CLASSES OF FUNCTIONS ON ALGEBRAS

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1. Introduction. Let $\mathfrak{A}$ be a finite-dimensional linear associative algebra over the real field R or the complex field C and let F be a function with domain and range in $\mathfrak{A}$.

Several classes of functions on $\mathfrak{A}$ have been discussed in the literature, and it is the purpose of this paper to discuss the relationships between these classes and to present some interesting examples. First we shall list the definitions of the classes we wish to consider here.

DEFINITION 1.1. *Let $f(z)$ be a single-valued function of the complex variable z, and $\mathfrak{A}$ an algebra over C. Let $\alpha \in \mathfrak{A}$ be an element the zeros of whose minimum polynomial lie in the domain of $f(z)$, with the repeated zeros being points of analyticity of $f(z)$. Let these zeros be $\lambda_1, \ldots, \lambda_t$ with respective multiplicities $s_1, \ldots, s_t$. Let $L_f(z)$ be the unique polynomial of degree less than the degree of the minimum polynomial of α such that*

$$d^iL_f(z)/dz^i]_{z=\lambda_j} = f^{(i)}(\lambda_j), \qquad i = 0, \ldots, s_j - 1, \quad j = 1, \ldots, t.$$

Then $F(\alpha)$ is defined to be $L_f(\alpha)$ and we shall call F the primary function with stem function f.

$L_f(z)$ is the so-called Lagrange–Hermite interpolation polynomial and is given explicitly by

$$L_f(z) = \sum_{j=1}^{t} \left\{ \prod_{i \neq j} (z - \lambda_i)^{s_i} \sum_{k=0}^{s_j - 1} \frac{1}{k!} \times \frac{d^k}{dz^k} \left[f(z) \Big/ \prod_{h \neq j} (z - \lambda_h)^{s_h} \right]_{z=\lambda_j} (z - \lambda_j)^k \right\}. \tag{1.1}$$

The above definition can be applied to algebras over R by extending the ground field to C and computing $F(\alpha)$ as in Definition 1.1. If $F(\alpha)$ is an element of the original algebra, then F is said to be defined at α with value $F(\alpha)$. A necessary and sufficient condition for this to occur is that $f(\bar{z}) = \overline{f(z)}$ at the zeros of the minimum polynomial of α **(5)**.

DEFINITION 1.2. *Let Ω be any automorphism or anti-automorphism of $\mathfrak{A}$ that leaves the ground field invariant. F is intrinsic on $\mathfrak{A}$ if whenever α is in the domain of F, so is $\Omega\alpha$ and further $F(\Omega\alpha) = \Omega F(\alpha)$.*

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We shall denote the set of all $n \times n$ matrices with elements from K by K_n. It is easy to see that F is intrinsic on R_n (or C_n) if and only if $F(A^t) = F(A)^t$ and $F(P^{-1}AP) = P^{-1}F(A)P$ for all non-singular P in R_n (or C_n).

The study of intrinsic functions on $\mathfrak{A}$ was introduced and motivated by Rinehart **(5)** and has led for $\mathfrak{A} = C_n$ or R_n to the third class of functions we wish to consider.

DEFINITION 1.3. *Let $F(z, \sigma_1, \ldots, \sigma_{n-1})$ be a function from C^n (complex n-space) to C. For $\alpha \in \mathfrak{A} = C_n$, with characteristic polynomial*

$$x^n - \sigma_1[\alpha]x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}[\alpha]x + (-1)^n\sigma_n[\alpha],$$

define $f_\alpha(z) = f(z, \sigma_1[\alpha], \ldots, \sigma_{n-1}[\alpha])$. The function F defined by $F(\alpha) = f_\alpha(\alpha)$, where $f_\alpha(\alpha)$ is computed from $f_\alpha(z)$ as in Definition 1.1, *is called the n-ary function with stem function $f(z, \sigma_1, \ldots, \sigma_{n-1})$.*

If $\mathfrak{A} = R_n$, we need to make the same type of comment made following Definition 1.1. In this case the necessary and sufficient conditions are that $\overline{f_\alpha(\lambda_i)} = f_\alpha(\bar{\lambda}_i)$ for each eigenvalue λ_i of α **(6)**.

