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ON 3-CONNECTED MATROIDS

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1. Introduction. This paper extends several graph-theoretic results to matroids. The main result of Tutte's paper [**10**] which introduced the theory of n-connection for matroids was a generalization of an earlier result of his [**9**] for 3-connected graphs. The latter has since been strengthened by Halin [**3**] and in Section 3 of this paper we prove a matroid analogue of Halin's result. Tutte used his result for 3-connected graphs to deduce a recursive construction of all simple 3-connected graphs having at least four vertices. In Section 4 we generalize this by giving a recursive construction of all 3-connected matroids of rank at least three. Section 2 contains a generalization to minimally n-connected matroids of a result of Dirac [**2**] for minimally 2-connected graphs.

The terminology used here for matroids and graphs will in general follow [**12**] and [**1**] respectively. If S is a set, then $S = X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_m$ indicates that S is the disjoint union of $X_1, X_2, \ldots, X_m$. The ground set of the matroid M will be denoted by $E(M)$ and, if $T \subseteq E(M)$, we denote the rank of T by $\text{rk}T$. We shall write $\text{rk}M$ for $\text{rk}(E(M))$. The restriction of M to $E(M)\backslash T$ will sometimes be denoted by $M\backslash T$ or, if $T = \{x_1, x_2, \ldots, x_m\}$, by $M\backslash x_1, x_2, \ldots, x_m$. Likewise, the contraction of M to $E(M)\backslash T$ will be written as M/T or $M/x_1, x_2, \ldots, x_m$. A 3-element circuit of M will be called a *triangle*, and a 3-element cocircuit, a *triad*.

If N and M are matroids on S and $S \cup e$ respectively, then M is an *extension* of N if $M\backslash e = N$, and M is a *lift* of N if M^* is an extension of N^*. We call M a *non-trivial extension* of N if e is neither a loop nor a coloop of M, and e is not in a 2-element circuit of M. Likewise, M is a *non-trivial lift* of N if M^* is a non-trivial extension of N^*.

Familiarity will be assumed with the concept of n-connection for graphs as defined, for example, in [**1**, p. 42]. We now recall the definition of n-connection for matroids. If k is a positive integer, the matroid M is *k-separated* if there is a subset T of $E(M)$ such that $|T| \geqq k$, $|E(M)\backslash T| \geqq k$ and

$$\text{rk}\, T + \text{rk}(E(M)\backslash T) - \text{rk}\, M = k - 1.$$

If there is a least positive integer j such that M is j-separated, it is called the *connectivity* $\lambda(M)$ of M. If there is no such integer, we say that $\lambda(M) = \infty$.

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(1.1) [**7**, Lemma 2; **4**, Theorem 1]. *Let M be a matroid having m elements. Then $\lambda(M) = \infty$ if and only if $M \cong U_{k,m}$ where $k = \left\lfloor \frac{m}{2} \right\rfloor$ or $\left\lceil \frac{m}{2} \right\rceil$.*

The matroid M is said to be *n-connected* for any positive integer n such that $n \leqq \lambda(M)$. It is routine to show [**10**, (12)] that

(1.2) $\lambda(M) = \lambda(M^*)$.

An element e of a 3-connected matroid M is *essential* if neither $M\backslash e$ nor M/e is 3-connected. A matroid or graph H is *minimally n-connected* if H is n-connected and, for all elements e of $E(H)$, $H\backslash e$ is not n-connected. The following two results are easy to check.

(1.3) *Let M be an n-connected matroid of rank r where $r, n \geqq 2$. Then either M is minimally n-connected or, for some element x of M, the restriction $M\backslash x$ is n-connected and has rank r.*

(1.4) [**13**, Lemma 3.1; **5**, Lemma 2.2]. *If M is an n-connected matroid and $|E(M)| \geqq 2(n - 1)$, then every circuit and every cocircuit of M contains at least n elements.*

The notions of n-connectedness of a graph G and n-connectedness of its cycle matroid $M(G)$ do not, in general, coincide (see [**11**, **12**, **6**]). However, [**11**, pp. 1–2]:

(1.5) *If G is a simple graph having at least 4 vertices, then G is 3-connected if and only if $M(G)$ is 3-connected.*

Let G be a simple 3-connected graph. An edge e of G is *essential* if neither $G\backslash e$ nor G/e is both simple and 3-connected. It is straightforward to show that e is an essential edge of G if and only if e is an essential element of $M(G)$.

