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EXTENSIONS OF CONTINUOUS FUNCTIONS: REFLECTIVE FUNCTORS

W.N. HUNSAKER AND S.A. NAIMPALLY

The purpose of this paper is to develop a general technique for attacking problems involving extensions of continuous functions from dense subspaces and to use it to obtain new results as well as to improve some of the known ones. The theory of structures developed by Harris is used to get some general results relating filters and covers. A necessary condition is derived for a continuous function $f: X \to Y$ to have a continuous extension $\overline{f}: \lambda X \to \lambda Y$ where λZ denotes a given extension of the space Z. In the case of simple extensions, $\overline{f}$ is continuous and in the case of strict extensions $\overline{f}$ is θ-continuous. In the case of strict extensions, sufficient conditions for uniqueness of $\overline{f}$ are derived. These results are then applied to several extensions considered by Banaschewski, Fomin, Katětov, Liu-Strecker, Blaszczyk-Mioduszewski, Rudolf, etc.

1. Introduction

Several mathematicians have studied problems concerning extensions of continuous functions from dense subspaces. Much of the research

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has been centred around the problem of characterising known extensions as (epi-)reflections: [4,5,12-20,25]. In addition to this the problem of θ-continuous extensions of continuous functions has been considered in [28] including the problem of uniqueness. Harris [12] developed a general theory of extensions using "structures" whereas using proximity or nearness, the well-known Taimanov's theorem has been generalised in [6,7, 11,19,26]. The main purpose of this paper is to develop a general technique for attacking problems involving extensions of continuous functions and to use it to obtain new results as well as improve some of the known ones.

In Section 2 we use techniques of Harris [12-15] to get some general results relating to filters and covers. Suppose, with each member Z of a given class of topological spaces, we associate an extension λZ. We derive a necessary condition that a continuous function $f: X \to Y$ has a continuous extension $\bar{f}: \lambda X \to \lambda Y$. This condition enables us to describe $\bar{f}$. In Section 3 and 4 we show that $\bar{f}$ is continuous in the case of simple extensions and θ-continuous in the case of strict extensions. Special cases are the results in [13] and improvements of these in [24] A new result concerning the α-closure of Liu [23] as a reflection is obtained. In the case of strict extensions we get sufficient conditions only for a unique θ-continuous extension $\bar{f}$ and our results include these in [2,9] for suitable perfect maps.

In Section 5 we generalise some theorems in [25] to certain simple extensions. Finally, in Section 6 we use the method of proximities to obtain the van der Slot realcompactification [29].

We have given a fairly representative bibliography wherein the interested reader will find further references to the topics discussed here.

2. Preliminaries

The motivation for the method outlined in this section is Harris' treatment of d-maps [12] and Fan and Gottesman's treatment of Freudenthal compactification [8]. Thus by specialising the general theory of Harris to a family satisfying some reasonable conditions, we get a leverage to get several old and new results effortlessly. Since the proofs can be patterned after those in [8] and [12], we give an outline in this section, omitting most of the proofs.

Let T denote a family of Hausdorff spaces. We assume that each X in T has an open base $B(X)$ for its topology and that $B(X)$ satisfies:

(2.1) (i) If A, B in $B(X)$, then $A \cap B \in B(X)$,

(ii) If $A \in B(X)$, then $(X-A)^{\circ} \in B(X)$

(see Fan and Gottesman [8])

Any cover of a space X in T which is formed from the members of $B(X)$ is called a <u>$B(X)$-cover</u>. A $B(X)$-cover of X which contains a finite subfamily whose union is dense in X is called a <u>λ-cover</u>. A <u>$B(X)$-filter</u> is a filter formed from the members of $B(X)$. We let X^{λ} denote the collection of all nonconvergent maximal $B(X)$-filters, and let

$$\lambda X = \{N(x) : x \in X\} \cup X^{\lambda}$$

where

$$N(x) = \{E \in B(X) : x \in E\} .$$

(In some cases such as the α-closure αX [23] or almost real compactification ρX [24], we have to consider certain subsets of X^{λ}. The necessary modifications are explained below.)

