \mathcal{S}_j. \end{cases}$$

Then the symplectic leaves of $\mathfrak{X}$ are the irreducible components of the locally closed subvarieties $(\mathcal{S}_j \setminus \mathcal{S}_{j+1})_{j \geqslant 0}$. Let $\mathrm{PSpec}(\mathbb{C}[\mathfrak{X}])$ denote the set of prime ideals that are Poisson. If $\mathcal{S}$ is a symplectic leaf of $\mathfrak{X}$, we denote by $\mathfrak{p}_{\mathcal{S}}$ the defining ideal of $\overline{\mathcal{S}}$ in $\mathbb{C}[\mathfrak{X}]$: it belongs to $\mathrm{PSpec}(\mathbb{C}[\mathfrak{X}])$. If $\mathfrak{X}$ has finitely many symplectic leaves, then the map

$$\begin{array}{ccc} \mathcal{S}ymp(\mathfrak{X}) & \longrightarrow & \mathrm{PSpec}(\mathbb{C}[\mathfrak{X}]) \\ \mathcal{S} & \longmapsto & \mathfrak{p}_{\mathcal{S}} \end{array} \tag{3-1}$$

is bijective [BrGo, Lemma 3.4]. The inverse is given as follows: if $\mathfrak{p} \in \mathrm{PSpec}(\mathbb{C}[\mathfrak{X}])$ corresponds to $\mathcal{S}$ through this bijection, then $\mathcal{S}$ is the smooth locus of the closed irreducible subvariety of $\mathfrak{X}$ defined by $\mathfrak{p}$.

LEMMA 3.5. *Assume that $\mathfrak{X}$ has finitely many symplectic leaves and that $\mathcal{Y}$ is a locally closed Poisson subvariety of $\mathfrak{X}$. Then $\mathcal{Y}$ has finitely many symplectic leaves.*

PROOF. Taking the closure of $\mathcal{Y}$, which is also Poisson, allows us to assume that $\mathcal{Y}$ is closed. Let $(\mathcal{S}_l)_{l \in L}$ be the family of symplectic leaves of $\mathfrak{X}$ (for some finite indexing set L). Let $\mathcal{I}$ be an irreducible component of $\mathcal{Y}$. Then $\mathcal{I}$ is also Poisson by Proposition 3.1(a), so it is the closure of a symplectic leaf thanks to the bijection (3-1). In particular, there exists a subset I of L such that $\mathcal{I}$ is the union of the $\mathcal{S}_i$, for $i \in I$. This proves that $\mathcal{Y}$ is a union of symplectic leaves of $\mathfrak{X}$, each of which is also a symplectic leaf of $\mathcal{Y}$. $\qquad\square$

Now, let G be a finite group acting on $\mathfrak{X}$ and preserving the Poisson bracket. Then $\mathfrak{X}/G$ is an affine Poisson variety (because $\mathbb{C}[\mathfrak{X}/G] = \mathbb{C}[\mathfrak{X}]^G$ is a Poisson subalgebra of $\mathbb{C}[\mathfrak{X}]$; see Remark 3.3). If H is a subgroup of G, we denote by $\mathfrak{X}(H)$ the set of elements $x \in \mathfrak{X}$ whose stabilizer is exactly H. Then $\mathfrak{X}(H)$ is a locally closed subvariety

of $\mathfrak{X}$ (it is open in $\mathfrak{X}^H$). The subgroup H is called *parabolic* if $\mathfrak{X}(H) \neq \varnothing$. Let Parab$(G)$ denote the set of parabolic subgroups of G.

If $\mathfrak{H}$ is a conjugacy class of parabolic subgroups of G, we denote by $(\mathfrak{X}/G)(\mathfrak{H})$ the image of $\mathfrak{X}(H)$ in $\mathfrak{X}/G$ for some (or any) $H \in \mathfrak{H}$. Then the group $\overline{\mathrm{N}}_G(H)/H$ acts freely on $\mathfrak{X}(H)$ and the natural map $\mathfrak{X}(H) \to (\mathfrak{X}/G)(\mathfrak{H})$ induces an isomorphism

$$\mathfrak{X}(H)/G(H) \xrightarrow{\sim} (\mathfrak{X}/G)(\mathfrak{H}). \tag{3-2}$$

Indeed, if $g \in G$ and x, $x' \in \mathfrak{X}(H)$ are such that $g \cdot x = x'$, then $H = G_{x'} = {}^g G_x = {}^g H$ and so $g \in \mathrm{N}_G(H)$. The next result generalizes [BrGo, Proposition 7.4] slightly.

PROPOSITION 3.6. *Assume that $\mathfrak{X}$ is smooth and symplectic. Then the symplectic leaves of $\mathfrak{X}/G$ are the irreducible components of the locally closed subvarieties $(\mathfrak{X}/G)(\mathfrak{H})$ where $\mathfrak{H}$ runs over* Parab$(G)/G$.

In particular, if all the subvarieties $(\mathfrak{X}/G)(\mathfrak{H})$ are irreducible, then

$$\mathfrak{X}/G = \bigsqcup_{\mathfrak{H} \in \mathrm{Parab}(G)/G} (\mathfrak{X}/G)(\mathfrak{H})$$

is the stratification of $\mathfrak{X}/G$ into symplectic leaves.

PROOF. Let $\mathfrak{H} \in \mathrm{Parab}(G)$ and let $H \in \mathfrak{H}$. Since $\mathfrak{X}$ is smooth and symplectic, the subvariety $\mathfrak{X}^H$ is also smooth and symplectic. So its open subset $\mathfrak{X}(H)$ is also smooth and symplectic as well as $(\mathfrak{X}/G)(\mathfrak{H})$ thanks to the isomorphism (3-2). And the morphism $\mathfrak{X}^H \to \mathfrak{X}/G$ is Poisson: this proves that any irreducible component of $(\mathfrak{X}/G)(\mathfrak{H})$ is contained in a unique symplectic leaf. In particular, $\mathfrak{X}/G$ has finitely many symplectic leaves.

It remains to show that any irreducible component $\mathcal{I}$ of $(\mathfrak{X}/G)(\mathfrak{H})$ is a symplectic leaf. But $\overline{\mathcal{I}}$ is a closed Poisson subvariety of $\mathfrak{X}/G$, so its smooth locus is a symplectic leaf of $\mathfrak{X}/G$ by the bijection (3-1). Since $\mathcal{I}$ is smooth, it remains to show that $\mathcal{I}$ is the smooth locus of $\overline{\mathcal{I}}$. Note that $\overline{\mathcal{I}}$ has finitely many symplectic leaves; so, by the discussion at the beginning of this subsection, it is sufficient to show that $\overline{\mathcal{I}} \setminus \mathcal{I}$ is a (closed) Poisson subvariety.

But $\overline{(\mathfrak{X}/G)(\mathfrak{H})} \setminus (\mathfrak{X}/G)(\mathfrak{H})$ is the union of the $\overline{(\mathfrak{X}/G)(\mathfrak{H}')}$, where $\mathfrak{H}'$ runs over the set of conjugacy classes of parabolic subgroups of G strictly containing at least one element of $\mathfrak{H}$; so it is Poisson. Since $\overline{\mathcal{I}} \setminus \mathcal{I} = \overline{\mathcal{I}} \cap ((\overline{\mathfrak{X}/G)(\mathfrak{H})} \setminus (\mathfrak{X}/G)(\mathfrak{H}))$, we get that $\overline{\mathcal{I}} \setminus \mathcal{I}$ is Poisson, as desired. □

COROLLARY 3.7. *Assume that $\mathfrak{X}$ has finitely many symplectic leaves. Then $\mathfrak{X}/G$ and $\mathfrak{X}^G$ have finitely many symplectic leaves.*

PROOF. Let $\mathcal{S}$ denote a symplectic leaf of $\mathfrak{X}$ and let $H = G_{\mathcal{S}}^{\mathrm{set}}$. As symplectic leaves form a partition of $\mathfrak{X}$, and since $g(\mathcal{S})$ is a symplectic leaf of $\mathfrak{X}$ for any $g \in G$, we get that

$$g(\mathcal{S}) \cap \mathcal{S} = \varnothing$$

for all $g \notin H$. So, the image $\mathcal{S}$ in $\mathcal{X}/G$ is isomorphic to $\mathcal{S}/H$ and is a locally closed Poisson subvariety of $\mathcal{X}/G$. But, by Proposition 3.6, $\mathcal{S}/H$ has finitely many symplectic leaves.

As $\mathcal{X}$ has finitely many symplectic leaves, this shows that $\mathcal{X}/G$ also has finitely many symplectic leaves. Now, $\mathcal{X}^G$ is a closed Poisson subvariety of $\mathcal{X}/G$, so it also admits finitely many symplectic leaves by Lemma 3.5. □

As a consequence of the above proof, we get the following corollary.

COROLLARY 3.8. *Assume that $\mathcal{X}$ has finitely many symplectic leaves and that G acts freely on $\mathcal{X}$. Then the map $\mathcal{S}ymp(\mathcal{X})/G \longrightarrow \mathcal{S}ymp(\mathcal{X}/G)$ sending the G-orbit of a symplectic leaf of $\mathcal{X}$ to its image in $\mathcal{X}/G$ is well defined and bijective.*

4. Lehrer–Springer theory

4.1. Reflection groups. Recall that $\tau \in N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is assumed to be W-full. This implies that [Spr]

$$\delta(\tau) = \dim(V/W)^\tau.$$

Therefore, since $(V/W)^\tau$ is irreducible (it is isomorphic to an affine space [Spr]), we get that

$$\text{the natural map } V^\tau \to (V/W)^\tau \text{ is onto.} \tag{4-1}$$

To simplify notation, we set $W_\tau = W_{V^\tau}^{\mathrm{set}}/W_{V^\tau}^{\mathrm{pt}}$. Note that $W^\tau \subset W_{V^\tau}^{\mathrm{set}}$. Moreover, W_τ acts faithfully on V^τ, so

$$\tau \text{ acts trivially on } W_\tau. \tag{4-2}$$

Lehrer–Springer theory [LeSp, Theorem 2.5 and Corollary 2.7] gives the following result.

THEOREM 4.1 (Springer, Lehrer–Springer). *Recall that τ is W-full. Then the following assertions hold.*

(a) *The group W_τ is a reflection group for its action on V^τ.*
(b) *The natural map*

$$i_\tau : V^\tau/W_\tau \longrightarrow (V/W)^\tau$$

is an isomorphism of varieties.
(c) *The reflecting hyperplanes of W_τ are exactly the intersections with V^τ of the reflecting hyperplanes of W that do not contain V^τ.*

Similarly, the natural map $i_\tau^\vee : V^{*\tau}/W_\tau \to (V^*/W)^\tau$ is an isomorphism of varieties.

EXAMPLE 4.2. If τ is W-regular (as defined in Example 2.2), then $W_\tau = W^\tau$ by [Spr].

4.2. τ-split parabolic subgroups. A parabolic subgroup P of W is called τ-*split* if it is the stabilizer of some point of V^τ (that is if $P = W^{\mathrm{pt}}_{V^P \cap V^\tau}$ or, in other words, if $\mathscr{V}(P) \cap V^\tau \neq \varnothing$). This is equivalent to saying that P is the stabilizer of some point of $V^{*\tau}$. Note the following easy fact.

<div align="center">The intersection of τ-split parabolic subgroups is τ-split.</div>

In this case, P is normalized by τ and τ is P-full, and we define the τ-*rank* of P to be the number $\dim(V^P)^\tau$. We denote by $\mathrm{Parab}_\tau(W)$ the set of τ-split parabolic subgroups of W. If $P \in \mathrm{Parab}_\tau(W)$, then $W^{\mathrm{pt}}_{V^\tau} \subset P$ and $P_\tau = (P \cap W^{\mathrm{set}}_{V^\tau})/W^{\mathrm{pt}}_{V^\tau}$ is a parabolic subgroup of W_τ. This shows that the map

$$\begin{array}{ccc} \mathrm{Parab}_\tau(W) & \longrightarrow & \mathrm{Parab}(W_\tau) \\ P & \longmapsto & P_\tau \end{array}$$

is well defined.

LEMMA 4.3. *The map*

$$\begin{array}{ccc} \mathrm{Parab}_\tau(W) & \longrightarrow & \mathrm{Parab}(W_\tau) \\ P & \longmapsto & P_\tau \end{array}$$

is bijective.

PROOF. First, if Q is a parabolic subgroup of W_τ, then there exists $v \in V^\tau$ such that $Q = (W_\tau)_v$ and so, if we set $P = W_v$, then P is τ-split and $P_\tau = Q$. This shows that the map is surjective.

