8

Domain walls as D-brane prototypes

D branes are extended objects in string theory on which strings can end [10]. Moreover, the gauge fields are the lowest excitations of open superstrings, with the endpoints attached to D branes. SU(N) gauge theories are obtained as a field-theoretic reduction of a string theory on the world volume of a stack of N D branes.

Our task is to see how the above assertions are implemented in field theory. We have already thoroughly discussed field-theoretic strings. Solitonic objects of the domain wall type were also extensively studied in supersymmetric gauge theories in 1+3 dimensions. The original impetus was provided by the Dvali–Shifman observation [11] of the critical (BPS-saturated) domain walls in $\mathcal{N} = 1$ gluodynamics, with the tension scaling as $N\Lambda^3$. The peculiar N dependence of the tension prompted [12] a D-brane interpretation of such walls. Ideas as to how flux tubes can end on the BPS walls were analyzed [213] at the qualitative level shortly thereafter. Later on, BPS-saturated domain walls and their junctions with strings were discussed [214, 215] in a more quantitative aspect in $\mathcal{N} = 2$ sigma models. Some remarkable parallels between field-theoretical critical solitons and the D-brane string theory construction were discovered.

In this and subsequent chapters we will review the parallel found between the field-theoretical BPS domain walls in gauge theories and D branes/strings. In other words, we will discuss BPS domain walls with the emphasis on localization of the gauge fields on their world volume. In this sense the BPS domain walls become D-brane prototypes in field theory.

As was mentioned, research on field-theoretic mechanisms of gauge field localization on the domain walls attracted much attention. The only viable mechanism of gauge field localization was outlined in Ref. [11] where it was noted that if a gauge field is confined in the bulk and is unconfined (or less confined) on the brane, this naturally gives rise to a gauge field on the wall (for further developments see Refs. [216, 217]). Although this idea seems easy to implement, in fact it requires a careful consideration of quantum effects (confinement is certainly such an effect) which is hard to do at strong coupling.

Building on these initial proposals models with localization of gauge fields on the world volume of domain walls at weak coupling in $\mathcal{N} = 2$ supersymmetric gauge theories were suggested in [142, 37, 218]. Using a dual language, the basic idea can be expressed as follows: the gauge group is completely Higgsed in the bulk while inside the wall the charged scalar fields almost vanish. In the bulk magnetic flux tubes are formed while inside the wall the magnetic fields can propagate freely. In Ref. [142] domain walls in the simplest $\mathcal{N} = 2$ SQED theory were considered while Refs. [218, 37, 219] deal with the domain walls in non-Abelian $\mathcal{N} = 2$ gauge theories (4.1.7), with the gauge group U(N). Below we will review some results obtained in these papers.

The moduli space of the multiple domain walls in $\mathcal{N} = 2$ supersymmetric gauge theories and corresponding sigma models were studied in [220, 221, 222, 223, 224]. Note that the domain walls can intersect [84, 85, 88]. In particular, in [86, 87] honeycomb webs of walls were obtained in Abelian and non-Abelian gauge theories, respectively. We briefly discussed this phenomenon in Part I, Section 3.1.5.

We start our discussion of the BPS domain walls as D-brane prototypes in the simplest Abelian theory – $\mathcal{N} = 2$ SQED with 2 flavors [142]. It supports both the BPS-saturated domain walls and the BPS-saturated ANO strings if the Fayet–Iliopoulos term is added to the theory.

8.1 $\mathcal{N} = 2$ supersymmetric QED

$\mathcal{N} = 1$ SQED (four supercharges) was discussed in Section 3.2. Now we will extend supersymmetry to $\mathcal{N} = 2$ (eight supercharges). Some relevant features of this model are summarized in Appendix C.

