J. Aust. Math. Soc. **79** (2005), 243–255

SOME NEW PERMUTABILITY PROPERTIES OF HYPERCENTRALLY EMBEDDED SUBGROUPS OF FINITE GROUPS

L. M. EZQUERRO and X. SOLER-ESCRIVÀ ✉

(Received 19 December 2003; revised 10 November 2004)

Communicated by E. A. O'Brien

Abstract

Hypercentrally embedded subgroups of finite groups can be characterized in terms of permutability as those subgroups which permute with all pronormal subgroups of the group. Despite that, in general, hypercentrally embedded subgroups do not permute with the intersection of pronormal subgroups, in this paper we prove that they permute with certain relevant types of subgroups which can be described as intersections of pronormal subgroups. We prove that hypercentrally embedded subgroups permute with subgroups of prefrattini type, which are intersections of maximal subgroups, and with $\mathscr{F}$-normalizers, for a saturated formation $\mathscr{F}$. In the soluble universe, $\mathscr{F}$-normalizers can be described as intersection of some pronormal subgroups of the group.

2000 *Mathematics subject classification*: primary 20D20, 20D30, 20E15; secondary 20D35, 20D40.
Keywords and phrases: permutability, factorizations.

1. Introduction

All groups considered in this paper are finite.

Within the general theory of subnormal subgroups, a considerable amount of effort has been addressed in the last few years to study certain conditions of permutability. In this context, subgroups which permute with all Sylow subgroups, or S-permutable subgroups, are of particular interest. S-permutable subgroups were introduced by Kegel in [14]. In this paper, Kegel proved that S-permutable subgroups form a sublattice of the lattice of subnormal subgroups. The influence of S-permutable subgroups in the structure of groups is quite clear. For instance, if each maximal

This work was supported by Proyecto BFM 2001-1667-C03-01.
© 2005 Australian Mathematical Society 1446-7887/05 $A2.00 + 0.00

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

subgroup of any Sylow subgroup of a group G is S-permutable in G, then G is supersoluble [17]. Another example is that, if A and B are two S-permutable soluble subgroups of a group G such that $G = AB$, then G is soluble [18]. We may find many other results on this line in the literature (see for example [5] or [1]).

A special type of S-permutable subgroup is a hypercentrally embedded subgroup. Recall that a subgroup H of a group G is in the hypercentre $Z_\infty(G)$ (that is, the last member of the ascending central series of G) if and only if H normalizes each Sylow subgroup of G. A subgroup T of a group G is said to be hypercentrally embedded in G if the section T^G/T_G lies in the hypercentre of the quotient group G/T_G:

$$T^G/T_G \leq Z_\infty(G/T_G),$$

where T_G denotes the core of T in G, that is, the largest normal subgroup of G contained in T and T^G denotes the normal closure of T in G, that is, the smallest normal subgroup containing T.

In [10] it is proved that hypercentrally embedded subgroups form a sublattice of the lattice of S-permutable subgroups. Carocca and Maier, in [6], and Schmid, in [16], characterized hypercentrally embedded subgroups in the following way.

PROPOSITION 1.1. *Let G be a group and T a subgroup of G. The following conditions are equivalent*:

(i) *T is a hypercentrally embedded subgroup of G.*

(ii) *T permutes with every pronormal subgroup of G.*

(iii) *T is a S-permutable subgroup which permutes with the normalizers of all Sylow subgroups of G.*

A short proof of these equivalences appears in [9].

It is clear that if a subgroup T of a group G permutes with two subgroups A and B, then T permutes with the join $\langle A, B\rangle$ but not, in general, with the intersection $A \cap B$. In general, hypercentrally embedded subgroups do not permute with the intersection of pronormal subgroups (see Example 1 of Section 5).

Our aim here is to prove that hypercentrally embedded subgroups permute with some types of relevant subgroups which, in general, are non-pronormal. These subgroups are the $\mathscr{F}$-normalizers, for a saturated formation $\mathscr{F}$, and subgroups of prefrattini type. The importance of $\mathscr{F}$-normalizers and prefrattini subgroups in soluble groups come from the fact that, defined in terms of intersection of some special pronormal subgroups, they localize some particular information of the normal structure of the whole group. The extension of these concepts to the general finite, non-necessarily soluble, universe, following the well-known program of Wielandt, was an important challenge succesfully solved in [2, 3, 4]. These generalizations, done

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

with no use of the classical Hall theory of soluble groups, give new light to non-arithmetical properties of maximal subgroups. It is surprising to realize that, in the general context, although $\mathscr{F}$-normalizers and subgroups of prefrattini type lose their cover and avoidance properties, they keep their excellent permutability properties, which are reinforced by the results we present in this paper.