Now, if P is a τ-split parabolic subgroup of W, then

$$(V^P)^\tau = (V^\tau)^{P_\tau}. \tag{4-3}$$

PROOF OF (4-3). Since P is τ-split, there exists $v \in (V^P)^\tau$ such that $P = W_v$ (and so $P_\tau = (W_\tau)_v$). Therefore, by Theorem 4.1(a),

$$(V^\tau)^{P_\tau} = \bigcap_{\substack{H \in \mathscr{A}(V^\tau, W_\tau) \\ v \in H}} H,$$

and so, by Theorem 4.1(c),

$$(V^\tau)^{P_\tau} = V^\tau \cap \left(\bigcap_{\substack{H \in \mathscr{A}(V,W) \\ v \in H}} H \right) = V^\tau \cap V^P,$$

as expected.   □

Since $P = W^{\mathrm{pt}}_{(V^P)^\tau}$, the group P_τ determines P. This means that the map of the lemma is injective.   □

If $\mathfrak{Q} \in \mathrm{Parab}(W_\tau)/W_\tau$ and $Q \in \mathfrak{Q}$, we denote by $\mathscr{V}_\tau(Q)$ and $\mathscr{U}_\tau(\mathfrak{Q})$ the analogues of $\mathscr{V}(P)$ and $\mathscr{U}(\mathfrak{P})$ for $\mathfrak{P} \in \mathrm{Parab}(W)/W$ and $P \in \mathfrak{P}$. We similarly also define $\mathscr{V}_\tau^*(Q)$ and

$\mathcal{U}_\tau^*(\mathfrak{Q})$. The same argument as in the above proof (using Theorem 4.1(c)) shows that if P is τ-split, then

$$\mathcal{V}_\tau(P_\tau) = \mathcal{V}(P)^\tau. \tag{4-4}$$

4.3. Normalizers. Fix a τ-split parabolic subgroup P of W. If $w \in W_\tau$, then ^wP does not depend on the representative of w in $W_{V^\tau}^{\mathrm{set}}$, because $W_{V^\tau}^{\mathrm{pt}} \subset P$ by definition. So we can define the *normalizer* $\mathrm{N}_{W_\tau}(P)$ of P in W_τ and, by the bijectivity proved in Lemma 4.3, it coincides with the normalizer $\mathrm{N}_{W_\tau}(P_\tau)$. The kernel of the well-defined composition

$$\mathrm{N}_{W_\tau}(P_\tau) = \mathrm{N}_{W_\tau}(P) \longrightarrow \overline{\mathrm{N}}_W(P)/P$$

is equal to P_τ, so we get a natural injective map

$$\overline{\mathrm{N}}_{W_\tau}(P_\tau) \lhook\joinrel\longrightarrow \overline{\mathrm{N}}_W(P). \tag{4-5}$$

Now, τ acts on $\overline{\mathrm{N}}_W(P)$. The next result describes the image of the above injective map.

LEMMA 4.4. *The image of the morphism (4-5) is equal to $\overline{\mathrm{N}}_W(P)^\tau$.*

PROOF. Let G denote the image of the morphism (4-5) and let $v \in \mathcal{V}(P)^\tau$ (so that $P = W_v$).

Let $w \in \mathrm{N}_{W_\tau}(P_\tau)$ and let $\dot{w}$ be a representative of w in $W_{V^\tau}^{\mathrm{set}}$. Then $\dot{w}(v) \in \mathcal{V}(P)^\tau$ by (4-4). So $\tau(\dot{w}(v)) = \dot{w}(v)$, that is, $\dot{w}^{-1}\tau(\dot{w}) \in P$. So the image of w in $\overline{\mathrm{N}}_W(P)$ is τ-invariant. This proves that $G \subset \overline{\mathrm{N}}_W(P)^\tau$.

Conversely, let $w \in \overline{\mathrm{N}}_W(P)^\tau$ and let $\dot{w}$ denote a representative of w in $\mathrm{N}_W(P)$. Then $\tau(\dot{w}(v)) = \dot{w}(\dot{w}^{-1}\tau(\dot{w}))(v)$. But $(\dot{w}^{-1}\tau(\dot{w}))(v) = v$ since $\dot{w}^{-1}\tau(\dot{w}) \in P$ by hypothesis. So v and $\dot{w}(v)$ belong to V^τ; so, by Lehrer–Springer Theorem 4.1(b), there exists $x \in W_\tau^{\mathrm{set}}$ such that $\dot{w}(v) = x(v)$. In other words, $x^{-1}\dot{w} \in P$. Moreover, v and $\dot{w}(v)$ both belong to $\mathcal{V}(P)$, so x normalizes P (and P_τ); so w is the image of x under the morphism (4-5). In other words, $\overline{\mathrm{N}}_W(P)^\tau \subset G$. □

Thanks to Lemma 4.4, we identify $\overline{\mathrm{N}}_{W_\tau}(P_\tau)$ with $\overline{\mathrm{N}}_W(P)^\tau$. Note that $\overline{\mathrm{N}}_{W_\tau}(P_\tau) = \overline{\mathrm{N}}_W(P)^\tau$ is the stabilizer of the set $\mathcal{V}(P)^\tau$ in $\overline{\mathrm{N}}_W(P)$.

4.4. Orbits of τ-split parabolic subgroups. We denote by $\mathrm{Parab}(W)_{\mathrm{spl}}^\tau$ the set of τ-split parabolic subgroups of W and by $(\mathrm{Parab}(W)/W)_{\mathrm{spl}}^\tau$ the set of W-orbits of parabolic subgroups of W containing a τ-split one. The group $W_{V^\tau}^{\mathrm{set}}$ acts on $\mathrm{Parab}(W)_{\mathrm{spl}}^\tau$ by conjugacy and, since any τ-split parabolic subgroup of W contains $W_{V^\tau}^{\mathrm{pt}}$, this action factorizes through an action of W_τ. If $\mathfrak{P} \in (\mathrm{Parab}(W)/W)_{\mathrm{spl}}^\tau$, we set $\mathfrak{P}_{\mathrm{spl}}^\tau = \mathfrak{P} \cap \mathrm{Parab}(W)_{\mathrm{spl}}^\tau$. Now, let $\mathscr{E}_P$ (respectively, $\tilde{\mathscr{E}}_P$) denote the set of elements $w \in \mathrm{N}_W(P)$ (respectively, $\overline{\mathrm{N}}_W(P)$) such that $\mathcal{V}(P)^{w\tau} \neq \varnothing$. Then $\overline{\mathrm{N}}_W(P)$ acts by conjugacy on the set $\tilde{\mathscr{E}}_P\tau$. If $w \in \mathscr{E}_P$, we denote by $[w\tau]$ the $\overline{\mathrm{N}}_W(P)$-orbit of the image of $w\tau$ in $\overline{\mathrm{N}}_W(P)\tau$.

PROPOSITION 4.5. *Let P be a τ-split parabolic subgroup and let $\mathfrak{P}$ denote its W-orbit. Then the following assertions hold.*

(a) *Let $x \in W$. Then ${}^x P$ is τ-split if and only if $x^{-1}\tau(x) \in \mathscr{E}_P$.*

(b) *The map $\mathfrak{P}_{\mathrm{spl}}^{\tau} \to \tilde{\mathscr{E}}_P \tau / \overline{\mathrm{N}}_W(P)$, ${}^x P \mapsto [x^{-1}\tau x]$ is well defined and induces a bijection*

$$\mathfrak{P}_{\mathrm{spl}}^{\tau}/W_{\tau} \xrightarrow{\ \sim\ } \tilde{\mathscr{E}}_P \tau / \overline{\mathrm{N}}_W(P).$$

PROOF. (a) Assume that ${}^x P$ is τ-split. In other words, there exists $v \in V^{\tau}$ such that ${}^x P = W_v$. Now let $w = x^{-1}\tau(x)$. Then $x^{-1}(v) \in \mathscr{V}(P)^{w\tau}$ and so $w \in \mathscr{E}_P$.

Conversely, assume that $w = x^{-1}\tau(x) \in \mathscr{E}_P$. Then there exists $v \in \mathscr{V}(P)$ such that $v \in V^{w\tau}$. Therefore, $P = W_v$ and so ${}^x P = W_{x(v)}$. But $\tau(x(v)) = xx^{-1}\tau(x)\tau(v) = xw\tau(v) = x(v)$, so $x(v) \in V^{\tau}$. This implies that ${}^x P$ is τ-split by definition.

(b) Let us first show that the map (let us denote it by ϕ) is well defined. For this purpose, let x and y be two elements of W such that ${}^x P = {}^y P$ is τ-split. Then there exists $u \in \mathrm{N}_W(P)$ such that $y = xu$. So $y^{-1}\tau y = u^{-1}x^{-1}\tau xu$ and so $[y^{-1}\tau y] = [x^{-1}\tau x]$, as expected.

Let us now prove that ϕ is constant on W_{τ}-orbits. For this purpose, let $w \in W_{\tau}$ and $x \in W$ be such that $\xi^{-1}\tau(x) \in \mathscr{E}_P$. Then $(xw)^{-1}\tau(wx) = x^{-1}w^{-1}\tau(w)xx^{-1}\tau(x)$. But $w^{-1}\tau(w) \in W_{V^{\tau}}^{\mathrm{pt}} \subset P$ by (4-2), so the images of $x^{-1}\tau(x)$ and $(wx)^{-1}\tau(wx)$ in $\tilde{\mathscr{E}}_P$ coincide. Therefore, ϕ factorizes through a map

$$\tilde{\phi} : \mathfrak{P}_{\mathrm{spl}}^{\tau}/W_{\tau} \longrightarrow \mathscr{E}_P \tau / \overline{\mathrm{N}}_W(P).$$

Let us prove that $\tilde{\phi}$ is injective. So let x and y be two elements of W such that ${}^x P$ and ${}^y P$ are τ-split and $[x^{-1}\tau x] = [y^{-1}\tau y]$. Then there exists $u \in \mathrm{N}_W(P)$ and $p \in P$ such that $y^{-1}\tau y = u^{-1}x^{-1}\tau xup$. In particular, $\mathscr{V}(P)^{y^{-1}\tau y} = \mathscr{V}(P)^{u^{-1}x^{-1}\tau xu}$. Since ${}^x P = {}^{xu} P$, we may (and do) assume that $u = 1$. As ${}^x P$ is τ-split, the set $\mathscr{V}({}^x P)^{\tau}$ is nonempty, so we may pick an element $v \in \mathscr{V}({}^x P)^{\tau}$. Then

$$\tau yx^{-1}(v) = yy^{-1}\tau yx^{-1}(v) = yx^{-1}\tau xpx^{-1}(v).$$

But $x^{-1}(v) \in \mathscr{V}(P)$, so $px^{-1}(v) = x^{-1}(v)$. Consequently, $\tau yx^{-1}(v) = yx^{-1}\tau(v) = yx^{-1}(v)$. In other words, $yx^{-1}(v) \in \mathscr{V}({}^y P)^{\tau} \subset V^{\tau}$. By Lehrer–Springer Theorem 4.1(b), there exists $a \in W_{\tau}$ such that $yx^{-1}(v) = a(v)$. Then

$${}^y P = {}^{yx^{-1}}({}^x P) = {}^{yx^{-1}} W_v = W_{yx^{-1}(v)} = W_{a(v)} = {}^a W_v = {}^a({}^x P),$$

which shows that ${}^y P$ and ${}^x P$ are W_{τ}-conjugate.

Let us now prove that $\tilde{\phi}$ is surjective. So let $w \in \mathscr{E}_P$. Then there exists $v \in \mathscr{V}(P)^{w\tau}$. So $W \cdot v \in (V/W)^{\tau}$. By Lehrer–Springer Theorem 4.1, there exists $x \in W$ such that $x(v) \in V^{\tau}$. Therefore, ${}^x P = W_{x(v)}$ is τ-split and $v \in V^{x^{-1}\tau x}$. So, if we set $p = w^{-1}x^{-1}\tau(x)$, then $p(v) = v$; so $p \in P$ and $\phi({}^x P) = [x^{-1}\tau(x)] = [wp] = [w]$, as desired. □

4.5. Stratification of $(V/W)^{\tau}$. Applying Section 2.3 to the pair (V^{τ}, W_{τ}), the variety V^{τ}/W_{τ} admits a stratification $(\mathscr{U}_{\tau}(\mathfrak{Q}))_{\mathfrak{Q} \in \mathrm{Parab}(W_{\tau})/W_{\tau}}$ while the variety $(V/W)^{\tau}$

admits a stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P}\in\mathrm{Parab}(W)/W}$. Both varieties are isomorphic and so both stratifications can be compared: through this isomorphism, the first is a refinement of the second, as shown in Corollary 4.7 below by using Proposition 4.5.