The field content of $\mathcal{N} = 2$ SQED is as follows. In the gauge sector we have the U(1) vector $\mathcal{N} = 2$ multiplet. In the matter sector we have N_f matter hypermultiplets. In this section we will limit ourselves to $N_f = 2$. This is the simplest case which admits domain wall interpolating between quark vacua. The bosonic part of the action of this theory is

$$
\begin{aligned}
S = \int d^4x \Bigg\{ & \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\partial_\mu a|^2 + \bar{\nabla}_\mu \bar{q}_A \nabla_\mu q^A + \bar{\nabla}_\mu \tilde{q}_A \nabla_\mu \bar{\tilde{q}}^A \\
& + \frac{g^2}{8} \left(|q^A|^2 - |\tilde{q}_A|^2 - \xi \right)^2 + \frac{g^2}{2} \left| \tilde{q}_A q^A \right|^2 + \frac{1}{2} \left(|q^A|^2 + |\tilde{q}^A|^2 \right) \left| a + \sqrt{2} m_A \right|^2 \Bigg\},
\end{aligned}
\tag{8.1.1}
$$

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where

$$\nabla_\mu = \partial_\mu - \frac{i}{2}A_\mu, \quad \bar{\nabla}_\mu = \partial_\mu + \frac{i}{2}A_\mu. \tag{8.1.2}$$

With this convention the electric charges of the matter fields are $\pm 1/2$ (in the units of g). Parameter ξ in Eq. (8.1.1) is the coefficient in front of the Fayet–Iliopoulos term. It is introduced as in Eq. (4.1.5) with $F_3 = D$ and $F_{1,2} = 0$. In other words, here we introduce the Fayet–Iliopoulos term as the D term. Furthermore, g is the U(1) gauge coupling. The index $A = 1, 2$ is the flavor index.

The mass parameters m_1, m_2 are assumed to be real. In addition we will assume

$$\Delta m \equiv m_1 - m_2 \gg g\sqrt{\xi}\,. \tag{8.1.3}$$

Simultaneously, $\Delta m \ll (m_1 + m_2)/2$. There are two vacua in this theory: in the first vacuum

$$a = -\sqrt{2}m_1, \quad q_1 = \sqrt{\xi}, \quad q_2 = 0, \tag{8.1.4}$$

and in the second one

$$a = -\sqrt{2}m_2, \quad q_1 = 0, \quad q_2 = \sqrt{\xi}. \tag{8.1.5}$$

The vacuum expectation value of the field $\tilde{q}$ vanishes in both vacua. Hereafter in the search for domain wall solutions we will stick to the *ansatz* $\tilde{q} = 0$.

Now let us discuss the mass spectrum in both quark vacua. Consider for definiteness the first vacuum, Eq. (8.1.4). The spectrum can be obtained by diagonalizing the quadratic form in (8.1.1). This is done in Ref. [35]; the result is as follows: one real component of the field q^1 is eaten up by the Higgs mechanism to become the third component of the massive photon. Three components of the massive photon, one remaining component of q^1 and four real components of the fields $\tilde{q}_1$ and a form one long $\mathcal{N} = 2$ multiplet (8 boson states + 8 fermion states), with mass

$$m_\gamma^2 = \frac{1}{2}g^2\xi. \tag{8.1.6}$$

The second flavor q^2, $\tilde{q}_2$ (which does not condense in this vacuum) forms one short $\mathcal{N} = 2$ multiplet (4 boson states + 4 fermion states), with mass Δm which is heavier than the mass of the vector supermultiplet. The latter assertion applies to the regime (8.1.3). In the second vacuum the mass spectrum is similar – the roles of the first and the second flavors are interchanged.

If we consider the limit opposite to that in Eq. (8.1.3) and tend $\Delta m \to 0$, the "photonic" supermultiplet becomes heavier than that of q^2, the second flavor field.

Therefore, it can be integrated out, leaving us with the theory of massless moduli from q^2, $\tilde{q}_2$, which interact through a nonlinear sigma model with the Kähler term corresponding to the Eguchi–Hanson metric. The manifold parametrized by these (nearly) massless fields is obviously four-dimensional. Both vacua discussed above lie at the base of this manifold. Therefore, in considering the domain wall solutions in the sigma model limit $\Delta m \to 0$ [220, 221, 215] one can limit oneself to the base manifold, which is, in fact, a two-dimensional sphere. In other words, classically, it is sufficient to consider the domain wall in the CP(1) model deformed by a twisted mass term (related to a nonvanishing Δm), see Fig. 3.11. This was first done in [221]. A more general analysis of the domain walls on the Eguchi–Hanson manifold can be found in [225]. An interesting $\mathcal{N} = 1$ deformation of the model (8.1.1) which was treated in the literature [226] in the quest for "confinement on the wall" automatically requires construction of the wall on the Eguchi–Hanson manifold, rather than the CP(1) wall, since in this case the two vacua of the model between which the wall interpolates do not lie on the base.