Suppose that $r \geqq 3$. The *wheel $\mathscr{W}_r$ of order r* is a graph having $r + 1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex (the *hub*) is joined by a single edge (a *spoke*) to each of the other vertices. The *whirl $\mathscr{W}^r$ of order r* is a matroid on $E(\mathscr{W}_r)$ having as its circuits all cycles of $\mathscr{W}_r$ other than the rim, as well as all sets of edges formed by adding a single spoke to the set of edges of the rim. The terms "rim" and "spoke" will be applied in the obvious way in both $M(\mathscr{W}_r)$ and $\mathscr{W}^r$, and we shall usually call $M(\mathscr{W}_r)$ a wheel rather than the cycle matroid of a wheel. Each of $M(\mathscr{W}_r)$ and $\mathscr{W}^r$ is a self-dual matroid of rank r [**10**, 4.7]. Tutte's main result in [**10**] is the following.

(1.6) THEOREM. [**10**, 8.3]. *Let M be a 3-connected matroid. Then every element of M is essential if and only if M is isomorphic to a wheel or a whirl.*

The next theorem strengthens Tutte's result. It will be proved in Section 3.

(1.7) THEOREM. *Let M be a minimally 3-connected matroid having at least four elements. If every element in a triad is essential, then M is isomorphic to a wheel or a whirl.*

2. Minimally n-connected matroids. The following lemma, which will be needed in both Sections 3 and 4, can also be used to strengthen Lemmas 2.6 and 4.2 of [**5**] and to prove other similar results.

(2.1) LEMMA. *Let M be a matroid having at least two elements and n be an integer exceeding one. Suppose that M/e is n-connected, but M is not. Then either e is a loop of M, or M has a cocircuit which contains e and has fewer than n elements.*

Proof. As M is not n-connected, M is $(n - j)$-separated for some positive integer j. That is, $E(M) = X \cup Y$ where $|X|, |Y| \geqq n - j$ and

(2.2) $$\text{rk } X + \text{rk } Y - \text{rk } M = n - j - 1.$$

Suppose, without loss of generality, that $e \in X$. Moreover, assume that e is not a loop of M. Then, if rk′ denotes the rank function of M/e,

(2.3) $$\text{rk}'(X\backslash e) + \text{rk}'(Y) - \text{rk}'(M/e) = (\text{rk } X + \text{rk}(Y \cup e) - \text{rk } M) - 1.$$

If $\text{rk}(Y \cup e) = \text{rk } Y$, then, by (2.2), we have

$$\text{rk}'(X\backslash e) + \text{rk}'(Y) - \text{rk}'(M/e) = n - j - 2.$$

But $|X\backslash e| \geqq n - j - 1$ and $|Y| \geqq n - j - 1$, hence M/e is $(n - j - 1)$-separated; a contradiction. We may therefore assume that

(2.4) $$\text{rk}(Y \cup e) = \text{rk } Y + 1.$$

It follows, by (2.2) and (2.3), that

$$\text{rk}'(X\backslash e) + \text{rk}'(Y) - \text{rk}'(M/e) = n - j - 1.$$

Thus, as M/e is n-connected, $|X\backslash e| < n - j$. But $|X| \geqq n - j$, hence $|X| = n - j$. Therefore, $\text{rk } X \leqq n - j$, and, by (2.4), $\text{rk } Y \leqq \text{rk } M - 1$. Hence, by (2.2), $\text{rk } Y = \text{rk } M - 1$. Thus, as $\text{rk}(Y \cup e) = \text{rk } Y + 1$, the set X contains a cocircuit containing e and having at most $n - j$ elements.

We now recall two results from [**5**] which will be needed later.

(2.5) LEMMA. [**5**, Theorem 3.2]. *Let M be a minimally n-connected matroid of rank r where $r, n \geqq 2$. If $n \leqq r$, then $|E(M)| \geqq r + n - 1$ with equality being attained if and only if $M \cong U_{r,r+n-1}$. If $n > r$, then $|E(M)| = 2r - 1$ and $M \cong U_{r,2r-1}$.*

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(2.6) THEOREM. [**5**, Theorem 2.9]. *Let M be a minimally 3-connected matroid having at least four elements. Then for all elements e such that e is not in a triad, M/e is minimally 3-connected.*

The next theorem generalizes a result of Dirac [**2**, Theorem 4] for minimally 2-connected graphs.