PROPOSITION 4.6. *Let $\mathfrak{P} \in \mathrm{Parab}(W)/W$. Then $\mathcal{U}(\mathfrak{P})^\tau$ is nonempty if and only if $\mathfrak{P}$ contains a τ-split parabolic subgroup.*

PROOF. If $\mathfrak{P}$ contains a τ-split parabolic subgroup P and if $v \in V^\tau$ is such that $P = W_v$, then the W-orbit of v belongs to $\mathcal{U}(\mathfrak{P})^\tau$ which is therefore nonempty. Conversely, if $\mathcal{U}(\mathfrak{P})^\tau$ is nonempty, it then follows from Theorem 4.1(b) that there exists $v \in V^\tau$ whose W-orbit belongs to $\mathcal{U}(\mathfrak{P})^\tau$. By construction, $W_v \in \mathfrak{P}$ and is τ-split. □

After we eliminate the empty pieces, Proposition 4.6 shows that $(V/W)^\tau$ admits a stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P}\in(\mathrm{Parab}(W)/W)^\tau_{\mathrm{spl}}}$. Let us decompose the pieces of this stratification into irreducible components. For this, fix a τ-split parabolic subgroup P and let $\mathfrak{P}$ denote its conjugacy class. Then $\mathcal{U}(\mathfrak{P})^\tau$ is smooth since $\mathcal{U}(\mathfrak{P})$ is smooth and τ has finite order. Now we have

$$\mathcal{U}(\mathfrak{P})^\tau = (\mathcal{V}(P)/\overline{\mathrm{N}}_W(P))^\tau = \Big(\bigcup_{w\in\tilde{\mathscr{C}}_P} \mathcal{V}(P)^{w\tau} \Big)/\overline{\mathrm{N}}_W(P).$$

By definition of $\mathcal{V}(P)$, $\mathcal{V}(P)^{w\tau} \cap \mathcal{V}(P)^{w'\tau} = \varnothing$ if $w \neq w'$. If E is a subset of $\tilde{\mathscr{C}}_P\tau$, we denote by $\mathcal{V}(P)^E$ the (disjoint) union of the $\mathcal{V}(P)^g$ for $g \in E$. Then

$$\mathcal{U}(\mathfrak{P})^\tau = \bigcup_{E\in\tilde{\mathscr{C}}_P\tau/\overline{\mathrm{N}}_W(P)} \mathcal{V}(P)^E/\overline{\mathrm{N}}_W(P). \qquad (4\text{-}6)$$

Then $\mathcal{V}(P)^E/\overline{\mathrm{N}}_W(P)$ is the image of some $\mathcal{V}(P)^g$ for some $g \in E$ and so $\mathcal{V}(P)^E/\overline{\mathrm{N}}_W(P)$ is closed (in $\mathcal{U}(\mathfrak{P})$) and irreducible. So the decomposition (4-6) is the decomposition of $\mathcal{U}(\mathfrak{P})^\tau$ into irreducible (that is, connected because disjoint) components.

So the stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P}\in(\mathrm{Parab}(W)/W)^\tau_{\mathrm{spl}}}$ of $(V/W)^\tau$ together with the decomposition (4-6) provides a finer stratification of $(V/W)^\tau$, indexed by the W_τ-orbits of τ-split parabolic subgroups (by using the bijection of Proposition 4.5(b)). On the other hand, V^τ/W_τ admits a stratification $(\mathcal{U}_\tau(\mathfrak{Q}))_{\mathfrak{Q}\in\mathrm{Parab}(W_\tau)/W_\tau}$. Both stratifications coincide through the isomorphism i_τ, as shown by the next result.

COROLLARY 4.7. *Let $\mathfrak{P} \in (\mathrm{Parab}(W)/W)^\tau_{\mathrm{spl}}$, let $P \in \mathfrak{P}$ and let $E \in \tilde{\mathscr{C}}_P\tau/\overline{\mathrm{N}}_W(P)$. Let $\mathfrak{P}_E$ denote the W_τ-orbit of τ-split parabolic subgroups of W associated with E through the bijection of Proposition 4.5(b). Let $\mathfrak{Q}_E$ denote the W_τ-orbit of parabolic subgroups of W_τ of the form Q_τ for $Q \in \mathfrak{P}_E$ (see Lemma 4.3). Then*

$$i_\tau(\mathcal{U}_\tau(\mathfrak{Q}_E)) = \mathcal{V}(P)^E/\overline{\mathrm{N}}_W(P).$$

PROOF. Let $g \in E$ and let $x \in W$ be such that $[x^{-1}\tau x] = [g]$ (the existence of such an x is guaranteed by Proposition 4.5(b)). We set $Q = {}^xP$. Then Q is τ-split by Proposition 4.5(a) and $\mathfrak{P}_E$ (respectively, $\mathfrak{Q}_E$) is the W_τ-orbit of Q (respectively, Q_τ) by construction.

Now $i_\tau(\mathcal{U}_\tau(\mathfrak{Q}_E))$ is the image of $\mathcal{V}(^xP)^\tau$ in $(V/W)^P$ and, through the isomorphism $\mathcal{V}(P)/\overline{N}_W(P) \simeq \mathcal{U}(\mathfrak{P})$, the result comes from the fact that x^{-1} induces an isomorphism between $\mathcal{V}(^xP)^\tau$ and $\mathcal{V}(P)^{x^{-1}\tau x}$. $\square$

5. The problem and the main result

5.1. Symplectic leaves. Let I_k denote the ideal of $\mathbf{Z}_k$ generated by $(\tau(z) - z)_{z \in \mathbf{Z}_k}$. It is τ-stable. Recall from Remark 3.3 that $\mathbb{C}[\mathcal{Z}_k^\tau] = \mathbf{Z}_k/\sqrt{I_k} = \mathbf{Z}_k^\tau/\sqrt{I_k^\tau}$, and that $\mathbf{Z}_k/I_k$ inherits a Poisson bracket which makes $\mathcal{Z}_k^\tau$ an affine Poisson variety. Therefore, $\mathcal{Z}_k^\tau$ admits a stratification into symplectic leaves [BrGo, Section S.5]. We denote by $\mathcal{Symp}(\mathcal{Z}_k^\tau)$ the set of symplectic leaves of $\mathcal{Z}_k^\tau$.

REMARK 5.1. Note that $\mathcal{Z}_k^\tau$ is generally not irreducible, not connected, not equidimensional and that its irreducible components might not coincide with its connected components.

Since $\mathcal{Z}_k$ has finitely many symplectic leaves [BrGo, Proposition 7.4], it follows from Corollary 3.7 that $\mathcal{Z}_k^\tau$ has finitely many symplectic leaves too. They are obtained as in Section 3.4.

REMARK 5.2. This description shows that the symplectic leaves of $\mathcal{Z}_k^\tau$ are $\mathbb{C}^\times$-stable.

If $\mathcal{S}$ is a symplectic leaf of $\mathcal{Z}_k^\tau$, we denote by $\mathfrak{p}_\mathcal{S}$ the defining ideal of $\overline{\mathcal{S}}$ in $\mathbf{Z}_k/I_k$: it belongs to $\mathrm{PSpec}(\mathbf{Z}_k/I_k)$. Since $\mathcal{Z}_k^\tau$ has finitely many symplectic leaves, the map

$$\begin{array}{ccc} \mathcal{Symp}(\mathcal{Z}_k^\tau) & \longrightarrow & \mathrm{PSpec}(\mathbf{Z}_k/I_k) \\ \mathcal{S} & \longmapsto & \mathfrak{p}_\mathcal{S} \end{array} \tag{5-1}$$

is bijective (see (3-1)).

5.2. τ-cuspidality. We define a τ-*cuspidal* symplectic leaf to be a zero-dimensional symplectic leaf of $\mathcal{Z}_k^\tau$. This definition coincides with the notion of *cuspidal* leaf of $\mathcal{Z}_k$ introduced by Bellamy [Bel1, Section 5] in the case where $\tau = 1$. We therefore also call it a τ-*cuspidal* point. Through the bijection (5-1), the set of τ-cuspidal points is naturally in bijection with the set $\mathrm{PMax}(\mathbf{Z}_k/\sqrt{I_k})$ of maximal ideals of the algebra $\mathbb{C}[\mathcal{Z}_k^\tau] = \mathbf{Z}_k/\sqrt{I_k}$ that are also Poisson ideals (note that $\mathrm{PMax}(\mathbf{Z}_k/I_k) = \mathrm{PMax}(\mathbf{Z}_k/\sqrt{I_k}) \subset \mathrm{PSpec}(\mathbf{Z}_k/\sqrt{I_k})$).

REMARK 5.3. It follows from Remark 5.2 that τ-cuspidal points are fixed under the action of $\mathbb{C}^\times$.

We denote by $\mathrm{Cus}_k^\tau(V, W)$ the set of pairs (P, p) where P is a τ-split parabolic subgroup of W and p is a τ-cuspidal point of $\mathcal{Z}_{k_P}(V_P, P)^\tau$, where k_P denotes the restriction of k to the parabolic subgroup P. The group W_τ acts on $\mathrm{Cus}_k^\tau(V, W)$ and we denote by $\mathrm{Cus}_k^\tau(V, W)/W_\tau$ the set of its orbits in $\mathrm{Cus}_k^\tau(V, W)$. If $(P, p) \in \mathrm{Cus}_k^\tau(V, W)$, we denote by $[P, p]$ its W_τ-orbit.

5.3. Main result. With the above notation, Theorem A can be restated (and made more precise) as follows.

THEOREM A. *There is a natural bijection (explicitly constructed in Section 9)*

$$\mathrm{Cus}_k^\tau(V, W)/W_\tau \longrightarrow \mathcal{Symp}(\mathfrak{X}_k^\tau)$$
$$[P, p] \longmapsto \mathcal{S}_{P,p}.$$

It satisfies that $\Upsilon_k(\overline{\mathcal{S}}_{P,p})$ *is the image of* $(V^P)^\tau \times (V^{*P})^\tau$ *in* $V/W \times V^*/W$. *In particular,*

$$\dim \mathcal{S}_{P,p} = 2 \dim(V^P)^\tau.$$

We prove Theorem A in the next sections. First, in Section 6, we recall the proof, essentially due to Brown and Gordon [BrGo, Proposition 7.4], of Theorem A whenever $k = 0$ and $\tau = 1$. In Section 7 we use Lehrer–Springer Theorem 4.1 to prove Theorem A whenever $k = 0$. In Section 8 we use a deformation argument to attach to each symplectic leaf a W_τ-orbit of τ-split parabolic subgroups: in some sense, this is half of the construction of the above bijection. The second half is constructed in Section 9, where the proof of Theorem A is completed.

Let us also restate Conjecture B.

CONJECTURE B. *Let* $(P, p) \in \mathrm{Cus}_k^\tau(V, W)$. *Then there exist* $l \in \nabla((V^P)^\tau, \overline{\mathrm{N}}_{W_\tau}(P_\tau))$ *and a* $\mathbb{C}^\times$-*equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}}_{P,p}^{\mathrm{nor}} \simeq \mathfrak{X}_l((V^P)^\tau, \overline{\mathrm{N}}_{W_\tau}(P_\tau)).$$

6. Symplectic leaves of $\mathfrak{X}_0 = (V \times V^*)/W$

The Poisson bracket on $\mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W$ is the one obtained by restriction from the usual Poisson bracket on $\mathbb{C}[V \times V^*]$. The symplectic leaves of $\mathfrak{X}_0$ are described in [BrGo, Proposition 7.4]: we recall their description in this section, and give some more precise details about the structure of their closure.

If $P \in \mathrm{Parab}(W)$, let $\mathcal{VV}^*(P)$ denote the set of elements $(v, v^*) \in V \times V^*$ such that $W_v \cap W_{v^*} = P$. Again, the family $(\mathcal{VV}^*(P))_{P \in \mathrm{Parab}(W)}$ is a stratification of $V \times V^*$ (the order between strata corresponds to the reverse order of the inclusion of parabolic subgroups). If $\mathfrak{P} \in \mathrm{Parab}(W)/W$, we let $\mathcal{UU}^*(\mathfrak{P})$ denote the image of $\mathcal{VV}^*(P)$ in $(V \times V^*)/W$, where P is any element of $\mathfrak{P}$. Then $(\mathcal{UU}^*(\mathfrak{P}))_{\mathfrak{P} \in \mathrm{Parab}(W)/W}$ is a stratification of $(V \times V^*)/W$ (the order between strata corresponds to the reverse order of the inclusion, up to conjugacy, of parabolic subgroups).

Now fix $\mathfrak{P} \in \mathrm{Parab}(W)/W$ and $P \in \mathfrak{P}$. Then

$$\mathcal{V}(P) \times \mathcal{V}^*(P) \subset \mathcal{VV}^*(P) \subset V^P \times V^{*P}.$$

Note that $\overline{\mathrm{N}}_W(P)$ acts on $V^P \times V^{*P}$ and that $\mathcal{VV}^*(P)$ is the open subset of $V^P \times V^{*P}$ on which $\overline{\mathrm{N}}_W(P)$ acts freely. The image of $\overline{\mathcal{VV}^*(P)} = V^P \times V^{*P}$ is equal to $\overline{\mathcal{UU}^*(\mathfrak{P})}$.