It is worth remarking that, in general, these factorizations are not valid for S-permutable subgroups instead of hypercentrally embedded subgroups (see Example 2 of Section 5). In fact, Schmid proved in [16] that, in the soluble universe, hypercentrally embedded subgroups can be characterized as those S-permutable subgroups which permute with some system normalizer of the group.

A related result appears in [10] involving $\mathscr{F}$-hypercentrally embedded subgroups of soluble groups. These subgroups are a natural extension to the concept of hypercentrally embedded subgroups by considering the $\mathscr{F}$-hypercentre of the group. In [10] it is proved that if $\mathscr{F}$ is a saturated formation containing all nilpotent groups and G is a soluble group, then an S-permutable subgroup of G is $\mathscr{F}$-hypercentrally embedded in G if and only if it permutes with all $\mathscr{F}$-normalizers of G. This result is an extension of the above mentioned Schmid theorem ([16]).

In Section 4, we extend the factorizations obtained in the two previous sections. In the soluble case, subgroups whose Sylow subgroups are also Sylow subgroups of hypercentrally embedded subgroups also permute with the $\mathscr{F}$-normalizers and with the subgroups of prefrattini type associated to a fixed Hall system of the group.

Let us summarize the main properties of hypercentrally embedded subgroups which are used in the sequel.

PROPOSITION 1.2 ([10]). *Let G be a group and T a hypercentrally embedded subgroup of G. Then, if K is a normal subgroup of G and $M \leq G$, we have*

(i) *the subgroup TK is a hypercentrally embedded subgroup of G and TK/K is a hypercentrally embedded subgroup of G/K;*

(ii) *if $K \leq M$ and M/K is a hypercentrally embedded subgroup of G/K, then M is a hypercentrally embedded subgroup of G;*

(iii) *the subgroup $M \cap T$ is a hypercentrally embedded subgroup of M;*

(iv) *the subgroup $T \cap K$ is a hypercentrally embedded subgroup of G;*

(v) *T is a subgroup with the cover and avoidance property in G.*

2. Factorizations with $\mathscr{F}$-normalizers

In this section we assume that $\mathscr{F}$ is a saturated formation.

Our next result shows that in fact hypercentrally embedded subgroups of finite not-necessarily soluble groups permute with $\mathscr{F}$-normalizers for any saturated formation $\mathscr{F}$.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

First we observed that $\mathscr{F}$-normalizers are not pronormal subgroups in general. In the symmetric group of degree 4, $G = \mathrm{Sym}(4)$, the conjugacy class of all Sylow normalizers, or $\mathscr{N}$-normalizers, is the set $\mathrm{Nor}_{\mathscr{N}}(G) = \{\langle t\rangle : t \text{ is a transposition}\}$. If $\langle t\rangle \in \mathrm{Nor}_{\mathscr{N}}(G)$ were pronormal in G, then $\langle t\rangle$ would be pronormal and subnormal, and then normal, in any Sylow 2-subgroup of G containing it, which is not true.

However, $\mathscr{F}$-normalizers can be described in soluble groups as intersections of some pronormal subgroups (see [7, V.2.2]).

THEOREM 2.1. *Let $\mathscr{F}$ be a saturated formation and let G be a group. If T is a hypercentrally embedded subgroup of G and D is an $\mathscr{F}$-normalizer of G, then $TD = DT$.*

PROOF. Suppose that the theorem is not true and let G be a minimal counterexample. In G we choose a hypercentrally embedded subgroup T such that T does not permute with some $\mathscr{F}$-normalizer D of G and T is of minimal order with these conditions.

Since $\mathscr{F}$-normalizers are preserved under epimorphic images and the same happens to hypercentrally embedded subgroups by Proposition 1.2 (i), it follows that $D(TN)$ is a subgroup of G for every non-trivial normal subgroup N of G, by minimality of G. This implies that T is a core-free subgroup of G and then $T \le Z_\infty(G)$. In particular, T is a nilpotent group.

Every Sylow subgroup of T is hypercentrally embedded in G and then, if T is not of prime-power order, every Sylow subgroup of T permutes with D, by minimality of T. But this implies that T permutes with D, a contradiction. Consequently, T is a subnormal p-subgroup of G, for some prime p. For any prime $q \neq p$, if $Q \in \mathrm{Syl}_q(G)$, then T normalizes Q. Clearly T is a normal Sylow p-subgroup of QT, thus Q centralizes T. Hence T is centralized by all p'-elements of G. This is to say that $O^p(G) \le C_G(T)$.

Since obviously D is a proper subgroup of G, there exists an $\mathscr{F}$-critical maximal subgroup M of G such that D is an $\mathscr{F}$-normalizer of M, by [2, Theorem 3.5]. Let H/K be any chief factor of G under T^G. Since $T^G \le Z_\infty(G)$, then H/K is a central (abelian) chief factor of G.