We recall two known methods of assigning topologies to λX.

(2.2) The <u>simple extension topology</u> is described by assigning to each $F \in \lambda X$, the neighbourhood base $\{V_* : V \in F\}$ where

$$V_* = \{F\} \cup \{N(x) : x \in V\} .$$

One of the most famous examples of this type is the Katětov extension τX [22], where $B(X)$ is the family of <u>all</u> open sets in X. Liu [23] considered the subspace αX of τX consisting of all F with the countable intersection property (c.i.p.) and Liu and Strecker [24] the subspace ρX of τX consisting of all F with the countable closure intersection property (c.c.i.p.).

(2.3) The strict extension topology has a base for the open sets consisting of all sets of the form

$$V^* = \{\mathcal{F} \in \lambda X : V \in \mathcal{F}\}.$$

Examples of this type abound:

(i) If $\mathcal{B}(X)$ consists of all open sets in X, then $\lambda X = FX$, the Fomin extension of X [9].

(ii) If X is semi-regular and $\mathcal{B}(X)$ consists of all regular-open subsets of X, then λX is the Banaschewski T_2-minimal extension σX [2].

(iii) We may consider extensions $\alpha_1 X$, $\rho_1 X$ (corresponding to αX, ρX in (2.2) above) as subspaces of FX and also $\alpha_2 X$, $\rho_2 X$ as subspaces of σX.

If $\mathcal{F}$ is a $\mathcal{B}(X)$-filter on X and $U \subset X$, we write $U \cap \mathcal{F} \neq \emptyset$ if $U \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and write $U \cap \mathcal{F} = \emptyset$ otherwise. We now state without proof the basic results:

(2.4) LEMMA: *A nonconvergent $\mathcal{B}(X)$-filter $\mathcal{F}$ is maximal if and only if for each $U \in \mathcal{B}(X)$, either $U \in \mathcal{F}$ or $(X-U)^\circ \in \mathcal{F}$.*

(2.5) LEMMA: *A nonconvergent $\mathcal{B}(X)$-filter $\mathcal{F}$ is maximal if and only if $\mathcal{F} \cap \alpha \neq \emptyset$ for each λ-cover α of X.*

(2.6) LEMMA: *A $\mathcal{B}(X)$-cover α is a λ-cover if and only if $\alpha \cap \mathcal{F} \neq \emptyset$ for each $\mathcal{F} \in X^\lambda$.*

(2.7) LEMMA: *Let f be a continuous function from X to Y. The following are equivalent:*

(a) For each λ-cover α of Y, $f^{-1}(\alpha)$ has a λ-refinement in X,

(b) For each λ-cover α of Y and each $\mathcal{F} \in X^\lambda$, there exists $U \in \alpha$, $V \in \mathcal{F}$ such that $f(V) \subset U$.

(2.8) DEFINITION. A continuous map f satisfying (2.7) (a) or (b) is called a *λ-map*. (In the case of τX such maps are called p-maps and in

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the case of σX they are called m-maps.)

(2.9) THEOREM. *A necessary condition that a continuous function* $f: X \to Y$ *has a continuous extension* $\bar{f}: \lambda X \to \lambda Y$ *(with simple or strict extension topologies) is that* f *is a* λ*-map.*

Naturally questions arise:

(i) does a λ-map f have an extension $\bar{f}: \lambda X \to \lambda Y$?

(ii) if $\bar{f}$ exists, is it continuous?

We show here that the first question has an answer and defer the second one to the next two sections.

(2.10) LEMMA. *If* $f: X \to Y$ *is a* λ*-map, then for each* $\mathcal{F} \in X^\lambda$,

$$f^\circ(\mathcal{F}) = \{\mathcal{U} \in \mathcal{B}(Y) : f^{-1}(\mathcal{U}) \in \mathcal{F}\}$$

either converges or belongs to Y^λ .