8.2 Domain walls in $\mathcal{N} = 2$ SQED

A BPS domain wall interpolating between the two vacua of the bulk theory (8.1.1) was explicitly constructed in Ref. [142]. Assuming that all fields depend only on the coordinate $z = x_3$, it is possible to write the energy by performing the Bogomol'nyi completion [5],

$$E = \int dx_3 \left\{ \left| \nabla_3 q^A \pm \frac{1}{\sqrt{2}} q^A (a + \sqrt{2} m_A) \right|^2 + \left| \frac{1}{g} \partial_3 a \pm \frac{g}{2\sqrt{2}} \left(|q^A|^2 - \xi \right) \right|^2 \pm \frac{1}{\sqrt{2}} \xi \, \partial_3 a \right\}. \tag{8.2.1}$$

Requiring the first two terms above to vanish gives us the BPS equations for the wall. Assuming that $\Delta m > 0$ we choose the upper sign in (8.2.1) to get

$$\nabla_z q^A = -\frac{1}{\sqrt{2}} q^A \left(a + \sqrt{2} m_A \right),$$
$$\partial_z a = -\frac{g^2}{2\sqrt{2}} \left(|q^A|^2 - \xi \right). \tag{8.2.2}$$

These first-order equations should be supplemented by the following boundary conditions:

$$q^1(-\infty) = \sqrt{\xi}, \quad q^2(-\infty) = 0, \quad a(-\infty) = -\sqrt{2} m_1;$$
$$q^1(\infty) = 0, \quad |q^2(\infty)| = \sqrt{\xi}, \quad a(\infty) = -\sqrt{2} m_2, \tag{8.2.3}$$

which show that our wall interpolates between the two quark vacua. Here we use a U(1) gauge rotation to make q^1 in the left vacuum real.

The tension is given by the total derivative term (the last one in Eq. (8.2.1)) which can be identified as the (1, 0) central charge of the supersymmetry algebra,

$$T_{\rm w} = \xi \, \Delta m. \tag{8.2.4}$$

We can find the solution to the first-order equations (8.2.2) compatible with the boundary conditions (8.1.3). The range of variation of the field a inside the wall is of the order of Δm (see Eq. (8.2.3)). Minimization of its kinetic energy implies that this field slowly varies. Therefore, we may safely assume that the wall is thick; its size $R \gg 1/g\sqrt{\xi}$. This fact will be confirmed shortly.

We arrive at the following picture of the domain wall at hand. The wall solution has a three-layer structure [142], see Fig. 8.1. In the two outer layers – let us call them edges, they have thickness $O((g\sqrt{\xi})^{-1})$ which means that they are thin – the squark fields drop to zero exponentially; in the inner layer the field a interpolates between its two vacuum values.

Then to the leading order we can put the quark fields to zero in (8.2.2) inside the inner layer. The second equation in (8.2.2) tells us that a is a linear function of z. The solution for a takes the form