If H/K is avoided by M, then M is a normal subgroup of G complementing H/K. In particular $|G : M| = p = |H/K|$, for the prime p dividing the order of T. Since M is $\mathscr{F}$-abnormal in G, it follows that $p \notin \mathrm{char}\,\mathscr{F}$. But D is an $\mathscr{F}$-group and then a p'-group, by [7, IV.4.3]. Therefore, $D \le O^p(G) \le C_G(T)$. In particular, $DT = TD$, which is a contradiction. Consequently, M covers every chief factor of G under T^G. This implies that $T \le T^G \le M$. By Proposition 1.2 (iii), it follows that T is hypercentrally embedded in M. Since D is an $\mathscr{F}$-normalizer of M, we have that TD is a subgroup of M, by minimality of G. This is the final contradiction. □

REMARK 1. The converse of Theorem 2.1 is not true in general. For this, see

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

Example 3 of Section 5.

REMARK 2. Theorem 2.1 is also true for $\mathscr{F}$-hypercentrally embedded subgroups.

Each $\mathscr{F}$-normalizer of a group G covers all $\mathscr{F}$-central chief factors of G, by [2, Theorem 4.3]. Thus the $\mathscr{F}$-hypercentre of G is a subgroup of any $\mathscr{F}$-normalizer of G. Since $\mathscr{F}$-normalizers are preserved under epimorphic images ([2, Proposition 4.2]), it is not difficult to prove that each $\mathscr{F}$-hypercentrally embedded subgroup of a finite group permute with every $\mathscr{F}$-normalizer of the group.

The converse is not true in general. In fact, in [10] it is proved that, to obtain the converse in the soluble case, the saturated formation $\mathscr{F}$ must contain all nilpotent groups and the subgroup which permutes with all $\mathscr{F}$-normalizers must be S-permutable in the group.

Hypercentrally embedded subgroups are subgroups with the cover and avoidance property (see [10, Theorem 5]). In the soluble universe, $\mathscr{F}$-normalizers are subgroups which cover the $\mathscr{F}$-central chief factors and avoid the $\mathscr{F}$-eccentric ones. The factorization of the previous theorem produces, in a soluble group, a new subgroup with the cover and avoidance property.

THEOREM 2.2. *Let G be a soluble group and T a hypercentrally embedded subgroup of G. Let $\mathscr{F}$ be a saturated formation and D an $\mathscr{F}$-normalizer of G.*

(i) *The subgroup DT possesses the cover and avoidance property in G. More precisely, DT avoids the $\mathscr{F}$-eccentric chief factors of G avoided by T and covers the rest.*

(ii) *The subgroup $D \cap T$ possesses the cover and avoidance property in G. More precisely, $D \cap T$ covers the $\mathscr{F}$-central chief factors of G covered by T and avoids the rest.*

PROOF. Using routine order arguments, the statement (ii) can be deduced from (i). Therefore, we have only to prove (i).

Assume that the result is false and let G be a minimal counterexample. Let H/K be a chief factor of G. If either H/K is covered by T or is $\mathscr{F}$-central, then H/K is covered by DT. Thus we may assume that there exists an $\mathscr{F}$-eccentric chief factor H/K of G which is avoided by T but is not avoided by DT.

Write $N = T_G$ and suppose that $N \neq 1$. Since T avoids H/K, then HN/KN is a chief factor of G which is G-isomorphic to H/K. Moreover,

$$T \cap HN = (T \cap H)N \leq KN.$$

Thus, if α is the epimorphism of G onto G/N, we have that H^α/K^α is an $\mathscr{F}$-eccentric chief factor of G^α and the hypercentrally embedded subgroup T^α of G^α avoids H^α/K^α.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

Therefore, by minimality of G, it follows that $(DT)^\alpha$ avoids H^α/K^α. This means that $DT \cap HN = DT \cap KN$. Thus $DT \cap H \leq (DT \cap H)N = (DT \cap K)N$. Since $H \cap N \leq K$, we have

$$DT \cap H = (DT \cap H) \cap (DT \cap K)N = (DT \cap K)(DT \cap H \cap N) = DT \cap K$$

and DT avoids H/K, which is a contradiction. Therefore, necessarily, $T_G = 1$. Thus T is a subgroup of $Z_\infty(G)$.

