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MATHEMATICAL NOTES

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ON THE GENERATING FUNCTION FOR PERMUTATIONS WITH REPETITIONS AND INVERSIONS

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Let $(a_1, a_2, \ldots, a_m)$, $a_i \in \{1, 2, \ldots, n\}$, be an m-permutation of n (repetitions allowed) with exactly k_j of the a's equal to j, $j=1, 2, \ldots, n$, $m=k_1+\cdots+k_n$, $k_1, \ldots, k_n$ fixed nonnegative integers. An inversion is a pair i, j such that $i<j$, $a_i>a_j$. Denote by $N(r; k_1, \ldots, k_n)$ the number of such permutations with exactly r inversions. In the case $k_1=k_2=\cdots=k_n=1$, $m=n$, then $N(r; 1, 1, \ldots, 1)$, denoted by $N(r, n)$, is the number of permutations (without repetition) of $1, 2, \ldots, n$ with exactly r inversions. D. Z. Djokovic [2], using a brief argument, has shown that

$$(1) \qquad \sum_{r=0}^{\frac{1}{2}(n)(n-1)} N(r, n)x^r = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$

$$= \frac{(1-x)(1-x^2)\cdots(1-x^n)}{(1-x)^n}.$$

Recently L. Carlitz [1] has established the more general result

$$(2) \qquad \sum_{r=0}^{M} N(r; k_1, \ldots, k_n)x^r = \frac{[k_1+\cdots+k_n]!}{[k_1]![k_2]!\ldots[k_n]!},$$

where

$$M = \sum_{1\le s<t\le n} k_s k_t, \qquad [k]! = (1-x)(1-x^2)\cdots(1-x^k), \qquad [0]! = 1.$$

The purpose of this note is to give an alternate derivation of (2) by showing that (2) follows from (1) directly by using a simple combinatorial argument.

It is easy to see that

$$(3) \qquad \sum_{r_1+\cdots+r_n+r=u} N(r_1, k_1)\cdots N(r_n, k_n)N(r; k_1, \ldots, k_n) = N(u, k_1+\cdots+k_n).$$

For suppose $S=(s_1, s_2, \ldots, s_{k_1+\cdots+k_n})$ is a particular sequence counted in $N(r; k_1, \ldots, k_n)$. Replace the k_1 1's in S (from left to right) by $a_1, \ldots, a_{k_1}$, $a_i<a_j$,

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$i \neq j$. Replace the 2's in S by $b_1, \ldots, b_{k_2}$ with $a_{k_1} < b_1, b_i < b_j, i \neq j$. Continue until the n's are replaced. Permute the a's among themselves in $N(r_1, k_1)$ ways, the b's in $N(r_2, k_2)$ ways, etc. Clearly each sequence counted in $N(r_1, k_1) \cdots N(r_n, k_n) \times N(r; k_1, \ldots, k_n)$ is a sequence of $k_1 + \cdots + k_n$ distinct ordered objects with exactly $r_1 + r_2 + \cdots + r_n + r$ inversions. Therefore the left side of (3) is $\leq$ the right side. Using the converse of the above argument the right side of (3) is $\leq$ the left side. Hence

$$\sum_{u=0} N(u, k_1 + \cdots + k_n) x^u$$
$$= \sum_{u=0} \left(\sum_{r_1 + \cdots + r_n + r = u} N(r_1, k_1) \cdots N(r_n, k_n) N(r; k_1, \ldots, k_n) \right) x^u$$
$$= \left(\sum_{r_1=0} N(r_1, k_1) x^{r_1} \right) \cdots \left(\sum_{r_n=0} N(r_n, k_n) x^{r_n} \right) \left(\sum_{r=0} N(r; k_1, \ldots, k_n) x^r \right),$$

and using (1), (2) follows. Also it is clear that

$$\sum_{m=0} \left(\sum_{k_1 + \cdots + k_n = m} \frac{[k_1 + \cdots + k_n]!}{[k_1]! \ldots [k_n]!} \right) \frac{z^m}{[m]!} = \left(\sum_{k=0} \frac{z^k}{[k]!} \right)^n,$$

and hence from (2),

$$\sum_{m=0} \sum_{r=0} \sum_{k_1 + \cdots + k_n = m} N(r; k_1, \ldots, k_n) x^r \frac{z^m}{[m]!} = \left(\sum_{k=0} \frac{z^k}{[k]!} \right)^n,$$

where $\sum_{k_1 + \cdots + k_n = m} N(r; k_1, \ldots, k_n)$ is the total number of m-permutations of $1, 2, \ldots, n$, repetitions allowed, with exactly r inversions, in agreement with the last expression in [1].

References

1. L. Carlitz, *Sequences and inversions*, Duke Math. J. **37** (1970), 193–198.
2. D. Z. Djokovic, *Solution to Aufgabe 558*, *Elemente der Mathematik*, **23** (1968), p. 114; Proposer, Heinz Lüneburg, Elemente der Mathematik **22** (1967).

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