Recall from Section 3.3 that $V^P \times V^{*P}$ is not a Poisson subvariety of $V \times V^*$ but inherits from $V \times V^*$ a Poisson structure. This Poisson structure is the natural one

endowed by the product of a vector space with its dual: it is $\overline{\mathrm{N}}_W(P)$-equivariant, so $(V^P \times V^{*P})/\overline{\mathrm{N}}_W(P)$ is also a Poisson variety. By definition, $\overline{\mathrm{N}}_W(P)$ acts freely on the open subset $\mathcal{V}\mathcal{V}^*(P)$, so the variety $\mathcal{V}\mathcal{V}^*(P)/\overline{\mathrm{N}}_W(P)$ is smooth and its Poisson bracket makes it a symplectic variety. The next proposition is a particular case of the discussion preceding Proposition 3.6.

LEMMA 6.1. *Let* $\mathfrak{P} \in \mathrm{Parab}(W)/W$ *and let* $P \in \mathfrak{P}$. *Then the following assertions hold.*

(a) *The closed subvariety* $\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}$ *is a Poisson subvariety of* $(V \times V^*)/W$.
(b) *The map* $\mathcal{V}\mathcal{V}^*(P) \to \mathcal{U}\mathcal{U}^*(\mathfrak{P})$ *induces an isomorphism*

$$\mathcal{V}\mathcal{V}^*(P)/\overline{\mathrm{N}}_W(P) \xrightarrow{\sim} \mathcal{U}\mathcal{U}^*(\mathfrak{P})$$

   of Poisson varieties.

COROLLARY 6.2. *Let* $\mathfrak{P} \in \mathrm{Parab}(W)/W$ *and let* $P \in \mathfrak{P}$. *Then the above isomorphism* $\mathcal{V}\mathcal{V}^*(P)/\overline{\mathrm{N}}_W(P) \xrightarrow{\sim} \mathcal{U}\mathcal{U}^*(\mathfrak{P})$ *extends to an isomorphism of Poisson varieties*

$$(V^P \times V^{*P})/\overline{\mathrm{N}}_W(P) \xrightarrow{\sim} \overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}^{\mathrm{nor}}.$$

PROOF. The surjective map $\varphi : (V^P \times V^{*P})/\overline{\mathrm{N}}_W(P) \to \overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}$ induces an injection $\mathbb{C}[\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}] \subset \mathbb{C}[V^P \times V^{*P}]^{\overline{\mathrm{N}}_W(P)}$ between algebras of regular functions, and both algebras have the same fraction fields by Lemma 6.1(b). But φ is finite and $\mathbb{C}[V^P \times V^{*P}]^{\overline{\mathrm{N}}_W(P)}$ is integrally closed, so $\mathbb{C}[V^P \times V^{*P}]^{\overline{\mathrm{N}}_W(P)}$ is the integral closure of $\mathbb{C}[\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}]$ in its fraction field. This completes the proof of the corollary. □

The next result follows immediately from Lemma 6.1 and is a particular case of Proposition 3.6 (see also [BrGo, Proposition 7.4]).

PROPOSITION 6.3. *The family* $(\mathcal{U}\mathcal{U}^*(\mathfrak{P}))_{\mathfrak{P}\in\mathrm{Parab}(W)/W}$ *of locally closed subvarieties is the stratification of* $\mathfrak{X}_0 = (V \times V^*)/W$ *by symplectic leaves.*

Let us interpret the results of this section in terms of Theorem A and Conjecture B for $k = 0$ and $\tau = \mathrm{Id}_V$. First, it follows from Corollary 6.2 that, if $\mathfrak{P} \in \mathrm{Parab}(W)/W$ and if $P \in \mathfrak{P}$, then $\dim \mathcal{U}\mathcal{U}^*(\mathfrak{P}) = 2 \dim V^P$. Therefore, $\mathcal{U}\mathcal{U}^*(\mathfrak{P})$ is Id_V-cuspidal (we say *cuspidal* for the sake of simplicity) if and only if $V^P = 0$. Therefore, there is at most one cuspidal leaf of $\mathfrak{X}_0$ and there is actually one if and only if $V^W = 0$ (in this case, this cuspidal leaf will be simply denoted by 0, as it is the W-orbit of $0 \in V \times V^*$). This shows that

$$\mathrm{Cus}_0^{\mathrm{Id}_V}(V, W) = \{(P, 0) \mid P \in \mathrm{Parab}(W)\} \xleftrightarrow{\sim} \mathrm{Parab}(W).$$

Consequently, the bijection $\mathrm{Cus}_0^{\mathrm{Id}_V}(V, W)/W \xrightarrow{\sim} \mathcal{S}ymp(\mathfrak{X}_0)$ predicted by Theorem A in the case where $k = 0$ and $\tau = \mathrm{Id}_V$ is simply given by the formula

$$\mathcal{S}_{P,0} = \mathcal{U}\mathcal{U}^*(\mathfrak{P})$$

for all $\mathfrak{P} \in \mathrm{Parab}(W)/W$ and all $P \in \mathfrak{P}$: this is the content of Proposition 6.3. Moreover, Corollary 6.2 proves Conjecture B in this case.

PROPOSITION 6.4. *Theorem A and Conjecture B hold if $k = 0$ and $\tau = \mathrm{Id}_V$.*

7. Symplectic leaves of $\mathfrak{X}_0^\tau$

We have $\mathfrak{X}_0^\tau = ((V \times V^*)/W)^\tau$. To study its symplectic leaves, the next consequence of Lehrer–Springer Theorem 4.1 is crucial.

PROPOSITION 7.1. *The natural map*

$$\ddot{u}_\tau : (V^\tau \times V^{*\tau})/W_\tau \longrightarrow ((V \times V^*)/W)^\tau = \mathfrak{X}_0^\tau$$

is a finite bijective morphism of Poisson varieties: it is the normalization of the variety $\mathfrak{X}_0^\tau$.

PROOF. Only the statement on the bijectivity needs to be proved, the others being obvious or immediate consequences.

Let us first prove that $\ddot{u}_\tau$ is injective. Let (v_1, v_1^*) and $(v_2, v_2^*) \in V^\tau \times V^{*\tau}$ be such that (v_2, v_2^*) belong to the W-orbit of (v_1, v_1^*). Then there exists $a \in W$ such that $(v_2, v_2^*) = a(v_1, v_1^*)$. By Theorem 4.1(b), there exists $b \in W_{V^\tau}^{\mathrm{set}}$ such that $v_2 = b(v_1)$. Therefore, $b^{-1}a(v_1) = v_1$ and $b^{-1}(v_2^*) = b^{-1}a(v_1^*)$. In other words, $b^{-1}a$ belongs to the stabilizer W_{v_1} of v_1 in W (it is a parabolic subgroup). Since $\tau(v_1) = v_1$, τ normalizes W_{v_1}. Hence, since τ is W_{v_1}-full by (4-1), we may apply Theorem 4.1(b) to the pair (W_{v_1}, τ) so that, by dualizing, there exists $c \in (W_{V^\tau}^{\mathrm{set}})_{v_1}$ such that $b^{-1}(v_2^*) = c(v_1^*)$. Therefore, $bc \in W_{V^\tau}^{\mathrm{set}}$ and $bc(v_1, v_1^*) = (v_2, v_2^*)$, as desired.

Let us now prove that $\ddot{u}_\tau$ is surjective. Let $(v, v^*) \in V \times V^*$ be such that its W-orbit is τ-stable. By Theorem 4.1(b), there exists $x \in W$ such that $\tau(x(v)) = x(v)$. So, by replacing (v, v^*) by $x(v, v^*)$ if necessary, we may, and do, assume that $\tau(v) = v$. Therefore, there exists $a \in W$ such that $(\tau(v), \tau(v^*)) = (a(v), a(v^*))$. In other words, $a(v) = v$ and $\tau(v^*) = a(v^*)$. So a belongs to the parabolic subgroup W_v, which is τ-stable. Again applying Theorem 4.1(b) to (W_v, τ) (since τ is W_v-full by (4-1)), and dualizing, one gets that there exists $b \in W_v$ such that $\tau(b(v^*)) = b(v^*)$. Therefore, $ab(v, v^*) \in V^\tau \times V^{*\tau}$, as desired.   □

REMARK 7.2. We do not know if there are examples of pairs (W, τ) such that the variety $\mathfrak{X}_0^\tau$ is not normal. By the above proposition, saying that $\mathfrak{X}_0^\tau$ is normal is equivalent to saying that any W_τ-invariant polynomial function on $V^\tau \times V^{*\tau}$ extends to a W-invariant polynomial function on $V \times V^*$.

A bijective morphism of Poisson varieties does not necessarily induce a bijection between symplectic leaves, but it turns out that this holds for our map $\ddot{u}_\tau$, as shown by Corollary 7.5 below. Before proving it, let us introduce some notation. If Q is a parabolic subgroup of W_τ, we denote by $\mathscr{VV}_\tau^*(Q)$ the set of pairs $(v, v^*) \in V^\tau \times V^{*\tau}$ such that $Q = W_v \cap W_{v^*}$. If $\mathfrak{Q}$ denotes the W_τ-orbit of Q, we denote by $\mathscr{UU}_\tau^*(\mathfrak{Q})$ the image of $\mathscr{VV}_\tau^*(Q)$ in $(V^\tau \times V^{*\tau})/W_\tau$. By Proposition 6.3 applied to the pair (V^τ, W_τ), the locally closed subvariety $\mathscr{UU}_\tau^*(\mathfrak{Q})$ is a symplectic leaf of $(V^\tau \times V^{*\tau})/W_\tau$ and all the symplectic leaves are obtained in this way. Note first the following easy fact.

LEMMA 7.3. *Let $\mathfrak{P} \in \mathrm{Parab}(W)/W$, let $P \in \mathfrak{P}$ and let $w \in \overline{\mathrm{N}}_W(P)$. Then the following assertions hold.*

(a) $\mathscr{V}\mathscr{V}^*(P)^{w\tau} \neq \varnothing$ *if and only if* $\mathscr{V}(P)^{w\tau} \neq \varnothing$.
(b) $\mathscr{U}\mathscr{U}^*(\mathfrak{P})^\tau \neq \varnothing$ *if and only if* $\mathscr{U}(\mathfrak{P})^\tau \neq \varnothing$.

PROOF. Note that (a) implies (b) by Lemma 6.1(b). On the other hand, if $\mathscr{V}(P)^{w\tau} \neq \varnothing$, then $\mathscr{V}^*(P)^{w\tau} \neq \varnothing$. So, if we pick $v \in \mathscr{V}(P)^{w\tau}$ and $v^* \in \mathscr{V}^*(P)^{w\tau}$, then $(v, v^*) \in \mathscr{V}\mathscr{V}^*(P)^{w\tau}$. This proves the 'if' part of (a).

Conversely, if $\mathscr{V}\mathscr{V}^*(P)^{w\tau} \neq \varnothing$, pick $(v, v^*) \in \mathscr{V}\mathscr{V}^*(P)^{w\tau}$. Then $\mathscr{V}^*(W_{v^*})^{w\tau} \neq \varnothing$ so $\mathscr{V}(W_{v^*})^{w\tau}$. Pick $v' \in \mathscr{V}(W_{v^*})^{w\tau}$ and let S denote the subspace of V generated by v and v'. Then $P = W_S^{\mathrm{pt}}$, so there exists $v'' \in S$ such that $W_{v''} = P$. But $v'' \in V^{w\tau} \cap \mathscr{V}(P)$, which proves the 'only if' part of (a).   □

The above lemma allows us to apply the same arguments as in Section 4.5 to the bijective morphism of varieties $\ddot{u}_\tau : (V^\tau \times V^{*\tau})/W_\tau \longrightarrow \mathfrak{X}_0^\tau$. For instance, if $\mathfrak{P} \in \mathrm{Parab}(W)/W$, then it follows from Lemma 7.3 and Proposition 4.6 that $\mathscr{U}\mathscr{U}^*(\mathfrak{P})^\tau \neq \varnothing$ if and only if $\mathfrak{P}$ contains a τ-split parabolic subgroup.

Moreover, if $\mathfrak{P} \in (\mathrm{Parab}(W)/W)_{\mathrm{spl}}^\tau$ and if $P \in \mathfrak{P}$ is τ-split, then the τ-equivariant isomorphism $\mathscr{U}\mathscr{U}^*(\mathfrak{P}) \simeq \mathscr{V}\mathscr{V}^*(P)/\overline{\mathrm{N}}_W(P)$ induces a decomposition into irreducible components

$$\mathscr{U}\mathscr{U}^*(\mathfrak{P})^\tau = \bigcup_{E \in \tilde{\mathscr{E}}_P\tau/\overline{\mathrm{N}}_W(P)} \mathscr{V}\mathscr{V}^*(P)^E/\overline{\mathrm{N}}_W(P), \tag{7-1}$$

where $\mathscr{V}\mathscr{V}^*(P)^E$ is defined in the same way as $\mathscr{V}(P)^E$. Similarly, the analogue of Corollary 4.7 is given as follows.