(2.11) COROLLARY. *Every* λ*-map* $f: X \to Y$ *has an extension* $\bar{f}: \lambda X \to \lambda Y$ *defined by*

$$\bar{f}(\mathcal{F}) = \begin{cases} N(f(x)) & \text{if} \quad \mathcal{F} = N(x) \\ N(y_0) & \text{if} \quad f^\circ(\mathcal{F}) \to y_0 \\ f^\circ(\mathcal{F}) & \text{otherwise.} \end{cases}$$

3. Simple extensions

If λX, λY are simple extensions, then the map defined by (2.11) is continuous. To see this, let $\mathcal{U}_*$ be a neighbourhood of $\bar{f}(\mathcal{F})$, where $\mathcal{U} \in \bar{f}(\mathcal{F})$. There is a $\mathcal{B}(Y)$-cover α of Y such that

$$\alpha \cap \bar{f}(\mathcal{F}) = \begin{cases} \{\mathcal{U}\} & \text{if} \quad \bar{f}(\mathcal{F}) = N(y_0),\ y_0 \in Y \\ \emptyset & \text{if} \quad \bar{f}(\mathcal{F}) \in Y^\lambda . \end{cases}$$

Then $\beta = \alpha \cap \{\mathcal{U}, (X-\mathcal{U})^\circ\}$ is a λ-cover of Y and since f is a λ-map, there exist $V \in \mathcal{F}$, $W \in \beta$ such that $f(V) \subset W$. Clearly this means that

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$W \in \bar{f}(F)$ and so $W = U$. Obviously $\bar{f}(V_*) \subset U_*$ and so $\bar{f}$ is continuous, and hence is the unique continuous extension of f.

Thus we have using (2.9),

(3.1) THEOREM. *A necessary and sufficient condition that* $f: X \to Y$ *have a continuous extension* $\bar{f}: \lambda X \to \lambda Y$ *(with simple extension topologies) is that* f *is a* λ*-map. (see Harris* [12,13]).

In categorical terminology ([16], [20]) (3.1) takes the form:

(3.2) THEOREM. *Let* C *be the category whose objects are Hausdorff spaces with bases satisfying (2.1), and whose morphisms are* λ*-maps. Let* $\mathcal{D}$ *be the full subcategory of* C *whose objects are the simple extensions* λX *of objects in* C *and whose morphisms are continuous functions.* $\mathcal{D}$ *is an epireflective subcategory of* C, *and for each object* X *in* C, λX *is the epireflection.*

We now consider the α-closure αX of X (Liu [23]). An open cover ξ of X is an α-cover if and only if there is a countable subcollection $\{U_n\}$ of ξ satisfying $X = \cup \bar{U}_n$. We also write $X^\alpha = \alpha X - \{N(x) : x \in X\}$. The following lemma characterises X^α; we omit the proof.

(3.3) LEMMA. *A nonconvergent open filter* F *with the finite intersection property is maximal if and only if* $F \cap \xi \neq \emptyset$ *for every* α*-cover* ξ *of* X.

(3.4) DEFINITION. *(see Lemma (2.7)). A continuous function* $f: X \to Y$ *is an* $\alpha(b)$*-map if and only if for each* α*-cover* ξ *of* Y *and* $F \in X^\alpha$, *there exist* $U \in \xi$, $V \in F$ *such that* $f(V) \subset U$. *It is called an* <u>$\alpha(a)$*-map*</u> *if and only if for each* α*-cover* ξ *of* Y, $f^{-1}(\xi)$ *has an* α*-refinement.*

The following is proved similar to (2.7).

(3.5) LEMMA. *Every* $\alpha(a)$*-map is an* $\alpha(b)$*-map.*

(3.6) THEOREM. *A necessary and sufficient condition that a function* $f: X \to Y$ *has a continuous extension* $\bar{f}: \alpha X \to \alpha Y$ *is that* f *is an* $\alpha(b)$*-map.*

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Proof. The unique continuous extension $\overline{f}$ of f is the function defined in (2.11). The proof of the converse follows as in Theorem (2.9).