Let us consider the Hall system Σ of G associated to D, and $G_{p'}$ the Hall p'-subgroup of G in Σ, where p is the prime number dividing the order of H/K. Since T normalizes every pronormal subgroup of G, then $T \leq N_G(G_{p'} \cap G^{F(p)})$, where F is the canonical local definition of $\mathcal{F}$. By [7, V.3.2 (c)], we have that $DT \leq DG_{p'}$. If D_p denotes a Sylow p-subgroup of D, then D_p is also a Sylow p-subgroup of $DG_{p'}$. Consequently, $DTK \cap H \leq DG_{p'}K \cap H = D_pK \cap H = DK \cap H = K$. This to say that DT avoids H/K and this is the final contradiction. □

3. Factorizations with subgroups of prefrattini type

The classical prefrattini subgroups of soluble groups, introduced by Gaschütz and extended by Hawkes (see [11, 12]), are defined as intersections of certain maximal subgroups into which a fixed Hall system reduces. Obviously this choice of maximal subgroups cannot be done in the general non-soluble universe. The extension of prefrattini subgroups to finite non-necessarily soluble groups presented in [3] is possible by a new approach which does not depend on the arithmetical properties which characterize solubility. This is the origin of the systems of maximal subgroups, introduced in [3]. Later, in [4], the same authors introduced the concept of a *weakly solid* (or simply *w-solid*) *set of maximal subgroups* following some ideas due to Tomkinson. Equipped with this new definition, the authors were able to extend the idea of subgroups of prefrattini type to a finite group. In a group G, given a w-solid set of maximal subgroups $\boldsymbol{X}$, the $\boldsymbol{X}$-prefrattini subgroup of G associated to a system of maximal subgroups $\mathcal{S}$ is just the intersection of all maximal subgroups in $\boldsymbol{X} \cap \mathcal{S}$.

DEFINITION 3.1. Let G be a group, $\boldsymbol{X}$ a w-solid set of maximal subgroups of G and $\mathcal{S}$ a system of maximal subgroups of G.

(a) ([4]) Suppose that $\boldsymbol{X} \cap \mathcal{S} \neq \emptyset$ and form the subgroup $W(G, \boldsymbol{X}, \mathcal{S}) = \bigcap_{M \in \boldsymbol{X} \cap \mathcal{S}} M$. We say that $W(G, \boldsymbol{X}, \mathcal{S})$ is the $\boldsymbol{X}$*-prefrattini* subgroup of G associated to $\mathcal{S}$. Moreover, if $\boldsymbol{X} \cap \mathcal{S} = \emptyset$, then we put $W(G, \boldsymbol{X}, \mathcal{S}) = G$.

(b) We will say that a subgroup W is a *subgroup of prefrattini type* of G if W is the $\boldsymbol{X}$-prefrattini subgroup of G associated to $\mathcal{S}$, for some system $\mathcal{S}$ of maximal subgroups of G and some w-solid set $\boldsymbol{X}$ of maximal subgroups of G.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

When X is the set of all maximal subgroups of a soluble group, the X-prefrattini subgroups are just the classical Gaschütz prefrattini subgroups associated to the Hall systems. In fact, as it was remarked in [4], this definition extends the classical ones due to Gaschütz, Hawkes, Förster and Kurzweil (see [7] for details of these constructions).

A maximal subgroup is pronormal into the group. Thus a subgroup of prefrattini type is always an intersection of pronormal subgroups. But, in general, it is not pronormal. In the symmetric group of degree 4, $G = \mathrm{Sym}(4)$, if t is a transposition and we consider a core-free maximal subgroup M of G such that $t \in M$ and a Sylow 2-subgroup P of G such that $t \in P$, then $\langle t \rangle = P \cap M$. Therefore, $\langle t \rangle$ is a subgroup of prefrattini type of G. But $\langle t \rangle$ is non-pronormal in G.

Our aim here is to obtain a factorization of subgroups of prefrattini type with hypercentrally embedded subgroups in a finite group. Observe that the new subgroup appearing is again a subgroup of prefrattini type.

THEOREM 3.2. *Let G be a group and X a w-solid set of maximal subgroups of G.*

(i) *For any subgroup H of G, the set $X_H = \{M \in X : H \leq M\}$ is a w-solid set of maximal subgroups of G.*

(ii) *Let T be a hypercentrally embedded subgroup of G. If $\mathcal{S}$ is a system of maximal subgroups of G, then the X_T-prefrattini subgroup of G associated to $\mathcal{S}$ is $W(G, X_T, \mathcal{S}) = TW(G, X, \mathcal{S})$.*

PROOF. It is an easy exercise to check that X_H is a w-solid set of maximal subgroups of G whenever X is.

To prove (ii) assume that the theorem is false and let G be a counterexample of minimal order. Then G has a w-solid set of maximal subgroups X, a system of maximal subgroups $\mathcal{S}$ and a hypercentrally embedded subgroup $T \neq 1$ such that $W(G, X_T, \mathcal{S}) \neq TW(G, X, \mathcal{S})$. Let us denote $W := W(G, X, \mathcal{S})$ and $W_T := W(G, X_T, \mathcal{S})$.