Q, the algebra of real quaternions, is the four-dimensional division algebra over R with basis elements $1, i, j, k = ij$ and with multiplication determined by $ij = -ji$ and $i^2 = j^2 = -1$. For $A \in Q_n$, Wiegmann **(7)** shows that one can always find a non-singular $P \in Q_n$ such that $P^{-1}AP$ has elements all in a fixed complex plane of Q, e.g. which are all of the form $a + bi$ where $a, b \in R$. In **(1)** this result was used to define n-ary functions on Q_n by using the relationship $F(A) = PF(P^{-1}AP)P^{-1}$ and computing $F(P^{-1}AP)$ as in Definition 1.3.

The study of the fourth and final class of functions we wish to consider here was again motivated by the study of intrinsic functions.

DEFINITION 1.4. *F is a poly-function on $\mathfrak{A}$ if, for every α in the domain of F, $F(\alpha)$ is a polynomial in α with coefficients from the ground field of $\mathfrak{A}$.*

It is not meant that F is a polynomial function. In general the coefficients will depend on α.

It is known that every primary function on $\mathfrak{A}$ is intrinsic and that the converse is not true in general. However, for the algebra Q of real quaternions, the classes of intrinsic functions and primary functions are identical **(5)**. For R_n, C_n, and Q_n, every n-ary function is intrinsic and every appropriately continuous intrinsic function is an n-ary function **(6; 1)**. It is clear that every primary function is a special n-ary function and that for algebras over C all n-ary functions are poly-functions. In **(1)**, there is given the first example of an intrinsic function that is not a poly-function. The example involves C_2 or Q_2, as algebras over R.

2. Examples. The four classes of functions are related as shown in Figure 1. We shall first give examples of each type of function.

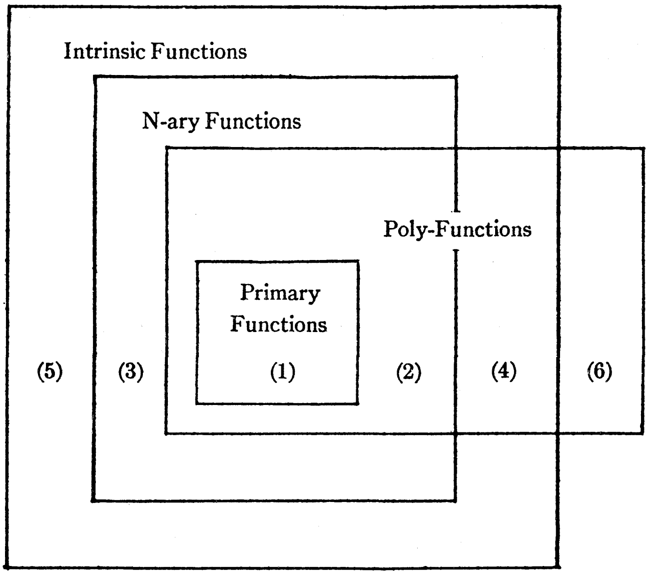

FIGURE 1

Example 2.1. Intrinsic, n-ary, primary, poly-function. Let $\mathfrak{A} = C_n$ and consider the primary function F_1 with stem function $f(z) = \cos z$. It is immediate from Definition 1.1 that the resulting primary function F_1 is a poly-function.

Example 2.2. Intrinsic, n-ary, non-primary, poly-function. Let $\mathfrak{A} = C_n$ and define $F_2(A) = \operatorname{tr}(A)I = \sigma_1[A]\cdot I$ for A in C_n. From the remark following Definition 1.2 it follows easily that F_2 is intrinsic on $\mathfrak{A}$ and it is clear that F_2 is 2-ary with stem function $f(z, \sigma_1) = \sigma_1$. That F_2 is not primary follows directly from Portmann's result **(4)** that if A is diagonal, then a necessary condition that F_2 be primary is that $F_2(A)_{ii}$ be a function only of A_{ii}. It is clear that F_2 is a poly-function (but not a polynomial function).