$$a = -\sqrt{2}\left(m - \Delta m \frac{z - z_0}{R}\right), \tag{8.2.5}$$

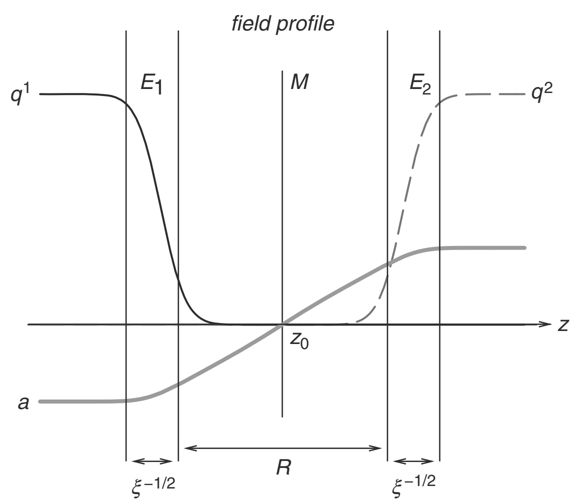

Figure 8.1. Internal structure of the domain wall: two edges (domains $E_{1,2}$) of the width $\sim (g\sqrt{\xi})^{-1}$ are separated by a broad middle band (domain M) of the width R, see Eq. (8.2.7).

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where the collective coordinate z_0 is the position of the wall center (and Δm is assumed positive). The solution is valid in a wide domain of z

$$|z-z_0| < \frac{R}{2}, \tag{8.2.6}$$

except narrow areas of size $\sim 1/g\sqrt{\xi}$ near the edges of the wall at $z-z_0=\pm R/2$.

Substituting the solution (8.2.5) in the second equation in (8.2.2) we get

$$R = \frac{4\Delta m}{g^2\xi} = \frac{2\Delta m}{m_\gamma^2}. \tag{8.2.7}$$

Since $\Delta m/g\sqrt{\xi} \gg 1$, see Eq. (8.1.3), this result shows that $R \gg 1/g\sqrt{\xi}$, which justifies our approximation. This approximation will be referred to as the thin-edge approximation.

Furthermore, we can now use the first relation in Eq. (8.2.2) to determine tails of the quark fields inside the wall. As was mentioned above, we fix the gauge imposing the condition that q^1 is real at $z \to -\infty$, see a more detailed discussion in [142].

Consider first the left edge (domain E_1 in Fig. 8.1) at $z-z_0=-R/2$. Substituting the above solution for a in the equation for q^1 we get

$$q^1 = \sqrt{\xi}\, e^{-\frac{m_\gamma^2}{4}\left(z-z_0+\frac{R}{2}\right)^2}, \tag{8.2.8}$$

where m_γ is given by (8.1.6). This behavior is valid in the domain M, at $(z-z_0+R/2) \gg 1/g\sqrt{\xi}$, and shows that the field of the first quark flavor tends to zero exponentially inside the wall, as was expected.

By the same token, we can consider the behavior of the second quark flavor near the right edge of the wall at $z-z_0=R/2$. The first equation in (8.2.2) for $A=2$ implies

$$q^2 = \sqrt{\xi}\, e^{-\frac{m_\gamma^2}{4}\left(z-z_0-\frac{R}{2}\right)^2 - i\sigma}, \tag{8.2.9}$$

which is valid in the domain M provided that $(R/2-z+z_0) \gg 1/g\sqrt{\xi}$. Here σ is an arbitrary phase which cannot be gauged away. Inside the wall the second quark flavor tends to zero exponentially too.

It is not difficult to check that the main contribution to the wall tension comes from the middle layer while the edge domains produce contributions of the order of $\xi^{3/2}$ which makes them negligibly small.

Now let us comment on the phase factor in (8.2.9). Its origin is as follows [142]. The bulk theory at $\Delta m \neq 0$ has the U(1)×U(1) flavor symmetry corresponding to two independent rotations of two quark flavors. In both vacua only one quark

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develops a VEV. Therefore, in both vacua only one of these two U(1)'s is broken. The corresponding phase is eaten by the Higgs mechanism. However, on the wall both quarks have nonvanishing values, breaking both U(1) groups. Only one of the corresponding two phases is eaten by the Higgs mechanism. The other one becomes a Goldstone mode living on the wall.

Thus, we have two collective coordinates characterizing our wall solution, the position of the center z_0 and the phase σ. In the effective low-energy theory on the wall they become scalar fields of the world volume (2+1)-dimensional theory, $z_0(t, x, y)$ and $\sigma(t, x, y)$, respectively. The target space of the second field is S_1.