(2.7) THEOREM. *Suppose that M is a minimally n-connected matroid where $n \geqq 2$. Let T be a subset of $E(M)$ such that $|T| \geqq 2$ and $M|T$ is n-connected. Then $M|T$ is minimally n-connected.*

Proof. We suppose first that $|T| < 2(n-1)$. Then, if $M|T$ is m-separated, $|T| \geqq 2m$ and so $m < n - 1$. But $M|T$ is n-connected, hence we have a contradiction. It follows that $\lambda(M|T) = \infty$.

Now let $\text{rk}\, M = r$. Then, by Lemma 2.5, if $n > r$, $M \cong U_{r,2r-1}$ and clearly no restriction of M other than $U_{1,1}$ is n-connected. If $r \geqq n$, then, by Lemma 2.5 again, $|E(M)| \geqq r + n - 1 \geqq 2n - 1$ and so, by (1.4), M has no circuit of size less than n. Hence either $M|T$ is free, or $\text{rk}(M|T) \geqq n - 1$. The first case is excluded because $M|T$ is n-connected and $|T| \geqq 2$. Hence $\text{rk}(M|T) \geqq n - 1$ and so, as $|T| < 2(n-1)$, we have by (1.1) that $M|T \cong U_{n-1,2n-3}$ and the required result follows.

We may now suppose that $|T| \geqq 2(n-1)$ and that $|E(M)| \geqq |T| + 1 \geqq 2n - 1$. Assume that for some element t of T, the matroid $M|(T\backslash t)$ is n-connected. Then, as $M\backslash t$ is not n-connected, $E(M\backslash t) = X \,\dot\cup\, Y$ where $|X|, |Y| \geqq n - 1$ and

$$\text{(2.8)} \quad \text{rk}\, X + \text{rk}\, Y - \text{rk}(M\backslash t) = n - 2.$$

Since M is n-connected, $\text{rk}(X \cup t) = \text{rk}\, X + 1$, and $\text{rk}(Y \cup t) = \text{rk}\, Y + 1$. Let $L = T\backslash t$. Then, by submodularity,

$$\begin{aligned} \text{(2.9)} \quad \text{rk}(X \cap L) + \text{rk}(Y \cap L) - \text{rk}\, L &\leqq (\text{rk}\, X + \text{rk}\, L - \text{rk}(X \cup L)) \\ &\quad + (\text{rk}\, Y + \text{rk}\, L - \text{rk}(Y \cup L)) - \text{rk}\, L \\ &= (\text{rk}\, X + \text{rk}\, Y - \text{rk}(M\backslash t)) \\ &\quad + (\text{rk}\, L + \text{rk}(M\backslash t) - \text{rk}(X \cup L) - \text{rk}(Y \cup L)). \end{aligned}$$

But

$$\begin{aligned} \text{rk}(X \cup L) + \text{rk}(Y \cup L) &\geqq \text{rk}\, L + \text{rk}(X \cup Y) \\ &= \text{rk}\, L + \text{rk}(M\backslash t). \end{aligned}$$

Therefore, by (2.8) and (2.9),

$$\text{rk}(X \cap L) + \text{rk}(Y \cap L) - \text{rk}\, L \leqq n - 2.$$

Since $M|L$ is n-connected, it follows that $|X \cap L| \leqq n - 2$ or $|Y \cap L| \leqq n - 2$. Assume, without loss of generality, that $|X \cap L| \leqq n - 2$. Now $M|L$ is certainly $(n-1)$-connected. Moreover, $|L| = |T| - 1 \geqq 2n - 3$, so, by (1.4), no cocircuit of $M|L$ has fewer than $n - 1$ elements. Thus

$X \cap L$ does not contain a cocircuit of $M|L$, and so $\text{rk}(Y \cap L) = \text{rk}\, L$. But, as $\text{rk}(Y \cup t) = \text{rk}\, Y + 1$, it follows that

$$\text{rk}((Y \cap L) \cup t) = \text{rk}(Y \cap L) + 1.$$

Thus

$$\text{rk}\, T = \text{rk}(L \cup t) \geqq \text{rk}((Y \cap L) \cup t) = \text{rk}(Y \cap L) + 1 = \text{rk}\, L + 1.$$

Hence t is a coloop of $M|T$; a contradiction.