Example 2.3. Intrinsic, n-ary, non-primary, non-poly-function. On the algebra $\mathfrak{A} = C_2$, as an algebra over R, consider the function $F_3(A) = \operatorname{tr}(A)\cdot I = \sigma_1[A]\cdot I$. Now

$$F_3\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} = (1 + i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not a real polynomial in

$$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix},$$

so F_3 is not a poly-function. Clearly F_3 is 2-ary with stem function $f(z, \sigma_1) = \sigma_1$

and hence is intrinsic and non-primary. (We wish to thank the referee for suggesting this example, which is much simpler than the one given in **(1)**.)

Example 2.4. Intrinsic, non-n-ary, non-primary, poly-function. For $\mathfrak{A} = R_2$ define

$$F_4(A) = \begin{cases} 0 & \text{if } \det A \text{ is rational,} \\ I & \text{if } \det A \text{ is irrational.} \end{cases}$$

It follows from the remark after Definition 1.2 that F_4 is intrinsic on $\mathfrak{A}$ and it is clear that it is a poly-function.

Assume now that F_4 is a 2-ary function with stem function $f_A(z) = f(z, \sigma_1[A])$. Consider

$$A_0 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \in \mathfrak{A}$$

so that $F_4(A_0) = 0$.

Now define

$$A_m = \begin{bmatrix} 2+\sqrt{\epsilon_m} & 1 \\ 0 & 2-\sqrt{\epsilon_m} \end{bmatrix}$$

where ϵ_m is irrational and

$$\lim_{m\to\infty} \epsilon_m = 0.$$

Now $F(A_m) = I$ for all m, since $\det A_m = 4 - \epsilon_m$, which is irrational, but $\operatorname{tr} A_m = \sigma_1[A_m] = 4$ so that $f_{A_m}(z) = f_{A_0}(z)$. By **(6**,Theorem 2.2**)**, the stem function $f_{A_0}(z)$ maps the roots of A_0 into the roots of $F_4(A_0)$, so we must have $f_{A_0}(2) = f(2, 4) = 0$ and also

$$f_{A_m}(2 - \sqrt{\epsilon_m}) = f(2 - \sqrt{\epsilon_m}, 4) = 1.$$

Thus

$$\lim_{m\to\infty} f(2 - \sqrt{\epsilon_m}, 4) = 1 \neq 0 = f(2, 4),$$

contradicting the required continuity of $f(z, 4)$ at the repeated root 2 of the minimum polynomial of A_0 (see Definition 1.1).

Example 2.5. Intrinsic, non-primary, non-n-ary, non-poly-function. Consider $\mathfrak{A} = C_2$ as an algebra over R and define

$$F_5(A) = \begin{cases} a_2 iI & \text{if } \det A = a_1 + a_2 i, \text{ where } a_1 \text{ and } a_2 \text{ are rational,} \\ 0 & \text{otherwise.} \end{cases}$$

By the remark following Definition 1.2, it is clear that F_5 is indeed an intrinsic function on $\mathfrak{A}$. To see that F_5 is not 2-ary with stem function $f(z, \sigma_1)$ we proceed as in Example 2.4 using

$$A_0 = \begin{bmatrix} 1+i & 1 \\ 0 & 1+i \end{bmatrix} \text{ and the sequence } A_m = \begin{bmatrix} 1+i+\sqrt{\epsilon_m} & 1 \\ 0 & 1+i-\sqrt{\epsilon_m} \end{bmatrix},$$

where ϵ_m is irrational and

$$\lim_{m\to\infty} \epsilon_m = 0.$$

To see that F_5 is not a poly-function consider

$$J = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

so that $F_5(J) = i\cdot I$. F_5 is not a poly-function since no real polynomial in J can yield $i = F_5(J)_{1,1}$ in the 1, 1 position.