PROPOSITION 7.4. *Let $\mathfrak{P} \in (\mathrm{Parab}(W)/W)_{\mathrm{spl}}^\tau$, let $P \in \mathfrak{P}$ and let $E \in \tilde{\mathscr{E}}_P\tau/\overline{\mathrm{N}}_W(P)$. Let $\mathfrak{P}_E$ denote the W_τ-orbit of τ-split parabolic subgroups of W associated with E through the bijection of Proposition 4.5(b). Let $\mathfrak{Q}_E$ denote the W_τ-orbit of parabolic subgroups of W_τ of the form Q_τ for $Q \in \mathfrak{P}_E$ (see Lemma 4.3). Then*

$$\ddot{u}_\tau(\mathscr{U}\mathscr{U}_\tau^*(\mathfrak{Q}_E)) = \mathscr{V}\mathscr{V}^*(P)^E/\overline{\mathrm{N}}_W(P).$$

COROLLARY 7.5. *The bijective morphism of varieties $\ddot{u}_\tau : (V^\tau \times V^{*\tau})/W_\tau \longrightarrow \mathfrak{X}_0^\tau$ induces a bijection between symplectic leaves.*

PROOF. Both varieties admit finitely many symplectic leaves so, by taking the closure, these leaves are, in both cases, in bijection with the set of irreducible closed Poisson subvarieties.

Now let $\mathcal{S}$ be an irreducible closed Poisson subvariety of $(V^\tau \times V^{*\tau})/W_\tau$. Since $\ddot{u}_\tau$ respects the Poisson bracket, $\ddot{u}_\tau(\mathcal{S})$ is also an irreducible closed Poisson subvariety of $\mathfrak{X}_0^\tau$: this shows that $\ddot{u}_\tau$ induces an injective map between the symplectic leaves of $(V^\tau \times V^{*\tau})/W_\tau$ and those of $\mathfrak{X}_0^\tau$.

Let us now show that this map is surjective. For this purpose, let $\mathcal{S}$ be a symplectic leaf of $\mathfrak{X}_0^\tau$. Then there exists $\mathfrak{P} \in \mathrm{Parab}(W)/W$ such that $\mathcal{S} \cap \mathscr{U}\mathscr{U}^*(\mathfrak{P})^\tau$ is open and

dense in $\mathcal{S}$. So $\mathfrak{P}$ contains a τ-split parabolic subgroup P and the decomposition of $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau \neq \varnothing$ into irreducible components is given by (7-1). But $\mathcal{U}\mathcal{U}^*(\mathfrak{P})$ is smooth and symplectic, so $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau$ is also smooth and symplectic; so $\mathcal{S} \cap \mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau$ is equal to one of these irreducible components. The result then follows from Proposition 7.4. $\qquad\square$

PROPOSITION 7.6. *If $k = 0$, then Theorem A and Conjecture B hold.*

PROOF. By Corollary 7.5, $\mathfrak{X}_0^\tau$ admits a τ-cuspidal point if and only if $(V^\tau)^{W_\tau} = 0$ and, in this case, there is only one τ-cuspidal point, namely the orbit of 0. So, still by Corollary 7.5 (and (4-3)), $\mathrm{Cus}_0^\tau(V, W)$ is in bijection with conjugacy classes of parabolic subgroups of W_τ and $\mathcal{Symp}(\mathfrak{X}_0^\tau)$ is also in bijection with conjugacy classes of parabolic subgroups of W_τ. This provides a natural bijection between $\mathrm{Cus}_0^\tau(V, W)$ and $\mathcal{Symp}(\mathfrak{X}_0^\tau)$ that satisfies the required properties of Theorem A.

Let us now prove Conjecture B in this case. So let $\mathcal{S}$ be a symplectic leaf of $\mathfrak{X}_0^\tau$. Let $\mathcal{L} = \ddot{u}_\tau^{-1}(\mathcal{S})$: it is a symplectic leaf of $(V^\tau \times V^{*\tau})^{W_\tau}$ and

$$\overline{\mathcal{M}} = \ddot{u}_\tau^{-1}(\overline{\mathcal{S}}).$$

Since $\ddot{u}_\tau$ is bijective, we have $\overline{\mathcal{L}}^{\mathrm{nor}} = \overline{\mathcal{S}}^{\mathrm{nor}}$; so Conjecture B now follows from Lemma 6.1 applied to the pair (V^τ, W_τ) instead of (V, W). $\qquad\square$

REMARK 7.7. If $k = 0$, then the parameter l involved in Conjecture B is equal to 0.

8. Parabolic subgroups attached to symplectic leaves

8.1. Definition. Let $\mathcal{S}$ be a symplectic leaf of $\mathfrak{X}_k^\tau$. We denote by $\mathfrak{p}_\mathcal{S}$ the prime ideal of $\mathbf{Z}_k$ defining $\overline{\mathcal{S}}$; then $\mathfrak{p}_\mathcal{S}^\tau \in \mathrm{PSpec}(\mathbf{Z}_k^\tau)$. Now, the isomorphism $\mathrm{gr}_\mathcal{F}\mathbf{Z}_k \simeq \mathbf{Z}_0$ is τ-equivariant and $(\mathrm{gr}_\mathcal{F}\mathbf{Z}_k)^\tau = \mathrm{gr}_\mathcal{F}(\mathbf{Z}_k^\tau)$. So $\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)$ is an ideal of $\mathbf{Z}_0^\tau$. The next important result follows mainly from [Mar1, Theorem 2.8].

LEMMA 8.1. *The ideal $\sqrt{\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)}$ of $\mathbf{Z}_0^\tau$ is prime, Poisson and contains I_0^τ.*

PROOF. First, the Poisson bracket $\{\ ,\ \}$ on $\mathbf{Z}_k$ is a *proto-Poisson bracket* of degree -2 in the sense of [Mar1, Definition 2.4] and its associated graded Poisson bracket on $\mathbf{Z}_0$ is also the natural Poisson bracket on $\mathbf{Z}_0$ (for a proof of both facts, see [EtGi, Lemma 2.26]).

The same facts also hold by taking fixed points under the τ-action, and so the fact that $\sqrt{\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)}$ is a prime ideal of $\mathbf{Z}_0^\tau$ that is Poisson is an application of [Mar1, Theorem 2.8] (because $\mathfrak{X}_k^\tau$ has finitely many symplectic leaves by Corollary 3.7).

Finally, τ acts trivially on $\mathbf{Z}_k/\mathfrak{p}_\mathcal{S}$, so it acts trivially on $\mathrm{gr}_\mathcal{F}(\mathbf{Z}_k/\mathfrak{p}_\mathcal{S}) = \mathrm{gr}_\mathcal{F}(\mathbf{Z}_k)/\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}) = \mathbf{Z}_0/\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S})$. This shows that $\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S})$ contains I_0 and so $\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)$ contains I_0^τ. $\qquad\square$

Lemma 8.1 shows that $\sqrt{\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)}$ defines a symplectic leaf $\mathcal{S}_0$ of $\mathfrak{X}_0^\tau$; so, by Corollary 7.5, there exists a unique W_τ-orbit $\mathfrak{Q}_\mathcal{S}$ of parabolic subgroups of W_τ such that $\sqrt{\mathrm{gr}_\mathcal{F}(\mathfrak{p}_\mathcal{S}^\tau)}$ is the defining ideal of $\ddot{u}_\tau(\mathcal{U}\mathcal{U}_\tau^*(\mathfrak{Q}_\mathcal{S}))$. Through the bijection of Lemma 4.3,

there exists a unique W_τ-orbit $\mathfrak{P}_\mathcal{S}$ of τ-split parabolic subgroups of W such that $\mathfrak{Q}_\mathcal{S} = \{P_\tau \mid P \in \mathfrak{P}_\mathcal{S}\}$.

DEFINITION 8.2. Let $\mathcal{S}$ be a symplectic leaf of $\mathcal{Z}_k^\tau$. The W_τ-orbit of τ-split parabolic subgroups $\mathfrak{P}_\mathcal{S}$ is called the W_τ-orbit *associated with* $\mathcal{S}$. Any element of $\mathfrak{P}_\mathcal{S}$ is called an *associated τ-split parabolic subgroup* (with $\mathcal{S}$).

If P is a τ-split parabolic subgroup of W associated with $\mathcal{S}$, then

$$\dim \mathcal{S} = 2\dim(V^P)^\tau. \tag{8-1}$$

Indeed, $\dim \mathcal{S} = \dim \mathcal{S}_0$.

8.2. Geometric construction.   Let

$$\pi : V/W \times V^*/W \to V/W \quad \text{and} \quad \pi^\vee : V/W \times V^*/W \longrightarrow V^*/W$$

denote the first and second projection, respectively. The next proposition gives another characterization of the W_τ-orbit of τ-split parabolic subgroups associated with a symplectic leaf:

PROPOSITION 8.3. *Let $\mathcal{S}$ be a symplectic leaf of $\mathcal{Z}_0^\tau$. Then:*

(a) $\Upsilon_k(\overline{\mathcal{S}}) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathfrak{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathfrak{Q}_\mathcal{S})})$;
(b) $\pi(\Upsilon_k(\overline{\mathcal{S}})) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathfrak{Q}_\mathcal{S})})$;
(c) $\pi^\vee(\Upsilon_k(\overline{\mathcal{S}})) = \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathfrak{Q}_\mathcal{S})})$.

PROOF. Let $\mathbf{H}_k^\#$ denote the $\mathbb{C}[\hbar]$-algebra that is obtained as the quotient of the algebra $\mathbb{C}[\hbar] \otimes (\mathrm{T}(V \oplus V^*) \rtimes W)$ by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \hbar^2 \displaystyle\sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H - 1} e_H(k_{H,i} - k_{H,i+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,i}, \end{cases} \tag{8-2}$$

for all y, $y' \in V$ and x, $x' \in V^*$. It follows from the comparison of the relations (2-1) and (8-2) that there is a well-defined morphism of $\mathbb{C}[\hbar]$-algebras $\theta : \mathbf{H}_k^\# \longrightarrow \mathrm{Rees}_{\mathcal{F}} \mathbf{H}_k$ such that

$$\theta(y) = \hbar y, \quad \theta(x) = \hbar x \quad \text{and} \quad \theta(w) = w$$

for all $y \in V$, $x \in V^*$ and $w \in W$. In fact,

$$\theta \text{ is an isomorphism of algebras.} \tag{8-3}$$

Indeed, surjectivity is immediate while injectivity follows from the PBW decomposition (2-2), which also holds for $\mathbf{H}_k^\#$, namely, the map

$$\mathbb{C}[\hbar] \otimes \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \longrightarrow \mathbf{H}_k^\#$$

induced by the multiplication is an isomorphism of $\mathbb{C}[\hbar]$-modules.

Let $\mathbf{Z}_k^{\#}$ denote the centre of $\mathbf{H}_k^{\#}$. Then it follows from (8-3) that θ induces an isomorphism of algebras

$$\mathbf{Z}_k^{\#} \longrightarrow \mathrm{Rees}_{\mathscr{F}}\mathbf{Z}_k.$$

Again, $\mathbf{Z}_k^{\#}$ is a flat family of deformations of $\mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W$. We denote by $\mathscr{X}_k^{\#}$ the affine variety such that $\mathbb{C}[\mathscr{X}_k^{\#}] = \mathbf{Z}_k^{\#}$. The inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k^{\#}$ induces a morphism $\Upsilon_k^{\#} : \mathscr{X}_k^{\#} \longrightarrow \mathscr{P} = V/W \times V^*/W$.

The action of τ and $\mathbb{C}^{\times}$ extends easily to $\mathbf{H}_k^{\#}$, by letting them act trivially on the indeterminate $\hbar$. However, $\mathbf{H}_k^{\#}$ (and so $\mathbf{Z}_k^{\#}$) inherits a further action of $\mathbb{C}^{\times}$. Namely, there is an action of $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ given by

$$\begin{cases} (\xi, \xi') \cdot x = \xi\xi'x & \text{if } x \in V^*, \\ (\xi, \xi') \cdot y = \xi^{-1}\xi'y & \text{if } y \in V, \\ (\xi, \xi') \cdot w = w & \text{if } w \in W, \\ (\xi, \xi') \cdot \hbar = \xi'\hbar. \end{cases}$$

The action of the first copy of $\mathbb{C}^{\times}$ extends the one already defined in Section 2.6.1. Through the isomorphism θ, this action on $\mathrm{Rees}_{\mathscr{F}}\mathbf{H}_k$ is just the restriction of the action on $\mathbb{C}[\hbar] \otimes \mathbf{H}_k$ given by

$$(\xi, \xi') \cdot (P(\hbar) \otimes h) = P(\xi'\hbar) \otimes (\xi \cdot h).$$

Then (2-4) can be retrieved thanks to (2-3) and the isomorphism θ, by specializing $\hbar$ to 0.

Specializing $\hbar$ to $\lambda \in \mathbb{C}$ gives the algebras $\mathbf{H}_{\lambda^2 k}$ and $\mathbf{Z}_{\lambda^2 k}$. Geometrically, the inclusion $\mathbb{C}[\hbar] \hookrightarrow \mathbf{Z}_k^{\#}$ induces a flat morphism $\mathscr{X}_k^{\#} \to \mathbb{C}$ whose fibre at λ is the Calogero–Moser space $\mathscr{X}_{\lambda^2 k}$.