Let N be any minimal normal subgroup of G. The set

$$\mathcal{S}/N = \{M/N : M \in \mathcal{S}, N \leq M\}$$

is a system of maximal subgroups of G/N (see [3, Proposition 2.3]) and $X/N = \{M/N : M \in X, N \leq M\}$ is a w-solid set of maximal subgroups of G/N (see [4, Lemma 2]). Moreover, $W(G/N, X/N, \mathcal{S}/N) = WN/N$, by [4, Theorem B]. Then in G/N all hypotheses hold for TN/N, $\mathcal{S}/N$ and X/N. By minimality of G, we obtain that $W_T N = (TN)(WN)$. Since $WT \subseteq W_T$, if $N \leq W$ or $N \leq T$, then $W_T = WT$, a contradiction. Hence, we can suppose $\mathrm{Core}_G(T) = \mathrm{Core}_G(W) = 1$.

Since $\mathrm{Core}_G(W) = 1$, we have that $\Phi(G) = 1$. This implies that

$$1 \neq T \leq Z_\infty(G) \leq F(G) = \mathrm{Soc}_a(G),$$

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

where $\mathrm{Soc}_a(G)$ denotes the abelian socle of G (see [7, A.10.6]). It follows that $Z_\infty(G)$ is generated by some minimal normal subgroups of G, all of which must be central, because they are covered by $Z_\infty(G)$. It follows that $T \leq Z_\infty(G) = Z(G)$.

In particular, T is a non-trivial normal subgroup of G. But this contradicts our assumption that $\mathrm{Core}_G(T) = 1$. This is the final contradiction. □

4. An extension of the previous results

Makan in [15] proved that, in the soluble universe, normally embedded subgroups also factorize with (Gaschütz-)prefrattini subgroups and $\mathcal{F}$-normalizers. Recall that normally embedded subgroups are subgroups whose Sylow subgroups are also Sylow subgroups of normal subgroups. In [15] all proofs depend heavily on the structure of normally embedded subgroups of finite soluble groups and in particular on the cover and avoidance properties.

In the soluble universe, our Theorems 2.1 and 3.2 can be extended to factorizations involving subgroups whose Sylow subgroups are also Sylow subgroups of hypercentrally embedded subgroups. However our proofs are essentially different to those of Makan and, for instance, do not use cover and avoidance properties.

DEFINITION 4.1. A subgroup V of a group G is said to be *local-hypercentrally embedded* in G if each Sylow subgroup of V is also a Sylow subgroup of a hypercentrally embedded subgroup of G.

In the soluble universe, we may describe the local-hypercentrally embedded subgroups in terms of hypercentrally embedded subgroups in the following way. The proof appears in [8].

PROPOSITION 4.2. *Let G be a soluble group. Denote by $\pi(G)$ the set of all primes dividing the order of G.*

A subgroup V is local-hypercentrally embedded in G if and only if for every Hall system $\Sigma = \{G_\pi : \pi \subseteq \pi(G)\}$ of G such that Σ reduces into V, there exists a family of hypercentrally embedded subgroups $\{T^p : p \in \pi(G)\}$ of G such that

$$V = \bigcap_{p \mid |G|} G_{p'} T^p.$$

Moreover, for every p, T^p is such that $V_p = V \cap G_p \in \mathrm{Syl}_p(V) \cap \mathrm{Syl}_p(T^p)$.

THEOREM 4.3. *Let V be a local-hypercentrally embedded subgroup of a soluble group G. Let Σ be a Hall system of G reducing into V and D the $\mathcal{F}$-normalizer of G associated to Σ, for a saturated formation $\mathcal{F}$. It follows that*

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

(1) *V has the cover and avoidance property in G.*

(2) *V permutes with D. Moreover, the product DV and the intersection $D \cap V$ are subgroups with the cover and avoidance property in G as in Theorem 2.2.*

PROOF. Statement (1) is a direct consequence of Proposition 1.2 (v).

Statement (2) is an easy corollary of Theorems 2.1–2.2 together with Proposition 4.2. □

Next we will see that, using the description in Proposition 4.2 for local-hypercentrally embedded subgroups, we may obtain that these subgroups permute with subgroups of prefrattini type.

In a soluble group G, for a given system of maximal subgroups $\mathcal{S}$ of G, there is always a Hall system Σ of G such that $\mathcal{S} = \mathcal{S}(\Sigma)$, the set of maximal subgroups of G such that Σ reduces into them (see [3]). If X is a w-solid set of maximal subgroups of G and $\mathcal{S} = \mathcal{S}(\Sigma)$, we write simply $W(G, X, \Sigma)$ instead of $W(G, X, \mathcal{S}(\Sigma))$.