Since $\alpha(a)$-maps are closed under composition, we get the following:

(3.7) THEOREM. *The category of α-closed spaces and continuous functions is an epireflective subcategory of the category of Hausdorff spaces and $\alpha(a)$-maps. For each Hausdorff space X, αX is the epireflection.*

A p-map or equivalently τ-proper map [5] $f: X \to Y$ has a unique continuous extension $\overline{f}: \tau X \to \tau Y$ given by (2.11) and it is not hard to show that $\overline{f}(\alpha X) \subset \alpha Y$. Thus we obtain another epireflective subcategory in (3.7) if we replace $\alpha(a)$-maps by p-maps.

We now consider the Liu-Strecker almost realcompactification ρX which is a subspace of τX consisting of all F in τX which have the c.c.i.p. We write $X^{\rho} = \rho X - \{N(x): x \in X\}$.

(3.8) DEFINITION. An open cover γ of X is called a *ρ-cover* if and only if γ is either a p-cover or γ has a countable subcover.

The following is an analogue of Lemma (2.5) and (3.3).

(3.9) LEMMA. *A nonconvergent open filter F is maximal and has c.c.i.p. if and only if for each ρ-cover γ of X , $\gamma \cap F \neq \emptyset$.*

Proof. Suppose $F \cap \gamma \neq \emptyset$ for each ρ-cover γ . Since every p-cover is a ρ-cover, it follows from (2.5) that $F \in X^{\tau}$. Also from Frolik [10, Lemma 1] if $F \in X^{\tau}$, then $F \in X^{\rho}$ if and only if $F \cap \gamma \neq \emptyset$ for every open cover γ which has a countable subcover. The converse follows from Lemma (2.5) and the above quoted result of Frolik.

(3.10) DEFINITION. A continuous function $f: X \to Y$ is a *$\rho(b)$-map* if and only if for each ρ-cover γ of Y and each $F \in X^{\rho}$, there is $U \in \gamma$, $V \in F$ such that $f(V) \subset U$. It is a *$\rho(a)$-map* if and only if for each ρ-cover γ of Y, $f^{-1}(\gamma)$ has a ρ-refinement.

The following results are analogues of (3.5), (3.6), (3.7).

(3.11) LEMMA. *Every $\rho(a)$-map is a $\rho(b)$-map.*

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(3.12) THEOREM. *A necessary and sufficient condition that a continuous function* $f: X \to Y$ *has a continuous extension* $\overline{f}: \rho X \to \rho Y$ *is that* f *is a* $\rho(b)$*-map.*

(3.13) THEOREM. *The category of almost realcompact spaces and continuous functions is an epireflective subcategory of the category of Hausdorff spaces and* $\rho(a)$*-maps. The epireflection of each Hausdorff space is* ρX .

As was remarked after Theorem (3.7), if f is a p-map or τ-proper map from X to Y, then f has a continuous extension $\overline{f}: \tau X \to \tau Y$. In fact, $\overline{f}(\rho X) \subset \rho Y$. For suppose $F \in X^{\rho}$ and suppose $\{U_n\}$ is a countable subcollection of $\overline{f}(F)$. $f^{-1}(U_n) \in F$ for each n and so $\cap \{f^{-1}(U_n)^-: n \in N\} \neq \emptyset$. Since f is continuous $f^{-1}(U_n)^- \subset f^{-1}(U_n^-)$ and hence $f^{-1}(\cap U_n^-) \neq \emptyset$. Thus we may replace $\rho(a)$-maps by p-maps in Theorem (3.13) to get another epireflective subcategory. This result also follows directly from the fact that every p-map is obviously a $\rho(a)$-map.