Now view $\mathcal{S}$ as a subvariety of $\mathscr{X}_k^{\#}$ and let $\overline{\mathcal{S}}_0 = \overline{(1 \times \mathbb{C}^{\times}) \cdot \mathcal{S}} \cap \mathscr{X}_0$, endowed with its reduced structure. Then, using the isomorphism θ, it follows from the definition of the action of the second copy of $\mathbb{C}^{\times}$ on $\mathrm{Rees}_{\mathscr{F}}(\mathbf{H}_k)$ that the defining ideal of $\overline{\mathcal{S}}_0$ is $\sqrt{\mathrm{gr}_{\mathscr{F}}(\mathfrak{p}_{\mathcal{S}})}$.

Since $\mathcal{S}$ is $(\mathbb{C}^{\times} \times 1)$-stable, we have

$$(1 \times \mathbb{C}^{\times}) \cdot \mathcal{S} = (\mathbb{C}^{\times} \times \mathbb{C}^{\times}) \cdot \mathcal{S} = \Delta(\mathbb{C}^{\times}) \cdot \mathcal{S},$$

where $\Delta(\mathbb{C}^{\times})$ is the diagonal in $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$. Note also that if $\xi \in \mathbb{C}^{\times}$ and $z \in \mathscr{X}_k^{\#}$, then $\pi \circ \Upsilon_k^{\#}((\xi, \xi) \cdot z) = \pi \circ \Upsilon_k^{\#}(z)$. Moreover, if $z \in \overline{\mathcal{S}}$, then $z_0 = \lim_{\xi \to 0}(\xi, \xi) \cdot z$ exists (because the action $\Delta(\mathbb{C}^{\times})$ has nonnegative weights) and $z_0 \in \overline{\mathcal{S}}_0$. So $(\pi \circ \Upsilon_k)(z) = (\pi \circ \Upsilon_0)(z_0)$ belongs to $\iota_{\tau}(\overline{\mathcal{U}_{\tau}(\mathfrak{Q}_{\mathcal{S}})})$, as expected. This shows that

$$\pi(\Upsilon_k(\overline{\mathcal{S}})) \subset \iota_{\tau}(\overline{\mathcal{U}_{\tau}(\mathfrak{Q}_{\mathcal{S}})}).$$

By exchanging the role of V and V^*, we have

$$\pi^{\vee}(\Upsilon_k(\overline{\mathcal{S}})) \subset \iota_{\tau}^{\vee}(\overline{\mathcal{U}_{\tau}^*(\mathfrak{Q}_{\mathcal{S}})}).$$

Therefore,

$$\Upsilon_k(\overline{\mathcal{S}}) \subset \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})}).$$

Since $\Upsilon_k(\overline{\mathcal{S}})$ is closed irreducible of dimension $2 \dim(V^P)^\tau$ (by (8-1) and the finiteness of the morphism Υ_k), we get that

$$\Upsilon_k(\overline{\mathcal{S}}) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})}).$$

In other words, this proves (a). Now, (b) and (c) follow from (a). □

Keep the notation introduced in the above proof ($\mathbf{H}_k^\#$, $\mathbf{Z}_k^\#$, $\mathcal{E}_k^\#$, ...) and let us explain how this proof provides a justification for Conjecture B as well as a possible strategy for proving it. Indeed, if $\mathcal{S}$ is a symplectic leaf of $\mathcal{E}_k$, let $\overline{\mathcal{S}}^\# = \overline{(1 \times \mathbb{C}^\times) \cdot \mathcal{S}}$. Then $\overline{\mathcal{S}}^\#$ comes equipped with a morphism $\varpi : \overline{\mathcal{S}}^\# \longrightarrow \mathbb{C}$ and we denote by $\nu : (\overline{\mathcal{S}}^\#)^{\mathrm{nor}} \longrightarrow \mathbb{C}$ the composition of the normalization morphism $(\overline{\mathcal{S}}^\#)^{\mathrm{nor}} \longrightarrow \overline{\mathcal{S}}^\#$ with ϖ. Then ν is flat [Har, Ch. III, Proposition 9.7]. Since $\varpi^{-1}(\mathbb{C}^\times) \simeq \mathbb{C}^\times \times \overline{\mathcal{S}}$, we have

$$\nu^{-1}(\mathbb{C}^\times) \simeq \mathbb{C}^\times \times \overline{\mathcal{S}}^{\mathrm{nor}}.$$

Let $\overline{\mathcal{S}}_0^\star$ denote the scheme-theoretic fibre of ν at 0. Assume that we are able to show the following two facts.

(1)   The reduced subscheme of $\overline{\mathcal{S}}_0^\star$ is the normalization of $\overline{\mathcal{S}}_0$.
(2)   The scheme $\overline{\mathcal{S}}_0^\star$ is generically reduced.

Then a theorem of Hironaka [Har, Ch. III, Theorem 9.11] would show that ν is a flat family of schemes, all of whose scheme-theoretic fibres are reduced, irreducible and normal varieties. As $\overline{\mathcal{S}}_0^\star = \overline{\mathcal{S}}_0^{\mathrm{nor}}$ is the normalization of $\overline{\mathcal{S}}_0$ by (1), this would imply that $\overline{\mathcal{S}}^{\mathrm{nor}}$ is a Poisson deformation of $\overline{\mathcal{S}}_0^{\mathrm{nor}}$. So Conjecture B would then follow from Propositions 7.4, 7.6 and a result of Bellamy [Bel2, Theorem 1.4] (which follows works by Ginzburg and Kaledin [GiKa] and Namikawa [Nam1, Nam2]).

9. τ-Harish-Chandra theory of symplectic leaves

Let P be a parabolic subgroup of W and let $\mathfrak{P}$ denote its conjugacy class. Let k_P denote the restriction of k to the hyperplane arrangement of P and let k_P° denote its 'extension by zero' to the hyperplane arrangement of $\mathrm{N}_W(P)$: in other words, if $H \in \mathcal{A}(V, \mathrm{N}_W(P))$ and $0 \leqslant i \leqslant e_H - 1$, we set

$$(k_P^\circ)_{H,i} = \begin{cases} k_{H,i} & \text{if } H \in \mathcal{A}(V, P), \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{X}$ is a locally closed subvariety of V/W, we denote by $\widehat{\mathcal{E}_k(V, W)}_\mathcal{X}$ the scheme equal to the completion of $\mathcal{E}_k(V, W)$ at its locally closed subvariety $(\pi \circ \Upsilon_k)^{-1}(\mathcal{X})$. Note

that it inherits a Poisson structure from that of $\mathscr{E}_k(V, W)$ (see, for instance, [Bel1, Lemma 3.5]).

Our construction of the τ-*Harish-Chandra theory of symplectic leaves* will follow from a forthcoming result of Bellamy and Chalykh [BeCh] which says that there is a natural isomorphism of Poisson schemes

$$\widehat{\mathscr{E}_k(V, W)}_{\mathcal{U}(\mathfrak{P})} \simeq \widehat{\mathscr{E}_{k_P^\circ}(V, \mathrm{N}_W(P))}_{\mathcal{U}(\mathfrak{P})}.$$

As [BeCh] is still not published, we just mention here that it is based on *Bezrukavnikov–Etingof-like* constructions of isomorphisms when completing at a single point of V/W. Note that $\mathcal{U}(\mathfrak{P}) \simeq \mathcal{V}(P)/\mathrm{N}_W(P)$ may be viewed as a locally closed subvariety of both V/W and $V/\mathrm{N}_W(P)$. The construction of this isomorphism implies that it is τ-equivariant.

The sheafified version of Proposition A.4 given in Remark A.6 implies that we can take fixed points under the action of τ in the above isomorphism and get an isomorphism

$$\widehat{\mathscr{E}_k(V, W)^\tau}_{\mathcal{U}(\mathfrak{P})^\tau} \simeq \widehat{\mathscr{E}_{k_P^\circ}(V, \mathrm{N}_W(P))^\tau}_{\mathcal{U}(\mathfrak{P})^\tau}, \tag{9-1}$$

with obvious notation. Moreover, this isomorphism is also Poisson: indeed, the Poisson structure on the left-hand side comes from the Poisson structure on the quotient scheme $(\widehat{\mathscr{E}_k(V, W)}_{\mathcal{U}(\mathfrak{P})})/\langle \tau \rangle$ and one can use Corollary A.3 (and similarly for the right-hand side).

Now, the irreducible (that is connected in this case) components of $\mathcal{U}(\mathfrak{P})^\tau$ have been described in Equation (4-6): this leads to a decomposition of the two schemes involved in isomorphism (9-1). We focus on the irreducible component $\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau))$ of $\mathcal{U}(\mathfrak{P})^\tau$ and get an isomorphism of Poisson schemes

$$\widehat{\mathscr{E}_k(V, W)^\tau}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau))} \simeq \widehat{\mathscr{E}_{k_P^\circ}(V, \mathrm{N}_W(P))^\tau}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau))}. \tag{9-2}$$

A sheafified version of [Bel1, Lemmas 3.3–3.5] provides a natural bijection between Poisson reduced irreducible subschemes of $\mathscr{E}_k(V, W)^\tau$ of dimension $2\dim(V^P)^\tau$ meeting $(\pi \circ \Upsilon_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau)))$ and Poisson reduced irreducible subschemes of $\widehat{\mathscr{E}_k(V, W)^\tau}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau))}$ of dimension $2\dim(V^P)^\tau$. A similar bijection is obtained with the right-hand side of (9-2). Using the isomorphism (9-2), one gets a bijection between the following two sets:

- the set $\mathcal{S}_{\mathfrak{P}_{\mathrm{spl}}^\tau}$ of symplectic leaves of $\mathscr{E}_k(V, W)^\tau$ of dimension $2\dim(V^P)^\tau$ meeting $(\pi \circ \Upsilon_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau)))$;
- the set $\mathcal{S}'_{\mathfrak{P}_{\mathrm{spl}}^\tau}$ of symplectic leaves of $\mathscr{E}_{k_P^\circ}(V, \mathrm{N}_W(P))^\tau$ of dimension $2\dim(V^P)^\tau$ meeting $(\pi' \circ \Upsilon'_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\mathrm{spl}}^\tau)))$.

Here, the maps π' and Υ'_k are the analogues of π and Υ_k for the Calogero–Moser space $\mathscr{E}_{k_P^\circ}(V, \mathrm{N}_W(P))$.

But it follows from Proposition 8.3 that $\mathcal{S}_{\mathfrak{P}^\tau_{\mathrm{spl}}}$ is exactly the set of symplectic leaves $\mathcal{S}$ of $\mathfrak{X}_k(V, W)^\tau$ such that $\mathfrak{P}_{\mathcal{S}} = \mathfrak{P}^\tau_{\mathrm{spl}}$. So Theorem A follows from the next lemma.

LEMMA 9.1. *The set* $\mathcal{S}'_{\mathfrak{P}^\tau_{\mathrm{spl}}}$ *is in natural bijection with the set of* $\mathrm{N}_{W_\tau}(P_\tau)$*-orbits of cuspidal points of* $\mathfrak{X}_{k_P}(V_P, P)^\tau$.