This wall is a $1/2$ BPS solution of the Bogomol'nyi equations. In other words, four out of eight supersymmetry generators of the $\mathcal{N} = 2$ bulk theory are broken. As was shown in [142], the four supercharges selected by the conditions

$$\bar{\varepsilon}^2_{\dot{2}} = -i\varepsilon^{21}, \quad \bar{\varepsilon}^1_{\dot{2}} = -i\varepsilon^{22},$$
$$\bar{\varepsilon}^1_{\dot{1}} = i\varepsilon^{12}, \quad \bar{\varepsilon}^2_{\dot{1}} = i\varepsilon^{11}, \tag{8.2.10}$$

act trivially on the wall solution. They become the four supersymmetries acting in the (2+1)-dimensional effective world volume theory on the wall. Here $\varepsilon^{\alpha f}$ and $\bar{\varepsilon}^f_{\dot{\alpha}}$ are eight supertransformation parameters.

8.3 Effective field theory on the wall

In this section we will review the (2+1)-dimensional effective low-energy theory of the moduli on the wall [142]. To this end we will make the wall collective coordinates z_0 and σ (together with their fermionic superpartners) slowly varying fields depending on x_n ($n = 0, 1, 2$), For simplicity let us consider the bosonic fields $z_0(x_n)$ and $\sigma(x_n)$; the residual supersymmetry will allow us to readily reconstruct the fermion part of the effective action.

Because $z_0(x_n)$ and $\sigma(x_n)$ correspond to zero modes of the wall, they have no potential terms in the world sheet theory. Therefore, in fact our task is to derive their kinetic terms, much in the same way as it was done for strings, see Section 4.4. For $z_0(x_n)$ this procedure is very simple. Substituting the wall solution (8.2.5), (8.2.8), and (8.2.9) in the action (8.1.1) and taking into account the x_n dependence of this modulus we immediately get

$$\frac{T_{\rm w}}{2} \int d^3x (\partial_n z_0)^2. \tag{8.3.1}$$

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As far as the kinetic term for $\sigma(x_n)$ is concerned more effort is needed. We start from Eqs. (8.2.8) and (8.2.9) for the quark fields. Then we will have to modify our *ansatz* introducing nonvanishing components of the gauge field,

$$A_n = \chi(z)\, \partial_n \sigma(x_n). \tag{8.3.2}$$

These components of the gauge field are needed to make the world volume action well-defined. They are introduced in order to cancel the x dependence of the quark fields far away from the wall (in the quark vacua at $z \to \infty$) emerging through the x dependence of $\sigma(x_n)$, see Eq. (8.2.9).

Thus, we introduce a new profile function $\chi(z)$. It has no role in the construction of the static wall solution *per se*. It is unavoidable, however, in constructing the kinetic part of the world sheet theory of the moduli. This new profile function is described by its own action, which will be subject to minimization procedure. This is quite similar to derivation of the world sheet effective theory for non-Abelian strings, see Section 4.4.

The gauge potential in Eq. (8.3.2) is pure gauge far away from the wall and is not pure gauge inside the wall. It does lead to a nonvanishing field strength.

To ensure proper vacua at $z \to \pm\infty$ we impose the following boundary conditions on the function $\chi(z)$

$$\begin{aligned} &\chi(z) \to 0, \quad z \to -\infty\,, \\ &\chi(z) \to -2, \quad z \to +\infty\,. \end{aligned} \tag{8.3.3}$$

Remember, the electric charge of the quark fields is $\pm 1/2$.

Next, substituting Eqs. (8.2.8), (8.2.9) and (8.3.2) in the action (8.1.1) we arrive at

$$\begin{aligned} S^{\sigma}_{2+1} = &\left[\int d^3x\, \frac{1}{2}(\partial_n \sigma)^2\right] \\ &\times \int dz \left\{\frac{1}{g^2}(\partial_z \chi)^2 + \chi^2 |q^1|^2 + (2+\chi)^2 |q^2|^2\right\}. \end{aligned} \tag{8.3.4}$$

The expression in the second line must be considered as an "action" for the χ profile function.