Example 2.6. Non-intrinsic, non-n-ary, non-primary, poly-function. For $\mathfrak{A} = R_2$ define, for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$F_6(A) = a_{11} A + a_{12} A^2$. Clearly F_6 is a poly-function. If F_6 is to be intrinsic on $\mathfrak{A}$, we must have $F_6(A^t) = F_6(A)^t$. However,

$$F_6(A^t) - F_6(A)^t = (a_{21} - a_{12})(A^t)^2 \neq 0$$

for non-symmetric A such that $A^2 \neq 0$; so F_6 is not intrinsic and hence non-n-ary and non-primary.

3. Poly-functions. The examples given above and the results cited in §1 justify the relative partitioning of the four classes of functions as shown in Figure 1 with the exception of the block representing the poly-functions.

THEOREM 3.1. *A primary function F on $\mathfrak{A}$ is a poly-function.*

Proof. If the ground field is C, then the assertion is immediate from Definition 1.1 and (1.1).

Consider $\alpha \in \mathfrak{A}$ with minimum polynomial of degree m and assume that the ground field of $\mathfrak{A}$ is R. From (1.1) we know that

$$F(\alpha) = \sum_{k=0}^{m-1} a_k \alpha^k, \qquad \text{where the } a_k \text{ are from } C.$$

Taking imaginary parts we have, since $F(\alpha)$ is in $\mathfrak{A}$,

$$0 = \mathrm{Im}(F(\alpha)) = \sum_{k=0}^{m-1} \mathrm{Im}(a_k)\alpha^k,$$

which yields a contradiction to the fact that the degree of the minimum polynomial is m unless $\mathrm{Im}\,(a_k) = 0$ for $k = 0, 1, \ldots, m - 1$. Thus F is a poly-function on $\mathfrak{A}$ and the proof is completed.

A second proof of the real case of Theorem 3.1 is of interest since it will be referred to later. For the complex field C it is well known that if $B \in C_n$ commutes with every matrix that commutes with A, then B is a polynomial in A with coefficients from C. This result also holds for any subfield K of C (**3**, p. 113).

Consider $\mathfrak{M} \subseteq R_m$, the first regular matrix representation of $\mathfrak{A}$, where $\alpha \in \mathfrak{A}$ corresponds to $\phi(\alpha) = A \in \mathfrak{M}$. The isomorphism ϕ induces a function G on $\mathfrak{M}$ defined by

$$G(A) = \phi(F(\phi^{-1}(A))) = \phi(F(\alpha)).$$

Let f be the stem function of F so that $f(\alpha) = L_f(\alpha)$ where $L_f(z)$ is the polynomial (1.1). Since F is primary on $\mathfrak{A}$, it is clear that G is a primary function on $\mathfrak{M}$ with stem function f and that $G(A) = L_f(A)$. Thus we see that G could have been obtained as the restriction to $\mathfrak{M}$ of the primary extension of f to R_m. Since G is primary on R_m, it is also intrinsic on R_m, and we have

$$G(B^{-1}AB) = B^{-1}G(A)B$$

for all non-singular B in R_m. Clearly, if G is a poly-function, then so is F.

We now consider any B that commutes with A and show that B commutes with $G(A)$. By the above result, this will show that $G(A)$ is a real polynomial in A. There are two cases: B singular and B non-singular.

In the first case $AB = BA$ implies that $A = BAB^{-1}$ and

$$G(A) = G(B^{-1}AB) = B^{-1}G(A)B$$

since G is intrinsic on $\mathfrak{M}$. Thus $G(A)B = BG(A)$, as desired.

If B is singular, then we can find ϵ such that $B + \epsilon I$ is non-singular. The first case then applies and we see at once that $G(A)B = BG(A)$.

We now turn our attention to necessary and sufficient conditions that n-ary and intrinsic functions be poly-functions.