THEOREM 4.4. *Let G be a soluble group, $\Sigma = \{G_\pi : \pi \subseteq \pi(G)\}$ a Hall system of G and X a w-solid set of maximal subgroups of G. Denote $W = W(G, X, \Sigma)$. Let V be a local-hypercentrally embedded subgroup of G such that Σ reduces into V. As in Proposition 4.2, let $\{T^p : p \in \pi(G)\}$ be a family of hypercentrally embedded subgroups of G such that $V = \bigcap_{p \in \pi(G)} G_{p'} T^p$. Then*

$$W(G, X_V, \Sigma) = VW = \bigcap_{p \mid |G|} G_{p'} T^p W.$$

PROOF. For any w-solid set of maximal subgroups Y of G, let us denote $Y(\Sigma)$ the set of maximal subgroups in Y such that Σ reduces into them. Since in soluble groups the index of any maximal subgroup is a prime power, we observe that, for each prime $p \in \pi(G)$, if M is a maximal subgroup of the group G and Σ reduces into M, then either $G_{p'} \le M$, if the index of M in G is a p-power, or $G_p \le M$, if the index of M in G is a p'-number. This is to say that for any w-solid set Y of maximal subgroups of G, we have that the set $Y(\Sigma)$ is the disjoint union set $Y(\Sigma) = Y_{G_{p'}}(\Sigma) \cup Y_{G_p}(\Sigma)$, for each prime p. Hence by the Dedekind law (see [7, A.1.6 (b)]), the following equalities hold:

$$\begin{aligned} G_{p'} W(G, Y, \Sigma) &= G_{p'} \left(\bigcap_{M \in Y(\Sigma)} M \right) \\ &= G_{p'} \left[\left(\bigcap_{M \in Y_{G_{p'}}(\Sigma)} M \right) \cap \left(\bigcap_{M \in Y_{G_p}(\Sigma)} M \right) \right] \end{aligned}$$

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

$$= \left(\bigcap_{M \in Y_{G_{p'}}(\Sigma)} M \right) \cap G_{p'} \left(\bigcap_{M \in Y_{G_p}(\Sigma)} M \right)$$
$$= W(G, Y_{G_{p'}}, \Sigma) \cap G = W(G, Y_{G_{p'}}, \Sigma).$$

Apply this result to the w-solid set $Y = X_{T^p}$ and, by Theorem 3.2, we have that $W(G, (X_{T^p})_{G_{p'}}, \Sigma) = G_{p'} W(G, X_{T^p}, \Sigma) = G_{p'}[T^p W(G, X, \Sigma)]$. Recall that T^p permutes with $G_{p'}$ since Hall subgroups are pronormal. So, in fact, we have

$$G_{p'} T^p W = W(G, X_{G_{p'} T^p}, \Sigma).$$

Now, by induction on the cardinal of $\pi(G)$ and using [7, A.1.6 (b)–(c)], it is easy to check that

$$VW = \left(\bigcap_{p \in \pi(G)} G_{p'} T^p \right) W = \bigcap_{p \in \pi(G)} G_{p'} T^p W = \bigcap_{p \in \pi(G)} W(G, X_{G_{p'} T^p}, \Sigma).$$

We know that, for each prime number p, $V_p = V \cap G_p \in \mathrm{Syl}_p(V) \cap \mathrm{Syl}_p(T^p)$ and then $V \leq G_{p'} T^p$. Therefore $X_{G_{p'} T^p} \subseteq X_V$, for each prime number p. Hence

$$W(G, X_V, \Sigma) \leq \bigcap_{p \in \pi(G)} W(G, X_{G_{p'} T^p}, \Sigma) = VW.$$

Since it is clear that $VW \leq W(G, X_V, \Sigma)$, the equality holds and the theorem is true. □

REMARK 3. It is not difficult to obtain a direct proof of Theorem 4.4 (without description of Proposition 4.2), using the ideas of Theorem 3.2.

5. Final examples

EXAMPLE 1. The following example shows a hypercentrally embedded subgroup which does not permute with the intersection of two pronormal subgroups.

Let D be a Sylow 2-subgroup of the symmetric group Sym(4) of degree 4. Write $D = \langle a, b \rangle$, where $a = (12)$, $b = (1324)$, and D is isomorphic to a dihedral group of order 8. If V is a 4-dimensional GF(3)-vector space, then Sym(4), and therefore D, acts on V by permuting the indices of a basis $\{v_1, v_2, v_3, v_4\}$. If we denote by $w_i = v_i - v_4$, for $i = 1, 2, 3$, then the subspace $W = \langle w_1, w_2, w_3 \rangle$ is an irreducible and faithful D-module over GF(3). Construct the semidirect product $G = [W]D$ and consider the element $w = w_1 + w_2$ (in abelian notation of vector space).