PROOF. Since k_P° is the extension by zero of k_P, we have

$$\mathfrak{X}_{k_P^\circ}(V, \mathrm{N}_W(P)) = \mathfrak{X}_{k_P}(V, P)/\overline{\mathrm{N}}_W(P) = (V^P \times V^{*P} \times \mathfrak{X}_{k_P}(V_P, P))/\overline{\mathrm{N}}_W(P).$$

Consequently,

$$\mathcal{U}(\mathfrak{P}) \times_{V/\mathrm{N}_W(P)} \mathfrak{X}_{k_P^\circ}(V, \mathrm{N}_W(P)) = (\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathfrak{X}_{k_P}(V_P, P)))/\overline{\mathrm{N}}_W(P).$$

As in (4-6) and (7-1), the τ-fixed points of $(\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathfrak{X}_{k_P}(V_P, P)))/\overline{\mathrm{N}}_W(P)$ decompose into pieces indexed by $\tilde{\mathscr{E}}_P \tau/\overline{\mathrm{N}}_W(P)$ as follows:

$$((\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathfrak{X}_{k_P}(V_P, P)))/\overline{\mathrm{N}}_W(P))^\tau$$
$$= \bigcup_{w \in [\tilde{\mathscr{E}}_P \tau/\overline{\mathrm{N}}_W(P)]} (\mathcal{V}(P)^{w\tau} \times (V^{*P})^{w\tau} \times \mathfrak{X}_{k_P}(V_P, P)^{w\tau})/\overline{\mathrm{N}}_W(P)^{w\tau}.$$

Here, $[\tilde{\mathscr{E}}_P \tau/\overline{\mathrm{N}}_W(P)]$ is a set of representatives of $\tilde{\mathscr{E}}_P \tau/\overline{\mathrm{N}}_W(P)$. We may, and do, assume that $1 \in [\tilde{\mathscr{E}}_P \tau/\overline{\mathrm{N}}_W(P)]$. Then, by construction, only the piece indexed by 1 meets $(\pi' \circ \Upsilon'_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}^\tau_{\mathrm{spl}})))$. Therefore, $\mathcal{S}'_{\mathfrak{P}^\tau_{\mathrm{spl}}}$ is in natural bijection with the set of symplectic leaves of

$$\mathfrak{X} = (\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathfrak{X}_{k_P}(V_P, P)^\tau)/\overline{\mathrm{N}}_W(P)^\tau$$

of dimension $2 \dim(V^P)^\tau$. But $\overline{\mathrm{N}}_W(P)^\tau = \overline{\mathrm{N}}_{W_\tau}(P_\tau)$ by Lemma 4.4, and it acts freely on $\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathfrak{X}_{k_P}(V_P, P)^\tau$. So it follows from Corollary 3.8 that the set of symplectic leaves of $\mathfrak{X}$ is in natural bijection with the set of $\overline{\mathrm{N}}_{W_\tau}(P_\tau)$-orbits of symplectic leaves of

$$\mathcal{Y} = \mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathfrak{X}_{k_P}(V_P, P)^\tau.$$

But any symplectic leaf of $\mathcal{Y}$ is of the form $\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathcal{S}$, where $\mathcal{S}$ is a symplectic leaf of $\mathfrak{X}_{k_P}(V_P, P)^\tau$. For dimension reasons, $\mathcal{S}'_{\mathfrak{P}^\tau_{\mathrm{spl}}}$ is in natural bijection with the set of $\overline{\mathrm{N}}_{W_\tau}(P_\tau)$-orbits of symplectic leaves of $\mathfrak{X}_{k_P}(V_P, P)^\tau$ of dimension 0, which is exactly the desired statement. □

10. Examples

10.1. Smooth case. Assume in this subsection, and only in this subsection, that $\mathfrak{X}_k$ is smooth and that τ is of the form ζw for some root of unity ζ and $w \in W$. We denote by d the order of ζ. Then

$$\mathfrak{X}_k^\tau = \mathfrak{X}_k^{\boldsymbol{\mu}_d}.$$

Since $\mathscr{Z}_k$ is smooth, it is symplectic by (5-1), and so $\mathscr{Z}_k^\tau$ is also smooth and symplectic: its symplectic leaves are exactly its irreducible (that is, connected) components.

In [BoMa], Maksimau and the author have described the irreducible components of $\mathscr{Z}_k^{\mu_d}$ as particular Calogero–Moser spaces, and the reader can check that this description is compatible with Conjecture B; however, it is not proved that the isomorphism preserves the Poisson structure. In other words, we have the following result [BoMa, Theorems 2.13 and 5.1].

THEOREM 10.1. *If $\mathscr{Z}_k$ is smooth and $\tau \in \mathbb{C}^\times W$, then Conjecture B holds, possibly up to the Poisson structure.*
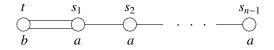
10.2. Type G_4. Thiel and the author [BoTh] have developed algorithms for computing presentations of $\mathbf{Z}_k$ that have been implemented in Magma [Mag] (more precisely, in the Champ package for Magma written by Thiel [Thi]). This allows computations for (very) small groups.

We assume in this subsection, and only in this subsection, that W is the group G_4, in the Shephard–Todd classification [ShTo]. Then a presentation of $\mathbf{Z}_k$ can be obtained with Magma (see, for instance, [BoMa, Section 5] or [BoTh, Theorem 5.2]), and it has been checked in [BoTh, Theorem 4.7] that Conjecture B holds in this case.

THEOREM 10.2. *If $W = G_4$, then Conjecture B holds.*

10.3. Type B. Assume in this subsection, and only in this subsection, that $W = W_n$ is a Coxeter group of type B_n for some $n \geqslant 2$ (that is, we may assume that $W = G(2, 1, n)$ in the Shephard–Todd classification) and that $\tau = \mathrm{Id}_V$. Let $t = \mathrm{diag}(-1, 1, \ldots, 1) \in W_n$ and, for $1 \leqslant j \leqslant n - 1$, let s_j denote the permutation matrix corresponding to the transposition $(j, j + 1)$.

There are two conjugacy classes of reflections: the class of t (which generates an elementary abelian normal subgroup of order 2^n) and that of s_1 (which generates a normal subgroup $W_n' = G(2, 2, n)$ of index 2 isomorphic to a Coxeter group of type D_n). We set $b = c_k(t)$ and $a = c_k(s_1)$ and we denote by I_n the set of $m \in \mathbb{Z}$ such that $|m| \leqslant n - 1$. The Dynkin diagram, together with the values of the parameter function c_k, is given as follows:

The case where $a = 0$ is somewhat uninteresting, as then $\mathscr{Z}_k \simeq (C_b)^n / \mathfrak{S}_n$, where C_b is the Calogero–Moser space associated with the cyclic group of order 2 whose equation is given by $C_b = \{(x, y, z) \in \mathbb{C}^3 \mid z^2 = xy + 4b^2\}$. So we assume throughout this subsection that $a \neq 0$. This implies that

$$\mathscr{Z}_k \text{ is smooth if and only if } b/a \notin I_n.$$

As $\tau = \mathrm{Id}_V$, the smooth case is uninteresting so we assume that $b/a = m \in I_n$. As the cases $b/a = m$ and $b/a = -m$ are equivalent, we also may assume that $m \geqslant 0$. The Calogero–Moser space $\mathfrak{X}_k$ is then denoted by $\mathfrak{X}_{a,ma}(n)$. Symplectic leaves were parametrized by Martino in his PhD Thesis [Mar2, Section 5.4]. Bellamy and Thiel then reinterpreted his result in terms of Bellamy parametrization *à la* Harish-Chandra [BeTh, Lemma 6.5]. This can be summarized as follows.

- $\mathfrak{X}_{a,ma}(n)$ admits a cuspidal point if and only if there exists $r \in \mathbb{Z}_{\geqslant 0}$ such that $n = r(r+m)$; if so, there is only one cuspidal point, which we denote by p_n.
- Therefore, $\mathrm{Cus}_{\mathrm{Id}_V}(\mathfrak{X}_{a,ma}(n)) = \{(W_{r(r+m)}, p_{r(r+m)}) \mid r(r+m) \leqslant n\}$, with the convention that $W_0 = 1$ and $W_1 = \langle t \rangle$.
- If $r(r+m) \leqslant n$, we denote by $\mathcal{S}_r^m(n)$ the symplectic leaf of $\mathfrak{X}_{a,ma}(n)$ associated with $(W_{r(r+m)}, p_{r(r+m)})$ through the bijection of Theorem A (since we are in the case where $\tau = \mathrm{Id}_V$, this bijection was established by Bellamy [Bel1] and Losev [Los]). We have

$$\dim \mathcal{S}_r^m(n) = 2(n - r(r+m)).$$

If $r(r+m) \leqslant n$, then

$$\overline{\mathrm{N}}_{W_n}(W_{r(r+m)}) \simeq W_{n-r(r+m)}.$$

Using the description of $\mathfrak{X}_{a,ma}(n)$ in terms of quiver varieties, Bellamy, Maksimau and Schedler proved the following result [BeSc].

THEOREM 10.3 (Bellamy–Maksimau–Schedler). *If $r(r+m) \leqslant n$, then there is a $\mathbb{C}^\times$-equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}_r^m(n)}^{\mathrm{nor}} \simeq \mathfrak{X}_{a,(m+2r)a}(n - r(r+m)).$$

COROLLARY 10.4. *Conjecture B holds if W is a Coxeter group of type B_n and $\tau = \mathrm{Id}_V$.*

10.4. Type D. Assume in this subsection, and only in this subsection, that $W = W_n'$ is a Coxeter group of type D_n for some $n \geqslant 4$ (that is, we may assume that $W = G(2,2,n)$ in the Shephard–Todd classification). We set $a = c_k(s_1)$, as in the previous subsection, and the Calogero–Moser space $\mathfrak{X}_k$ is denoted by $\mathfrak{X}_a'(n)$. The case where $a = 0$ being treated in Section 7, we assume throughout this subsection that $a \neq 0$. The following facts are proved in [BeTh, Theorem 7.2]. Note that there is a little mistake in [BeTh, Theorem 7.2], which can be easily corrected to give the statement written here. More precisely, and keeping the notation of [BeTh, Sections 6 and 7], the statement of [BeTh, Lemma 7.1] is false for $k = 1$ (but true for $k = 0$) because [BeTh, Theorem 6.24] cannot be applied to the case $(k,m) = (1,0)$ for going from type B_1 to $D_1 = B_1$ (!). So the correct statement of [BeTh, Theorem 7.2] is to replace the set $\{k \geqslant 1 \mid k^2 \leqslant n\}$ by the set $\{0\} \cup \{k \geqslant 2 \mid k^2 \leqslant n\}$: then, for instance, $\mathcal{L}_0$ is the smooth symplectic leaf.

- $\mathfrak{X}_a'(n)$ admits a cuspidal point if and only if there exists $r \in \mathbb{Z}_{\geqslant 0} \setminus \{1\}$ such that $n = r^2$; if so, there is only one cuspidal point, which we denote by p_n'.

- Therefore, $\operatorname{Cus}_{\operatorname{Id}_V}(\mathcal{Z}'_a(n)) = \{(W'_{r^2}, p'_{r^2}) \mid 0 \leqslant r^2 \leqslant n \text{ and } r \neq 1\}$, with the convention that $W'_0 = 1$.
- If $0 \leqslant r^2 \leqslant n$ and $r \neq 1$, we denote by $\mathcal{S}'_r(n)$ the symplectic leaf of $\mathcal{Z}'_a(n)$ associated with (W'_{r^2}, p'_{r^2}). We have

$$\dim \mathcal{S}'_r(n) = 2(n - r^2).$$

Let us give another description, coming from the link between $\mathcal{Z}'_a(n)$ and the Calogero–Moser space $\mathcal{Z}_{a,ma}(n)$ of type B_n defined in the previous subsection for the special value $m = 0$. Indeed, $\mathcal{Z}'_a(n)$ admits an action of the element $t \in W_n \subset \mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W'_n)$ defined in the previous subsection and [BeTh, Proposition 4.17]

$$\mathcal{Z}'_a(n)/\langle t \rangle \simeq \mathcal{Z}_{a,0}(n),$$

as Poisson varieties endowed with a $\mathbb{C}^\times$-action. Denote by $\gamma_n : \mathcal{Z}'_a(n) \to \mathcal{Z}_{a,0}(n)$ the quotient morphism.

PROPOSITION 10.5. *We have*

$$\mathcal{S}'_0(n) = \gamma_n^{-1}(\mathcal{S}^0_0(n) \cup \mathcal{S}^0_1(n)) \quad and \quad \mathcal{S}'_r(n) = \gamma_n^{-1}(\mathcal{S}^0_r(n))$$

for all $r \geqslant 2$ such that $r^2 \leqslant n$.

PROOF. The symplectic leaves of $\mathcal{Z}'_a(n)$ are characterized by their dimension, so every symplectic leaf is t-stable (so is the inverse image, under γ_n, of its image in $\mathcal{Z}_{a,0}(n)$). But if $0 \leqslant r^2 \leqslant n$ and $r \neq 1$, then $\gamma_n(\overline{\mathcal{S}'_r(n)})$ is a closed irreducible Poisson subvariety of $\mathcal{Z}_{a,0}(n)$, so it is the closure of a symplectic leaf. For dimension reasons, it must be equal to $\overline{\mathcal{S}^0_r(n)}$. The result follows. $\square$

COROLLARY 10.6. *We have*

$$\mathcal{Z}'_a(n)^t = \gamma_n^{-1}(\overline{\mathcal{S}^0_1(n)}).$$

In particular, if $4 \leqslant r^2 \leqslant n$, then t acts trivially on $\mathcal{S}'_r(n)$.

PROOF. First, t does not act trivially on $\mathcal{Z}'_a(n)$, so it does not act trivially on the open leaf $\mathcal{S}'_0(n)$. Since $\mathcal{S}'_0(n)$ is smooth and symplectic, the description of the symplectic leaves of $\mathcal{S}'_0(n)/\langle t \rangle$ is given by Proposition 3.6. But $\mathcal{S}'_0(n)/\langle t \rangle = \mathcal{S}^0_0(n) \cup \mathcal{S}^0_1(n)$ by Proposition 10.5. Comparing both descriptions shows that t acts freely on $\mathcal{S}^0_0(n)$ and trivially on $\mathcal{S}^0_1(n)$.

Therefore, t acts trivially on the closure of $\gamma_n^{-1}(\mathcal{S}^0_1(n))$ and freely on $\mathcal{S}^0_0(n)$. But the closure of $\mathcal{S}^0_1(n)$ is the union of the $\mathcal{S}^0_r(n)$ for $r \geqslant 1$ (see [BeTh, Lemma 6.5]). So the corollary now follows directly from Proposition 10.5. $\square$

Assume now that $4 \leqslant r^2 \leqslant n$. Then Corollary 10.6 shows that $\mathcal{S}'_r(n) \simeq \mathcal{S}^0_r(n)$. Moreover, $\overline{\mathrm{N}}_{W'_n}(W'_{r^2}) \simeq W_{n-r^2}$. So Theorem 10.3 shows the following result.