A continuous function $f: X \to Y$ is demi-open (semi-open) [24] if and only if for each $A \subset X$, $A^\circ \neq \emptyset$ implies $(f(A))^{-0} \neq \emptyset$ (respectively $f(A)^\circ \neq \emptyset$) . Harris [13] has shown that all semi-open and demi-open maps are p-maps and so we have

$$\textit{semi-open} \Rightarrow \textit{demi-open} \Rightarrow p\textit{-map} \Rightarrow \rho(a) \Rightarrow \rho(b)$$

The following results of Liu-Strecker [24] follows from these remarks and Theorem (3.12)

(3.14) THEOREM. *The category of almost realcompact spaces and continuous functions is an epireflective subcategory of the category of Hausdorff spaces and semi-open (respectively demi-open) maps.*

4. Strict extensions

In this section we study strict extensions. In general, a λ-map need not have a continuous extension; an example is provided by Banaschewski [2] in the case of semi-regular spaces X,Y and Banaschewski

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T_2-minimal extensions σX, σY respectively. We show below that nevertheless a λ-map does have a θ-continuous extension between strict extensions; this part is analogous to the work of Rudolf [28].

We recall the definition of a θ-continuous function $f\colon X \to Y$: for each x in X and for each open set U containing $f(x)$, there exists an open set V containing x such that $f(V^-) \subset U^-$. The notion of θ-continuity dates back at least to Fomin [9]; however, several recent authors now use that term to describe a slightly weaker condition $(f(V) \subset U^-)$ called weak θ-continuity by Rudolf [28].

(4.1) THEOREM. Suppose λX, λY have strict extension topologies, $f\colon X \to Y$ is a λ-map and let $\bar{f}$ be defined by (2.11). Then $\bar{f}\colon \lambda X \to \lambda Y$ is θ-continuous.

Proof. Suppose U^* is a neighbourhood of $\bar{f}(F)$ for $F \in X^\lambda$ and $U \in B(Y)$. There exists a $B(Y)$-cover α of Y such that

$$\alpha \cap \bar{f}(F) = \{U\} \text{ if } \bar{f}(F) = N(y_0),\ y_0 \in Y$$

$$= \emptyset \quad \text{if } \bar{f}(F) \in Y^\lambda$$

Then $\beta = \alpha \cup \{U, (X-U)^\circ\}$ is a λ-cover of Y and $\bar{f}(F) \cap \beta = \{U\}$. Since f is a λ-map, there exists a V in F, $W \in \beta$ such that $f(V) \subset W$. Clearly this means $W = U$. Note also that if $F = N(x)$ for some $x \in X$, then for any $U \in N(f(x_0))$, there is a $V \in N(x)$ such that $f(V) \subset U$.

Claim: $\bar{f}(Cl\ V^*) \subset Cl\ U^*$.

If $G \in Cl\ V^*$, then we have shown that if $U_1 \in \bar{f}(G)$ there exists a $V_1 \in G$ such that $f(V_1) \subset U_1$. Since $G \in Cl\ V^*$, there exists $H \in V^*$ such that $V_1 \in H$ and hence $V_1 \cap V \in H$. Let $x_0 \in V_1 \cap V$; then $f(x_0) \in f(V_1 \cap V) \subset U_1 \cap U$ and so $N(f(x_0)) \in (U_1 \cap U)^*$. Since $(U_1 \cap U)^* = U_1^* \cap U^*$, we have $\bar{f}(G) \in Cl\ U^*$. This completes the proof.

The proof of Theorem (4.1) shows that for each neighbourhood U^* of

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$f(F)$, $U \in B(Y)$, there exists $V \in F$ such that $f(V) \subset U$. Consequently, if $G \in V^*$, then $U \in f^\circ(G)$. Hence $U \in \bar{f}(G)$ provided $\bar{f}(G) \in Y^\lambda$. This shows that if $\bar{f}(X^\lambda) \subset Y^\lambda$, then $\bar{f}$ is indeed continuous! A necessary and sufficient condition for this to happen is given in the following lemma; this is a generalisation of the one given by Blaszczyk and Mioduszewski [5] for the case of the Katětov extension.