Our next task is to explicitly find the function χ. To this end we have to minimize (8.3.4) with respect to χ. This gives the following equation:

$$-\partial_z^2 \chi + g^2 \chi |q^1|^2 + g^2 (2+\chi)|q^2|^2 = 0. \tag{8.3.5}$$

The equation for χ is of the second order. This is because the domain wall is no longer BPS state once we switch on the dependence of the moduli on the "longitudinal" variables x_n.

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To the leading order in $g\sqrt{\xi}/\Delta m$ the solution of Eq. (8.3.5) can be obtained in the same manner as it was done previously for other profile functions. Let us first discuss what happens outside the inner part of the wall. Say, at $z - z_0 \gg R/2$ the profile $|q^1|$ vanishes while $|q^2|$ is exponentially close to $\sqrt{\xi}$ and, hence,

$$\chi \to -2 + \text{const}\, e^{-m_\gamma (z-z_0)}. \tag{8.3.6}$$

At $z_0 - z \gg R/2$ the profile function χ falls off exponentially to zero. Thus, outside the inner part of the wall, at $|z - z_0| \gg R/2$, the function χ approaches its boundary values with the exponential rate of approach.

Of most interest, however, is the inside part, the middle domain M (see Fig. 8.1). Here both quark profile functions vanish, and Eq. (8.3.5) degenerates into $\partial_z^2 \chi = 0$. As a result, the solution takes the form

$$\chi = -1 - 2\frac{z - z_0}{R}\,. \tag{8.3.7}$$

In the narrow edge domains $E_{1,2}$ the exact χ profile smoothly interpolates between the boundary values, see Eq. (8.3.6), and the linear behavior (8.3.7) inside the wall. These edge domains give small corrections to the leading term in the action.

Substituting the solution (8.3.7) in the χ action, the second line in Eq. (8.3.4), we finally arrive at

$$S^{\sigma}_{2+1} = \frac{\xi}{\Delta m} \int d^3x \, \frac{1}{2} \, (\partial_n \sigma)^2. \tag{8.3.8}$$

As well-known [227], the compact scalar field $\sigma(t, x, y)$ can be reinterpreted to be dual to the (2+1)-dimensional Abelian gauge field living on the wall. The emergence of the gauge field on the wall is easy to understand. The quark fields almost vanish inside the wall. Therefore the U(1) gauge group is restored inside the wall while it is Higgsed in the bulk. The dual U(1) is in the confinement regime in the bulk. Hence, the dual U(1) gauge field is localized on the wall, in full accordance with the general argument of Ref. [11]. The compact scalar field $\sigma(x_n)$ living on the wall is a manifestation of this magnetic localization.

The action in Eq. (8.3.8) implies that the coupling constant of our effective U(1) theory on the wall is given by

$$e^2 = 4\pi^2 \frac{\xi}{\Delta m}. \tag{8.3.9}$$

In particular, the definition of the (2+1)-dimensional gauge field takes the form

$$F^{(2+1)}_{nm} = \frac{e^2}{2\pi} \, \varepsilon_{nmk} \, \partial^k \sigma\,. \tag{8.3.10}$$

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This finally leads us to the following effective low-energy theory of the moduli fields on the wall:

$$S_{2+1} = \int d^3x \left\{ \frac{T_w}{2} \left(\partial_n z_0\right)^2 + \frac{1}{4\,e^2} \left(F_{nm}^{(2+1)}\right)^2 + \text{fermion terms} \right\}. \quad (8.3.11)$$

The fermion content of the world volume theory is given by two three-dimensional Majorana spinors, as is required by $\mathcal{N} = 2$ in three dimensions (four supercharges, see (8.2.10)). The full world volume theory is a U(1) gauge theory in (2+1) dimensions, with four supercharges. The Lagrangian and the corresponding superalgebra can be obtained by reducing four-dimensional $\mathcal{N} = 1$ SQED (with no matter) to three dimensions.