3. Wheels and whirls. Theorem 1.7 is motivated by Halin's result [**3**, Satz 7.3] that a minimally 3-connected graph is a wheel if every edge incident with a vertex of degree 3 is essential. In this section we prove Theorem 1.7. The following result of Tutte will be needed.

(3.1) LEMMA. [**10**, 7.3]. *Let M be a 3-connected matroid having at least four elements. Suppose $\{a, b, c\}$ is a triad of M such that neither M/a nor M/b is 3-connected. Then M has a triangle containing a and just one of b and c.*

Proof of Theorem 1.7. In view of Theorem 1.6, it suffices to show that every element of M is essential. We argue by induction on $|E(M)|$. Since there is no minimally 3-connected matroid having 4 elements, the result is vacuously true for $|E(M)| = 4$. Assume that the required result holds for $|E(M)| = k - 1$ and let $|E(M)| = k \geqq 5$. If every element of M is in a triad, then every element is essential. We may therefore assume that M has an element e which is not in a triad. Then, by Theorem 2.6, M/e is minimally 3-connected.

Now let x be an element in a triad of M/e. Then x is in a triad of M, so x is essential in M. Therefore M/x is not 3-connected. If $M/x/e$ is 3-connected, then, by Lemma 2.1, e is a loop of M/x, or M/x has a cocircuit containing e and having fewer than 3 elements. In both cases, (1.4) is contradicted and so $M/x, e$ is not 3-connected. Therefore every element of M/e which is in a triad is essential, and so, by the induction assumption, M/e is isomorphic to a wheel or a whirl. Thus every element of M/e is in a triad of M/e and hence is in a triad of M. Therefore every element of M/e is essential for M. As M/e is 3-connected, e is not essential for M and so e is not in a triad of M.

By Lemma 3.1, since every element of M other than e is in a triad and is essential, every such element is in a triangle of M. We now distinguish two cases:

(I) $\text{rk}(M/e) \geqq 4$; and (II) $\text{rk}(M/e) < 4$.

(I) Suppose that $\text{rk}(M/e) \geqq 4$. If T is a triangle of M/e, then T contains a unique element t of the rim of M/e. Moreover, T is the only triangle of M/e containing t, and t is in some triangle of M. Therefore, since e is not in a triangle of M, every triangle of M/e is a triangle of M. Now label $E(M/e)$ as shown in Figure 1 and let C^* be a cocircuit of M which contains e and is of minimum size among such cocircuits.

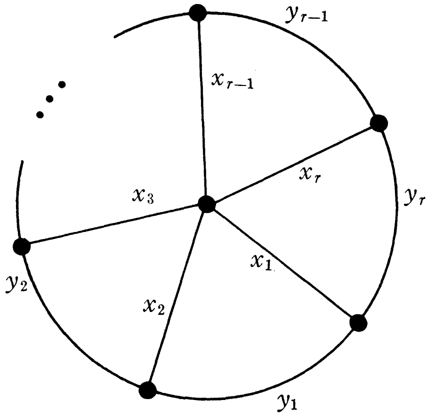

FIGURE 1

We shall now show that we can assume that C^* contains an element of the rim of M/e. For, if C^* does not contain such an element, then C^* contains a spoke, say x_1, of M/e. Since $\{x_1, x_2, y_1\}$ is a circuit of M, it follows that $|C^* \cap \{x_1, x_2, y_1\}| \neq 1$. Hence, as $y_1 \notin C^*$, $x_2 \in C^*$. Similarly, since $x_2 \in C^*$ and $y_2 \notin C^*$, $x_3 \in C^*$. By repeated application of this argument, we obtain that C^* contains $\{e, x_1, x_2, \ldots, x_r\}$. Thus $|C^*| \geqq r + 1$. But, as M has corank r, no cocircuit of M has more than $r + 1$ elements. Since M certainly has a cocircuit containing e and some element of the rim of M/e, it follows, by the choice of C^*, that we may indeed assume that C^* contains an element of the rim of M/e.