Clearly, D and D^w are Sylow 2-subgroups of G, and therefore pronormal subgroups, of G.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

Consider the subgroup $T = \langle ab \rangle W$ of G. The core of T in G is $T_G = W$ and then G/T_G is nilpotent. Thus T is a hypercentrally embedded subgroup of G.

It is clear that T permutes with D and with D^w. However, T does not permute with $D \cap D^w = \langle a \rangle$ since the subgroups $\langle a \rangle$ and $\langle ab \rangle$ do not permute in D.

EXAMPLE 2. It is worth remarking that the factorizations of Theorems 2.1 and 3.2 are no longer valid if we use S-permutable subgroups instead of hypercentrally embedded subgroups, as we may see in the following example.

Let $D = \langle c, b : c^7 = b^2 = 1, c^b = c^6 \rangle$ be the dihedral group of order 14. Denote by $C = \langle c \rangle \cong C_7$. There exists an irreducible C-module U over $GF(2)$ of dimension 3, such that the minimal polynomial of the action of c over U is $x^3 + x^2 + 1$. Consider the induced module $V = U^D$. Then, the restricted module V_C is $V_C = U \oplus U^b$. Since the minimal polynomial of the action of c over U^b is $x^3 + x + 1$, we have that U and U^b are non-isomorphic irreducible C-modules. Therefore, the inertia subgroup is $I_D(U) = C$. This implies that V is an irreducible D-module over $GF(2)$, by [13, VII 9.6].

Construct the semidirect product $G = [V]D$. Then $V = U \times U^b$. If $Q \in \mathrm{Syl}_7(G)$, then $Q = C^v$, for some element $v \in V$. Since UC is a subgroup of G, we have that $(UC)^v = UC^v = UQ$ is a subgroup of G. Moreover $U \leq V = O_2(G)$ and then U is contained in all Sylow 2-subgroups of G. Hence U is an S-permutable subgroup of G.

(i) The subgroup D is maximal in G. Therefore, D is pronormal in G. However, U does not permute with D. Thus the subgroup U is not hypercentrally embedded in G. Notice that $U_G = 1$ and $Z_\infty(G) = 1$.

(ii) Since V is a minimal normal subgroup of G and $1 \neq U = U \cap V \neq V$, the subgroup U does not possess the cover and avoidance property in G.

(iii) The subgroup $\langle b \rangle$ is a system normalizer of G, that is, an $\mathcal{N}$-normalizer of G, for $\mathcal{N}$ the saturated formation of all nilpotent groups. The subgroup U does not permute with $\langle b \rangle$. Hence, Theorem 2.1 does not hold for S-permutable subgroups in general.

(iv) If we consider the w-solid set of maximal subgroups $X := \{D, P\}$ of G, where $P = V\langle b \rangle$, then $X_U = \{P\}$. Let Σ be a Hall system of G such that $P \in \Sigma$, then $W(G, X, \Sigma) = \langle b \rangle$ and $W(G, X_U, \Sigma) = P$. Nevertheless, the product $U\langle b \rangle$ is not a subgroup of G and obviously $U\langle b \rangle \neq P$. Hence Theorem 3.2 is not true for S-permutable subgroups.

EXAMPLE 3. The converse of Theorem 2.1 is not true in general. In [10] appears a counterexample using the saturated formation of all 3-groups. Here we present a counterexample, suggested by the referee, using a saturated formation of full characteristic.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

Let $G = \mathrm{Alt}(5)$ be the alternating group of degree 5. Let us consider $A_2(G)$, the 2-Frattini G-module and $E = E_2(G)$, the maximal 2-elementary Frattini extension of G. Then $E/\Phi(E)$ is isomorphic to Alt(5) and $\Phi(E) \cong A_2(G)$ is an elementary abelian normal 2-subgroup of E. (See [7, Appendix β] for details about Frattini extensions).

Let $\mathscr{F}$ be the class composed of all finite groups with no epimorphic image isomorphic to Alt(5). The class $\mathscr{F}$ is a saturated formation containing all soluble groups. Clearly, $E \notin \mathscr{F}$.

Let us see that the set of all $\mathscr{F}$-normalizers of E coincides exactly with the set of all maximal subgroups of E. For this, let us take a maximal subgroup M of E. The quotient $M/\Phi(E)$ is a maximal subgroup of $E/\Phi(E) \cong \mathrm{Alt}(5)$. Therefore, $M/\Phi(E)$ is soluble and so is M. In particular, $M \in \mathscr{F}$. Since $E/\Phi(E)$ is a simple group, it follows that $E = F'(E)$, where $F'(E) = \mathrm{Soc}(E \mod \Phi(E))$. Thus M is an $\mathscr{F}$-normalizer of E.