The field z_0 in (8.3.11) is the $\mathcal{N} = 2$ superpartner of the gauge field A_n. To make it more transparent we make a rescaling, introducing a new field

$$a_{2+1} = 2\pi\xi\, z_0. \quad (8.3.12)$$

In terms of a_{2+1} the action (8.3.11) takes the form

$$S_{2+1} = \int d^3x \left\{ \frac{1}{2e^2} \left(\partial_n a_{2+1}\right)^2 + \frac{1}{4e^2} \left(F_{mn}^{(2+1)}\right)^2 + \text{fermions} \right\}. \quad (8.3.13)$$

The gauge coupling constant e^2 has dimension of mass in three dimensions. A characteristic scale of massive excitations on the world volume theory is of the order of the inverse thickness of the wall $1/R$, see (8.2.7). Thus, the dimensionless parameter that characterizes the coupling strength in the world volume theory is $e^2 R$,

$$e^2 R = \frac{16\pi^2}{g^2}. \quad (8.3.14)$$

This can be interpreted as a feature of the bulk–wall duality: the weak coupling regime in the bulk theory corresponds to strong coupling on the wall and *vice versa* [142, 228]. Of course, finding explicit domain wall solutions and deriving the effective theory on the wall assumes weak coupling in the bulk, $g^2 \ll 1$. In this limit the world volume theory is in the strong coupling regime and is not very useful.

The fact that each domain wall has two bosonic collective coordinates – its center and the phase – in the sigma model limit was noted in [214, 221].

To summarize, we showed that the world volume theory on the domain wall is the U(1) gauge theory (8.3.13) with extended supersymmetry, $\mathcal{N} = 2$. Thus, the domain wall in the theory (8.1.1) presents an example of a field-theoretic D brane: it localizes a gauge field on its world volume. In string theory gauge fields are localized on D branes because fundamental open strings can end on D branes. It turns out that this is also true for field-theoretic "D branes." In fact, various junctions of field-theoretic strings (flux tubes) with domain walls were found explicitly [215, 142, 37]. We will review 1/4-BPS junctions in Chapter 9. Meanwhile, in Section 8.4 we will consider non-Abelian generalizations of the localization effect for the gauge fields.

8.4 Domain walls in the U(N) gauge theories

In this section we will review the domain walls in $\mathcal{N}=2$ SQCD (see Eq. (4.1.7)) with the U(N) gauge group. We assume that the number of the quark flavors in this theory $N_f > N$, so the theory has many vacua of the type (4.1.11), (4.1.14) depending on which N quarks out of N_f develop VEVs. We can denote different vacua as $(A_1, A_2, \ldots, A_N)$ specifying which quark flavors develop VEVs. Mostly, we will consider a general case assuming all quark masses to be different.

Let us arrange the quark masses as follows:

$$m_1 > m_2 > \cdots > m_{N_f}. \tag{8.4.1}$$

In this case the theory (4.1.7) has

$$\frac{N_f!}{N!(N_f-N)!} \tag{8.4.2}$$

isolated vacua.

Domain walls interpolating between these vacua were classified in [218]. Below we will briefly review this classification.

The Bogomol'nyi representation of the action (4.1.7) leads to the first-order equations for the wall configurations [229], see also [37],

$$\begin{aligned}
\partial_z \varphi^A &= -\frac{1}{\sqrt{2}}\left(a_a \tau^a + a + \sqrt{2} m_A\right)\varphi^A,\\
\partial_z a^a &= -\frac{g_2^2}{2\sqrt{2}}\left(\bar{\varphi}_A \tau^a \varphi^A\right),\\
\partial_z a &= -\frac{g_1^2}{2\sqrt{2}}\left(|\varphi^A|^2 - 2\xi\right),
\end{aligned} \tag{8.4.3}$$

where we used the *ansatz* (4.2.1) and introduced a single quark field φ^{kA} instead of two fields q^{kA} and $\tilde{q}_{Ak}$. These walls are 1/2 BPS saturated. The wall tensions are given by the surface term

$$T_{\mathrm{w}} = \sqrt{2}\xi \int dz\, \partial_z a\,. \tag{8.4.4}$$

They can be written as [218]