Suppose, without loss of generality, that $y_1 \in C^*$. Then, as $|C^* \cap \{x_1, x_2, y_1\}| \neq 1$, $x_1 \in C^*$ or $x_2 \in C^*$, so say $x_1 \in C^*$. Since C^* does not contain the triad $\{x_1, y_1, y_r\}$, $y_r \notin C^*$. Now consider the cocircuits $\{x_1, y_1, y_r\}$ and C^*. By exchange, M has a cocircuit D_1^* such that

$$e \in D_1^* \subseteq (C^* \cup y_r)\backslash x_1.$$

By the choice of C^*, it follows that $|D_1^*| = |C^*|$ and so

$$D_1^* = (C^* \cup y_r)\backslash x_1.$$

But $\{x_1, y_1, x_2\}$ is a triangle of M, so

$$|D_1^* \cap \{x_1, y_1, x_2\}| \neq 1.$$

Hence $x_2 \in D_1^*$. Since $\{x_2, y_1, y_2\} \nsubseteq D_1^*$, it follows that $y_2 \notin D_1^*$.

Now consider D_1^* and $\{x_2, y_1, y_2\}$. By exchange, M has a cocircuit D_2^* such that

$$e \in D_2^* \subseteq (D_1^* \cup y_2)\backslash x_2.$$

Since $|D_1^*| = |C^*|$, it follows by the choice of C^* that

$$D_2^* = (D_1^* \cup y_2) \setminus x_2.$$

But $D_1^* = (C^* \cup y_r)\backslash x_1$, so

$$D_2^* = (C^* \cup \{y_2, y_r\})\backslash\{x_1, x_2\},$$

and therefore

$$|D_2^* \cap \{x_1, x_2, y_1\}| = 1;$$

a contradiction, since $\{x_1, x_2, y_1\}$ is a circuit of M.

(II) Suppose that $\text{rk}(M/e) < 4$. Then, as M/e is isomorphic to a wheel or a whirl, $\text{rk}(M/e) \geqq 3$ and hence $M/e \cong M(\mathscr{W}_3)$ or $\mathscr{W}^3$. Therefore $M^*\backslash e \cong M(\mathscr{W}_3)$ or $\mathscr{W}^3$. Moreover, as e is not in a triad of M, it follows that e is not in a triangle of M^*. It is now straightforward to check that $M^*/e \cong U_{2,6}$. Thus $M\backslash e \cong U_{4,6}$. But the latter is 3-connected and this is a contradiction to the fact that M is minimally 3-connected.

4. Constructing 3-connected matroids. In this section we give a matroid generalization of Tutte's recursive construction [**9**, § 5] of all simple 3-connected graphs having at least 4 vertices.

(4.1) THEOREM. *A matroid of rank at least three is 3-connected if and only if it is a wheel, a whirl or* $U_{3,5}$, *or is obtainable from one of these matroids by a sequence of the following operations:*

(i) *non-trivial extensions; and*

(ii) *non-trivial lifts.*

Proof. To show that every 3-connected matroid of rank at least 3 is obtainable as described, we argue by induction on $\text{rk}\, M$. Suppose that M is a 3-connected matroid having rank 3. Then, by (1.3), M has a restriction N which is minimally 3-connected of rank r. Moreover, either $N = M$, or $N = M\backslash x_1, x_2, \ldots, x_m$ where $M\backslash x_1, x_2, \ldots, x_i$ is 3-connected for all $i < m$. Thus $M = N$ or M can be obtained from N by a sequence of non-trivial extensions.

Now by Lemma 2.5, as $\text{rk}\, N = 3$, $|E(N)| \geqq 5$ with equality being attained if and only if $N \cong U_{3,5}$. Moreover, one can check that $|E(N)| \leqq 6$ with equality being attained here if and only if $N \cong M(\mathscr{W}_3)$ or $\mathscr{W}^3$ (see [**5**, Theorems 4.7 and 5.2]). The required result follows for $\text{rk}\, M = 3$.

Next assume that the required result holds for $\text{rk}\, M = r - 1$ and let $\text{rk}\, M = r \geqq 4$. Then, by (1.3) again, M has a restriction N which is minimally 3-connected of rank r and such that M is obtained from N by a sequence of non-trivial extensions. Since N is minimally 3-connected, either every element of N is essential or else N has a non-essential element f. In the first case, by Theorem 1.6, N is isomorphic to a wheel or a whirl and hence the required result holds. In the second case, N/f is 3-connected and has rank $r - 1$. Thus, by the induction assumption, N/f is obtainable from a wheel, a whirl or $U_{3,5}$ by a sequence of the operations (i) and (ii). Since N is 3-connected, N is a non-trivial lift of N/f, and hence N, and therefore M, is obtainable in the prescribed way.

The converse follows without difficulty by combining Lemma 2.1 with (1.2).

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