Recall that $\mathrm{Soc}(A_2(G))$ is a completely reducible G-module. If all irreducible submodules of $\mathrm{Soc}(A_2(G))$ were trivial, then $C_E(\mathrm{Soc}(A_2(G))) = E$. But, by the Griess-Schmid theorem, we know that $C_G(\mathrm{Soc}(A_2(G))) = O_{2'2}(G) = 1$. Hence $C_E(\mathrm{Soc}(A_2(G))) = \Phi(E)$. Therefore, there exists a non-central minimal normal subgroup N of E such that $N \leq \Phi(E)$.

Clearly, N is not of order 2. Thus we may consider a subgroup T of order 2 such that $T < N$. It follows that T is a core-free subgroup such that $T \cap Z_\infty(E) \leq N \cap Z_\infty(E) = 1$. Hence, T is not a hypercentrally embedded subgroup of E.

Finally, since $T \leq N \leq \Phi(E)$ and the $\mathscr{F}$-normalizers of E are exactly the maximal subgroups of E, it is clear that T permutes with all $\mathscr{F}$-normalizers of E.

Acknowledgement

The authors thank the referees for their suggestions which have improved our exposition.

References

[1] M. Asaad and A. A. Heliel, 'On S-quasinormally embedded subgroups of finite groups', *J. Pure Appl. Algebra* **165** (2001), 129–135.

[2] A. Ballester-Bolinches, '$\mathscr{H}$-normalizers and local definitions of saturated formations of finite groups', *Israel J. Math.* **67** (1989), 312–326.

[3] A. Ballester-Bolinches and L. M. Ezquerro, 'On maximal subgroups of finite groups', *Comm. Algebra* **19** (1991), 2373–2394.

[4] ——, 'The Jordan-Hölder theorem and prefrattini subgroups of finite groups', *Glasgow Math. J.* **37** (1995), 265–277.

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press

[5] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, 'Sufficient conditions for supersolubility of finite groups', *J. Pure Appl. Algebra* **127** (1998), 118–134.

[6] A. Carocca and R. Maier, 'Hypercentral embedding and pronormality', *Arch. Math.* **71** (1998), 433–436.

[7] K. Doerk and T. O. Hawkes, *Finite soluble groups* (De Gruyter, Berlin, 1992).

[8] L. M. Ezquerro, M. Gómez-Fernández and X. Soler-Escrivà, 'On lattice properties of S-permutably embedded subgroups of finite soluble groups', *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **8** (2005), 505–517.

[9] L. M. Ezquerro and X. Soler-Escrivà, 'Some factorizations involving hypercentrally embedded subgroups in finite soluble groups', *Proc. Groups St. Andrews-Oxford 2001* **304** (2003), 190–196.

[10] ———, 'Some permutability properties related to $\mathcal{F}$-hypercentrally embedded subgroups of finite groups', *J. Algebra* **264** (2003), 279–295.

[11] W. Gaschütz, 'Praefrattinigruppen', *Arch. Math.* **13** (1962), 418–426.

[12] T. O. Hawkes, 'Analogues of prefrattini subgroups', in: *Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra (1965)* (Gordon and Breach Science, 1967) pp. 145–150.

[13] B. Huppert and N. Blackburn, *Finite groups II* (Springer, Berlin, 1982).

[14] O. H. Kegel, 'Sylow-Gruppen und Subnormalteiler endlicher Gruppen', *Math. Z.* **78** (1962), 205–221.

[15] A. Makan, 'Another characteristic conjugacy class of subgroups of finite soluble groups', *J. Austral. Math. Soc.* **11** (1970), 395–400.

[16] P. Schmid, 'Subgroups permutable with all Sylow subgroups', *J. Algebra* **207** (1998), 285–293.

[17] S. Srinivasan, 'Two sufficient conditions for supersolvability of finite groups', *Israel J. Math.* **25** (1980), 210–214.

[18] R. Yong Cai, 'Notes on π-quasi-normal subgroups in finite groups', *Proc. Amer. Math. Soc.* **117** (1993), 631–636.

Departamento de Matemática e Informática
Universidad Pública de Navarra
Campus de Arrosadía
31006 Pamplona
Spain
e-mail: ezquerro@unavarra.es

Departament de Matemàtica Aplicada
Universitat d'Alacant
Campus de Sant Vicent
ap. Correus 99
03080 Alacant
Spain
e-mail: xaro.soler@ua.es

https://doi.org/10.1017/S1446788700010478 Published online by Cambridge University Press