$$T_{\rm w} = \xi\, \vec{g}\, \vec{m}, \tag{8.4.5}$$

where we use Eq. (4.1.11) and define $\vec{m} = (m_1, \ldots, m_{N_f})$, while

$$\vec{g} = \sum_{i=1}^{N_f-1} k_i \vec{\alpha}_i\,. \tag{8.4.6}$$

Here k_i are integers while α_i are simple roots of the U(N_f) algebra,[1]

$$\begin{aligned} \vec{\alpha}_1 &= (1, -1, 0, \ldots, 0), \\ \vec{\alpha}_2 &= (0, 1, -1, \ldots, 0), \\ &\ldots\,, \\ \vec{\alpha}_{N_f-1} &= (0, \ldots, 0, 1, -1). \end{aligned} \tag{8.4.7}$$

Elementary walls arise if one of the k_i's reduces to unity while all other integers in the set vanish. The tensions of the elementary walls are

$$T^i_{\rm w} = \xi\, (m_i - m_{i+1}). \tag{8.4.8}$$

The ith elementary wall interpolates between the vacua (..., i, ...) and (..., $i+1$, ...). All other walls can be considered as composite states of elementary walls.

As an example let us consider the theory (4.1.7) with the gauge group U(2) and $N_f = 4$. Explicit solutions for the elementary walls in the limit

$$(m_i - m_{i+1}) \gg g\sqrt{\xi} \tag{8.4.9}$$

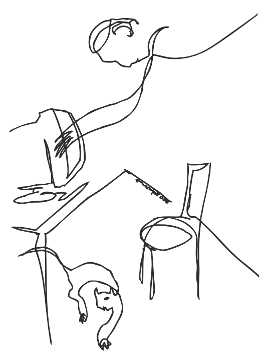
were obtained in [37]. They have the same three-layer structure as in the Abelian case, see Section 8.2. Say, the elementary wall interpolating between the vacua $(1, 2)$ and $(1, 3)$ has the following structure. At the left edge the quark φ^2 varies from its VEV $\sqrt{\xi}$ to zero exponentially, while at the right edge the quark φ^3 evolves from zero to its VEV $\sqrt{\xi}$. In the broad middle domain the fields a and a^3 linearly interpolate between their VEVs in two vacua. A novel feature of the domain wall solution as compared to the Abelian case (see Section 8.2) is that the quark field φ^1 does not vanish both outside and inside the wall.

[1] Each $\vec{\alpha}$ in Eq. (8.4.7) is an N_f-component vector, rather than $(N_f - 1)$-component vector of SU(N_f). The Cartan generators H_i $(i = 1, 2, \ldots, N_f)$ are $N_f \times N_f$ diagonal matrices, $(H_i)_{kl} = \delta_{ki}\delta_{li}$, while the relevant non-Cartan generators $E_{\vec{\alpha}_i}$ are defined as $(E_{\vec{\alpha}_i})_{i,i+1} = 1$, with all other entries vanishing.

The solution for the elementary wall has two real moduli much in the same way as in the Abelian case: the wall center z_0 and a compact phase. The phase can be rewritten as a U(1) gauge field. Therefore, the effective theory on the elementary wall is of the type (8.3.13), as in the Abelian case. The physical reason behind the localization of the U(1) gauge field on the wall world volume is easy to understand. Since the quark φ^1 does not vanish inside the wall only an appropriately chosen U(1) field, namely $(A_\mu - A_\mu^3)$, which does not interact with this quark field can propagate freely inside the wall.

In the case of generic quark masses the effective world volume theory for composite domain walls contains U(1) gauge fields associated with each elementary wall. However, the metric on the moduli space can be more complicated. For example the metric for the $\vec{\alpha}_1 + \vec{\alpha}_2$ composite wall was shown [221, 230] to have a cigar-like geometry.

We conclude this section noting that the case of the degenerate quark masses was considered in [37, 219]. In particular, in [37] the $N = 2$ case was studied and it was argued that the composite wall made of two elementary walls localizes a non-Abelian U(2) gauge field. In [219] non-localized zero modes which were called "non-Abelian clouds" were found on the composite wall.



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