(4.2) LEMMA: *For a λ-map $f: X \to Y$, $\bar{f}(X^\lambda) \subset Y^\lambda$ if and only if for each $F \in X^\lambda$, there exists $G \in Y^\lambda$ such that for every $U \in F$, $V \in G$, $f(U) \cap V \neq \emptyset$.*

Proof. In the proof of Lemma (2.10), for $F \in X^\lambda$, $f^\circ(F) \subset f(F)$ and the condition $f(U) \cap V \neq \emptyset$ for every $U \in F$, $V \in G$ implies that $f^\circ(F) \subset G$. So $f^\circ(F)$ cannot converge and $f^\circ(F) = G$. Conversely, if $\bar{f}(F) \in Y^\lambda$ for $F \in X^\lambda$, then set $\bar{f}(F) = G$. Clearly, $f^\circ(F) = G \subset f(F)$ and the result follows trivially.

(4.3) DEFINITION. A continuous function $f: X \to Y$ is called *λ-perfect* if and only if f has a continuous extension $\bar{f}: \lambda X \to \lambda Y$ such that $\bar{f}(X^\lambda) \subset Y^\lambda$.

(4.4) THEOREM. *A continuous function $f: X \to Y$ is λ-perfect if and only if f is a λ-map and for each $F \in X^\lambda$, there is a $G \in Y^\lambda$ such that for each $U \in F$, for each $V \in G$, $f(U) \cap V \neq \emptyset$.*

(4.5) THEOREM. *The category of H-closed spaces (respectively minimal Hausdorff spaces) and continuous maps is an epireflective subcategory of the category of Hausdorff spaces (respectively semi-regular spaces) and λ-perfect maps. The epireflection of X is its Fomin extension $F(X)$ (respectively the Banaschewski T_2-minimal extension σX).*

Since all the spaces are Hausdorff it is known that whenever f has a continuous extension, this extension is unique. However, if f has a θ-continuous extension (as in Theorem (4.1)), then the extension need not be unique (Rudolf [28]). We now generalise a condition of Rudolf to ensure that the θ-continuous extension is unique.

(4.6) DEFINITION. A continuous function $f\colon X \to Y$ is called a *Urysohn map* if and only if for every pair $\mathcal{F}$, $\mathcal{G}$ of distinct members of Y^λ, there exists $U_1 \in \mathcal{F}$, $U_2 \in \mathcal{G}$ such that $[f^{-1}(U_1^- \cap U_2^-)]^\circ = \emptyset$.

(4.7) THEOREM. *A Urysohn λ-map $f\colon X \to Y$ has a unique θ-continuous extension $\bar{f}\colon \lambda X \to \lambda Y$ given by (2.11).*

Proof: Theorem (4.1) yields the existence of a θ-continuous extension $\bar{f}\colon \lambda X \to \lambda Y$. We now show uniqueness. Suppose on the contrary that f_1 and f_2 are two distinct θ-continuous extension of f . Then there exists $\mathcal{F} \in X^\lambda$ such that $f_1(\mathcal{F}) \neq f_2(\mathcal{F})$. Since λY is T_2 , there exist $W_i \in f_i(\mathcal{F})$ $i = 1,2$ such that $W_1^* \cap W_2^* = \emptyset$. Since f is Urysohn, there exist $V_i \in f_i(\mathcal{F}),, i = 1,2$ such that $[f^{-1}(V_1^- \cap V_2^-)]^\circ = \emptyset$. Set $U_i = V_i \cap W_i$, $i = 1,2$. Then $f^{-1}(U_1^- \cap U_2^-) \subset f^{-1}(V_1^- \cap V_2^-)$. Since f_i is θ-continuous, there exists $V \in \mathcal{F}$ such that $f_i(Cl\ V^*) \subset Cl\ U_i^*$, $i = 1,2$. Since the f_i's are extensions of f and $V^- = Cl\ V^* \cap X$, it follows that $f(V^-) \subset U_i^-$, $i = 1,2$. So $V \subset V^- \subset f^{-1}(U_1^- \cap U_2^-)$, a contradiction.