COROLLARY 10.7. *Conjecture B holds if W is a Coxeter group of type D_n and $\tau \in \{\operatorname{Id}_V, t\}$.*

PROOF. For the case $\tau = \mathrm{Id}_V$, the work has already been done. For the case where $\tau = t$, one must observe that τ is W'_n-regular and thus W'_n-full (see Example 2.2), that $(W'_n)_\tau = (W'_n)^\tau \simeq W_{n-1}$ (see Example 4.2), and that

$$\overline{\mathrm{N}}_{(W'_n)_\tau}((W'_{r^2})_\tau) = \overline{\mathrm{N}}_{W_{n-1}}(W_{r^2-1}) \simeq W_{n-r^2}.$$

Then the result follows from Theorem 10.3 and Corollary 10.6.   □

10.5. Dihedral groups at equal parameters. Let d be a natural number and let ξ denote a primitive $2d$ th root of unity. For $j \in \mathbb{Z}/2d\mathbb{Z}$, we set

$$s_j = \begin{pmatrix} 0 & \xi^j \\ \xi^{-j} & 0 \end{pmatrix}.$$

We assume in this subsection, and only in this subsection, that $W = \langle s_0, s_2 \rangle$ is dihedral of order $2d$ and that $\tau = s_1$; note that $\tau^2 = \mathrm{Id}_V$, that $\tau s_0 \tau^{-1} = s_2$, that $\tau s_2 \tau^{-1} = s_0$ and that τ is W-full. We set $a = c_k(s_0)$ and, since k is τ-stable by hypothesis, we have $c_k(s_2) = a$. In other words, we are in the equal parameter case studied by the author in [Bon1]. Moreover, in [Bon1, Section 4], the author determined the structure of $\mathfrak{Z}_k^\tau$. This gives the following proposition.

PROPOSITION 10.8. *If W is dihedral of order $2d$ and if τ is as above, then Conjecture B holds.*

Acknowledgements

Appendix A. Completion and finite group actions

HYPOTHESIS AND NOTATION. In this appendix we fix a commutative noetherian $\mathbb{C}$-algebra R, an ideal I of R and a finite group G acting on the $\mathbb{C}$-algebra R, and we assume that I is G-stable. We set $J = \langle I^G \rangle_R$.

Let $\mathfrak{r}_G$ be the ideal of R generated by the family $(r - g(r))_{r \in R, g \in G}$ and set $R(G) = R/\sqrt{\mathfrak{r}_G}$. We denote by $I(G)$ the image of I in $R(G)$. Note that $R(G)$ is the biggest quotient algebra of R that is reduced and on which G acts trivially.

Finally, we denote by $\hat{R}_I$ the I-adic completion of R, that is,

$$\hat{R}_I = \varprojlim_j R/I^j,$$

and by $\iota : R \to \hat{R}_I$ the canonical map.

The results of this appendix do not pretend to any originality, and might certainly be written in greater generality. We nevertheless cannot find appropriate references containing all of them, and decided to state them in terms that are suitable for our purposes.

LEMMA A.1. *There exists an integer m such that $I^m \subset J$.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of R containing I^G. We first wish to prove that $\mathfrak{p}$ contains I. For this purpose, let $r \in I$. Then $\prod_{g \in G} g(r) \in I^G$, and so there exists $g_r \in G$ such that $g_r(r) \in \mathfrak{p}$ because $\mathfrak{p}$ is prime. This shows that $I \subset \bigcup_{g \in G} g(\mathfrak{p})$. By the prime avoidance lemma, we get that there exists $g \in G$ such that $I \subset g(\mathfrak{p})$. Since, moreover, I is G-stable, we get that $I \subset \mathfrak{p}$. In other words, I is contained in any prime ideal containing J. So $I \subset \sqrt{J}$. As R is noetherian, the result follows from Levitsky's theorem [Lam, Theorem 10.30]. $\qquad\square$

LEMMA A.2. *Let $j \geqslant 0$. Then $(J^j)^G = (I^G)^j$.*

PROOF. The inclusion $(I^G)^j \subset (J^j)^G$ is obvious. Conversely, let $r \in (J^j)^G$. Then there exist a finite set E, a family $(r_e)_{e \in E}$ of elements of R and a family $(i_e^{(1)}, \ldots, i_e^{(j)})$ of j-tuples of elements of I^G such that

$$r = \sum_{e \in E} r_e i_e^{(1)} \cdots i_e^{(j)}.$$

Since r is G-invariant, we have $r = (1/|G|) \sum_{g \in G} g(r)$, so

$$r = \sum_{e \in E} \left(\frac{1}{|G|} \sum_{g \in G} g(r_e) \right) i_e^{(1)} \cdots i_e^{(j)}.$$

Hence, $r \in (I^G)^j$. $\qquad\square$

Since I and J are G-stable, the completions $\hat{R}_I$ and $\hat{R}_J$ inherit a G-action.

COROLLARY A.3. *The $\mathbb{C}$-algebras $(\hat{R}_I)^G$ and $\widehat{(R^G)}_{I^G}$ are canonically isomorphic.*

PROOF. As $I^m \subset J \subset I$ for some m by Lemma A.1, the completions $\hat{R}_I$ and $\hat{R}_J$ are canonically isomorphic, and the isomorphism is G-equivariant. This gives an isomorphism $(\hat{R}_I)^G \simeq (\hat{R}_J)^G$. So the result follows directly from Lemma A.2, because $(R/J^j)^G = R^G/(J^j)^G$ since we work in characteristic 0. $\qquad\square$

PROPOSITION A.4. *Assume that R is Nagata. Then*

$$\widehat{R(G)}_I = \hat{R}_I(G).$$

PROOF. Let $\hat{\mathfrak{r}}_G$ denote the completion of $\mathfrak{r}_G$ at I. Since R is noetherian, $\hat{\mathfrak{r}}_G$ is the ideal of $\hat{R}_I$ generated by $\mathfrak{r}_G$ and

$$(\widehat{R/\mathfrak{r}_G})_I = \hat{R}_I/\hat{\mathfrak{r}}_G$$

(see, for instance, [GrSa, Section 4]). This shows that G acts trivially on $\hat{R}_I/\hat{\mathfrak{r}}_G$ and so $\hat{\mathfrak{r}}_G$ is the ideal of $\hat{R}_I$ generated by $(g(r) - r)_{r \in \hat{R}_I, g \in G}$.

Moreover, as R is Nagata, we have that $\sqrt{\hat{\mathfrak{r}}_G} = \widehat{\sqrt{\mathfrak{r}_G}}$ by [GrSa, Corollary 14.8]. The proposition follows.   □

EXAMPLE A.5. Assume that R is a localization or a completion of a finitely generated algebra. Then R is Nagata.

REMARK A.6. Let $\mathfrak{X}$ be a quasi-projective variety acted on by the finite group G and let $\mathcal{U}$ be a G-stable locally closed subvariety. Then we have an isomorphism of formal schemes

$$(\hat{\mathfrak{X}}_{\mathcal{U}})^G \simeq \widehat{(\mathfrak{X}^G)}_{\mathcal{U}^G}. \tag{A-1}$$

Indeed, by replacing $\mathfrak{X}$ by the G-stable open subset $\mathfrak{X} \setminus (\overline{\mathcal{U}} \setminus \mathcal{U})$, we may assume that $\mathcal{U}$ is closed in $\mathfrak{X}$ and we denote by $\mathcal{F}$ its sheaf of ideals in $\mathcal{O}_{\mathfrak{X}}$. The underlying topological space of both sides of (A-1) is equal to $\mathcal{U}^G$ and there is a natural morphism of sheaves of algebras

$$\mathcal{O}_{\widehat{(\mathfrak{X}^G)}_{\mathcal{U}^G}} \longrightarrow \mathcal{O}_{(\hat{\mathfrak{X}}_{\mathcal{U}})^G}.$$

We need to prove that it is an isomorphism and this can be checked locally. Since $\mathfrak{X}$ is quasi-projective, it can be covered by G-stable open affine subsets, and for each of these open affine subsets the expected isomorphism follows from Proposition A.4.

References

[BeCh]   G. Bellamy and O. Chalykh, 'Motivic invariants of Calogero–Moser spaces', in preparation.

[Bel1]   G. Bellamy, 'Cuspidal representations of rational Cherednik algebras at $t = 0$', *Math. Z.* **269** (2011), 609–627.

[Bel2]   G. Bellamy, 'Counting resolutions of symplectic quotient singularities', *Compos. Math.* **152** (2016), 99–114.

[BeSc]   G. Bellamy, R. Maksimau and T. Schedler, in preparation.

[BeTh]   G. Bellamy and U. Thiel, 'Cuspidal Calogero–Moser and Lusztig families for Coxeter groups', *J. Algebra* **462** (2016), 197–252.

[BMM]   M. Broué, G. Malle and J. Michel, 'Generic blocks of finite reductive groups', in: *Représentations Unipotentes Génériques et Blocs des Groupes Réductifs Finis*, Astérisque, 212 (1993), 7–92.

[BoMa]   C. Bonnafé and R. Maksimau, 'Fixed points in smooth Calogero–Moser spaces', *Ann. Inst. Fourier (Grenoble)* **71** (2021), 643–678.

[Bon1]   C. Bonnafé, 'On the Calogero–Moser space associated with dihedral groups II: The equal parameter case', Preprint, 2022, arXiv:2112.12401.

[Bon2]   C. Bonnafé, 'Calogero–Moser spaces vs unipotent representations', *Pure Appl. Math. Q.*, to appear, Preprint, 2021, arXiv:2112.13684.

[BoRo]   C. Bonnafé and R. Rouquier, 'Cherednik algebras and Calogero–Moser cells', Preprint, 2022, arXiv:1708.09764.

[BoTh]   C. Bonnafé and U. Thiel, 'Computational aspects of Calogero–Moser spaces', Preprint, 2022, arXiv:2112.15495.

[BrGo]   K. Brown and I. Gordon, 'Poisson orders, symplectic reflection algebras and representation theory', *J. reine angew. Math.* **559** (2003), 193–216.

[Dix]   J. Dixmier, *Enveloping Algebras*, North-Holland Mathematical Library, 14 (North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977), xvi + 375 pages.

[EtGi]   P. Etingof and V. Ginzburg, 'Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism', *Invent. Math.* **147** (2002), 243–348.

[GiKa]   V. Ginzburg and D. Kaledin, 'Poisson deformations of symplectic quotient singularities', *Adv. Math.* **186** (2004), 1–57.

[Gor]   I. Gordon, 'Baby Verma modules for rational Cherednik algebras', *Bull. Lond. Math. Soc.* **35** (2003), 321–336.

[GrSa]   S. Greco and P. Salmon, *Topics in m-adic Topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge, 58 (Springer-Verlag, Berlin, 1971), vii + 74 pages.

[Har]   R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52 (Springer-Verlag, New York, 1977), xvi + 496 pages.

[Kal]   D. Kaledin, 'Normalization of a Poisson algebra is Poisson', *Proc. Steklov Inst. Math.* **264** (2009), 70–73.

[Lam]   T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, 131 (Springer-Verlag, New York, 1991), xvi + 397 pages.

[LeSp]   G. I. Lehrer and T. A. Springer, 'Intersection multiplicities and reflection subquotients of unitary reflection groups I', in: *Geometric Group Theory Down Under (Canberra, 1996)* (de Gruyter, Berlin, 1999), 181–193.

[Los]   I. Losev, 'Completions of symplectic reflection algebras', *Selecta Math. (N.S.)* **18** (2012), 179–251.

[Mag]   W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system. I. The user language', *J. Symbolic Comput.* **24** (1997), 235–265.

[Mar1]   M. Martino, 'The associated variety of a Poisson prime ideal', *J. Lond. Math. Soc. (2)* **72** (2005), 110–120.

[Mar2]   M. Martino, 'Symplectic reflection algebras and Poisson geometry', PhD Thesis, University of Glasgow, 2006.

[Nam1]   Y. Namikawa, 'Poisson deformations of affine symplectic varieties', *Duke Math. J.* **156** (2011), 51–85.

[Nam2]   Y. Namikawa, 'Poisson deformations of affine symplectic varieties, II', *Kyoto J. Math.* **50** (2010), 727–752.

[ShTo]   G. C. Shephard and J. A. Todd, 'Finite unitary reflection groups', *Canad. J. Math.* **6** (1954), 274–304.

[Spr]   T. A. Springer, 'Regular elements of finite reflection groups', *Invent. Math.* **25** (1974), 159–198.

[Thi]   U. Thiel, 'Champ: a Cherednik algebra Magma package', *LMS J. Comput. Math.* **18** (2015), 266–307.

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