N-ary functions are well defined for R_n, C_n, and Q_n. Since n-ary function values on R_n and C_n are computed as primary function values on these algebras, we have, as a consequence of Definition 1.3 and Theorem 3.1, the following theorem.

THEOREM 3.2. *All n-ary functions on the total matrix algebras R_n and C_n are poly-functions.*

Example 2.3 shows that not all n-*ary* functions on Q_n (or on C_n as an algebra over R) are poly-functions and further that not all intrinsic functions on Q_n are poly-functions. Our next theorem provides some insight into this situation.

THEOREM 3.3. *Let F be an n-ary function on Q_n with stem function*

$$f(z, \sigma_1, \ldots, \sigma_{n-1})$$

and consider J in the domain of F. $F(J)$ is a real polynomial in J if and only if $f_J(z)$ is an intrinsic function of the complex variable z at the roots of the real minimum polynomial of J.

Proof. By Wiegmann's result, which we mentioned following Definition 1.3, it suffices to consider $J \in C_n$ (more precisely in that subspace of Q_n whose elements have all their entries of the form $a + bi$). By definition, $F(J)$ is the primary function value of $f_J(z) = f(z, \sigma_1[J], \ldots, \sigma_{n-1}[J])$ at J.

Now let $\mathfrak{A}_1 \subseteq R_m$ be an algebra of real matrices isomorphic to Q_n, in which J corresponds to J_1, and let G be the primary extension of $f_J(z)$ to R_m. Now $F(J) = p_1(J)$ and $G(J_1) = p_2(J_1)$ where $p_1(z)$ and $p_2(z)$ are polynomials in z. Since the minimum polynomial of J (possibly with complex coefficients) divides the real minimum polynomial of J_1, it follows (**2**, Theorem 2.1) that $p_1(J) = p_2(J)$ and hence that $F(J)$ and $G(J_1)$ correspond under the above isomorphism.

Now $G(J_1)$ is an element of $\mathfrak{A}_1$, and hence by Theorem 3.1 a real polynomial in J_1, if and only if $f_J(z)$ is an intrinsic function of $z(f_J(\bar{z}) = \overline{f_J(z)})$ at the zeros of the real minimum polynomial of J_1. Our result now follows since J and J_1 have the same real minimum polynomial, and by isomorphism $G(J_1)$ is a real polynomial in J_1 if and only if $F(J)$ is a real polynomial in J.

We shall now consider the special case in which $\mathfrak{A}$ is simple, and determine under what conditions an intrinsic function is a poly-function.

Suppose $\mathfrak{M}$ is an algebra isomorphic to $\mathfrak{A}$ under a mapping ϕ and that for a given function F on $\mathfrak{A}$ we define a function G on $\mathfrak{M}$ by

$$G(A) = \phi \cdot F(\phi^{-1}A) = \phi \cdot F(\alpha)$$

where $A = \phi(\alpha)$. The following lemma is immediate.

Lemma. *If F is intrinsic on $\mathfrak{A}$, then G is intrinsic on $\mathfrak{M}$.*

It is well known that $\mathfrak{A}$ simple over C implies $\mathfrak{A}$ is isomorphic to C_n. The lemma and an argument like that of the second proof of Theorem 3.1 yields the following theorem.

Theorem 3.4. *If $\mathfrak{A}$ is simple over C, then every intrinsic function on $\mathfrak{A}$ is a poly-function.*

It is also well known that $\mathfrak{A}$ simple over R implies $\mathfrak{A}$ is isomorphic to R_n, C_n, or Q_n. Using the lemma and Theorem 3.3 we have the following theorem.

Theorem 3.5. *Let $\mathfrak{A}$ be simple over R and let F be an intrinsic function on $\mathfrak{A}$. If $\mathfrak{A} = R_n$, then F is a poly-function on $\mathfrak{A}$. If $\mathfrak{A} = C_n$ or Q_n and the induced function G, of the lemma, is n-ary with its stem function satisfying the conditions of Theorem* 3.3, *then F is a poly-function on $\mathfrak{A}$.*

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