5. A realcompactification

In this section we characterise a special case of the van der Slot realcompactification [29] as an epireflection. Let $\mathcal{L}$ be a normal base for the closed sets ([1], [30]) in a Hausdorff space X . We also assume that $\mathcal{L}$ satisfies the countability condition: every countable cover of X by complements of members of $\mathcal{L}$ has a countable refinement by members of $\mathcal{L}$. The realcompactification $\nu(\mathcal{L})$ of X consists of all $\mathcal{L}$-ultrafilters with c.i.p. with strict extension topology whose basis for closed sets is $\{L^*\colon L \in \mathcal{L}\}$ where $L^* = \{\mathcal{F} \in \nu(\mathcal{L})\colon L \in \mathcal{F}\}$. Van der Slot's construction is more general than the above in that he assumes that $\mathcal{L}$ satisfies the conditions of subbase-regularity, subbase-normality and the countability condition.

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Let L_1 and L_2 be two normal bases satisfying the countability condition on X, Y respectively and let $\nu_1 = \nu(L_1)$, $\nu_2 = \nu(L_2)$ be their respective van der Slot realcompactifications. Our problem is to find necessary and sufficient conditions for a continuous function $f\colon X \to Y$ to have a continuous extension $\overline{f}\colon \nu_1 \to \nu_2$.

We assign the LO-proximity [27] δ_o on each of ν_1 and ν_2, respectively that is, for subsets A, B in ν_i, $A \,\delta_o\, B$ if and only if $Cl\, A \cap Cl\, B \neq \emptyset$ and the subspace proximities on X and Y denoted by δ_1, δ_2, respectively. Thus

(6.1) $\qquad A \,\delta_1\, B$ if and only if $Cl_{\nu_1} A \cap Cl_{\nu_1} B \neq \emptyset$

It follows that $A \,\delta_1\, B$ if and only if there exists an $F \in \nu_1$ such that if $A \subset L_A^* \in L_1$, $B \subset L_B^* \in L_1$, then $L_A, L_B \in F$. Obviously, if $f\colon X \to Y$ has a continuous extension $\overline{f}\colon \nu_1 \to \nu_2$, then f is proximally continuous. We now proceed to show that this condition is also sufficient:

If $f\colon X \to Y$ is proximally continuous, then f has a continuous extension $f_\Sigma\colon \nu_1 \to \Sigma Y$ where $f_\Sigma(F) = \{E \subset Y\colon F \in Cl_{\nu_1} f^{-1}(E^-)\}$ ([11], 3.7, 3.8), and ΣY is the space of all bunches in Y with the absorption topology. Obviously, $f_\Sigma(F) \neq \emptyset$ and for $L_1, L_2 \in L_2$, $L_1 \cup L_2 \in f_\Sigma(F)$ if and only if $L_1 \in f_\Sigma(F)$ or $L_2 \in f_\Sigma(F)$.

(6.2) LEMMA. *$f_\Sigma(F) \cap L_2$ has the countable intersection property.*

Proof. Suppose on the contrary that $L_n \in f_\Sigma(F) \cap L_2$, $n = 1,2,\ldots$ and $\cap L_n = \emptyset$. By the countability condition, there exist $L_n' \in L_2$ such that $\cup L_n' = Y$ and for each $n \in \mathbb{N}$ there exist $L_{\alpha_n} \in L_2$ such that $L_n' \cap L_{\alpha_n} = \emptyset$. Clearly, $X = \cup f^{-1}(L_n')$, and since F has the countable intersection property there exists an $m \in \mathbb{N}$ such that $f^{-1}(L_m') \cap F \neq \emptyset$. This implies that $L_m' \in f_\Sigma(F)$, and contradicts the fact that $L_{\alpha_m} \in f_\Sigma(F)$ since

$L'_m \, \delta_2^- \, L_{\alpha_m}$.

(6.3) LEMMA. *If* L *is a normal base on* X *, every prime* L*-filter* F *is contained in a unique* L*-ultrafilter on* X .

Proof. Suppose F is contained in distinct L-ultrafilters F_1, F_2 . Then there exist $F_i \in F_i$ such that $F_1 \cap F_2 = \emptyset$. Since L is normal, there exist $F_1', F_2' \in L$ such that, $F_1 \cap F_1' = \emptyset$, $F_2 \cap F_2' = \emptyset$ and $F_1' \cup F_2' = X$. Since F is prime, either F_1' or F_2' must be in F a contradiction, since $F_i \cap F_i' = \emptyset$, $i = 1,2$.

By Lemma 2.10 of Gagrat and Naimpally [11], there exists a prime L_2-filter $F' \subset f_\Sigma(F) \cap L_2$. By the above Lemma (6.2), F' has the countable intersection property. It follows from the proof of Proposition 2 [29], that F' is contained in a filter $F'' \in \nu_2$. It follows from Lemma 6.3 that F'' is unique. Since $f_\Sigma(F) \cap L_2$ has the finite intersection property there exists an L_2-ultrafilter G containing $f_\Sigma(F) \cap L_2$. Since G must also contain F' , we have $G = F''$. Thus we have a well defined map g from $f_\Sigma(\nu_1) \to \nu_2$.

(6.4) LEMMA. g *is continuous.*

Proof. A basic neighbourhood of $F'' \in \nu_2$ is $\nu(L_2)-L^*$ where $L \in L_2 - F''$. Then $L \cap F = \emptyset$ for some $F \in F''$. Since L_2 is normal, there exist $F', L' \in L_2$ such that $F \cap F' = \emptyset$, $L \cap L' = \emptyset$ and $L' \cup F' = Y$. Clearly $F' \notin f_\Sigma(F)$ and so $L' \in f_\Sigma(F)$.

Claim. $g[f_\Sigma(\nu_1) - F'_*] \subset \nu_2 - L^*$ where $F'_* = \{\sigma \in f_\Sigma(\nu_1) : F' \in \sigma\}$. Note that F'_* is a closed set in $\Sigma Y \cap f_\Sigma(\nu_1)$, and $f_\Sigma(F) \in f_\Sigma(\nu_1) - F'_*$. Suppose $G \in \nu_1$ and $f_\Sigma(G) \in f_\Sigma(\nu_1) - F'_*$. Then $F' \subset f_\Sigma(G)$ and $F' \cup L' = Y$ implies $L' \in f_\Sigma(G)$. If G'' is the unique element of ν_2 containing $f_\Sigma(G) \cap L_2$, then $L' \in G''$ and so $L \in G''$. Thus $G'' \in \nu_2 - L^*$ and the proof is complete.

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The above lemmas and remarks yield the following theorem:

(6.5) THEOREM. *A necessary and sufficient condition that a continuous function* $f\colon\ X \to Y$ *has a continuous extension* $\bar{f}\colon \nu_1 \to \nu_2$ *is that* f *is proximally continuous with respect to the LO-proximities induced by* δ_0 *on* ν_1, ν_2 .

(6.6) THEOREM. *The category of realcompact spaces and continuous functions is an epireflective subcategory of the category of Tihonov spaces with normal bases satisfying the countability condition, and proximally continuous maps. The epireflection of* (X, L) *is* $\nu(L)$.

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Department of Mathematics
Southern Illinois,
University at Carbondale
Carbondale,
Illinois 62901
United States of America

Department of Mathematicsl Sciences
Lakehead University,
Thunder Bay,
Ontario,
P7B 5E1 Canada.