J. Austral. Math. Soc. (*Series A*) **28** (1979), 346–352

MEASURABILITY OF CROSS SECTION MEASURE OF A PRODUCT BOREL SET

ROY A. JOHNSON

(Received 15 September 1978; revised 5 January 1979)
Communicated by E. Strzelecki

Abstract

Suppose μ and ν are Borel measures on locally compact spaces X and Y, respectively. A product measure λ can be defined on the Borel sets of $X\times Y$ by the formula $\lambda(M) = \int \nu(M_x)\, d\mu$, provided that vertical cross section measure $\nu(M_x)$ is a measurable function in x. Conditions are summarized for $\nu(M_x)$ to be measurable as a function in x, and examples are given in which the function $\nu(M_x)$ is not measurable. It is shown that a dense, countably compact set fails to be a Borel set if it contains no nonempty zero set.

Subject classification (*Amer. Math. Soc* (*MOS*) 1970): 28 A 20, 28 A 35.

1. Introduction

Let μ and ν be Borel measures on locally compact spaces X and Y, respectively. For undefined notation and terminology, the reader is referred to Berberian (1965). If M is a Borel set in the product space $X\times Y$ and $x\in X$, then the vertical cross section of M at x is given by $M_x = \{y\colon (x,y)\in M\}$. Each vertical cross section M_x is a Borel set in Y and $\nu(M_x)$ is the ν-measure of the vertical cross section of M at x. We shall also use $\nu(M_x)$ to denote that *function* on X whose value at $x\in X$ is given by $\nu(M_x)$.

The main purpose of this paper is to give examples in Section 3 to show that $\nu(M_x)$ can fail to be measurable as a function in x. Indeed, it does not help to assume that μ is complete since $\nu(M_x)$ can fail to be almost everywhere measurable (that is, almost everywhere equal to a measurable function).

If $\nu(M_x)$ is measurable for all Borel sets M in $X\times Y$, then a product measure λ can be defined on the Borel sets of $X\times Y$ by the formula $\lambda(M) = \int \nu(M_x)\, d\mu$ for all Borel sets M in $X\times Y$. A straightforward proof yields a Fubini theorem for the measure λ. That is, if $\nu(M_x)$ is measurable (almost everywhere measurable) as a

https://doi.org/10.1017/S1446788700012301 Published online by Cambridge University Press

function in x for all Borel sets M in $X\times Y$ and if h is a nonnegative Borel measurable (respectively, almost everywhere measurable) function on $X\times Y$, then

(1) $\int h(x,\cdot)\,d\nu$ is a measurable (respectively, almost everywhere measurable) function in x, and

(2) $\int[\int h(x,\cdot)\,d\nu]\,d\mu$ exists and equals $\int h\,d\lambda$.

The existence of the other iterated integral depends on the measurability of horizontal cross section measure. Moreover, it is possible for both iterated integrals to exist and be unequal (Johnson (1966), Theorem 5.6 and §6).

2. Sufficient conditions for measurability of vertical cross section measure

The function $\nu(M_x)$ is measurable for all Borel sets M in $X\times Y$ if and only if $\nu(K_x)$ is measurable for all compact sets K in $X\times Y$ (Johnson (1966), pp. 118–119). In this section we summarize conditions which make $\nu(K_x)$ measurable for each compact set K in $X\times Y$, and everything in this section leads to that summary. Our approach is to examine the set $A=\{x\in X\colon \nu(K_x)\geqslant\alpha\}$ for compact K and positive α in order to give conditions which make A a Borel set. For example, Theorem 1 shows that this set A is countably compact if ν is purely atomic; this fact will be used to assist with one of the examples of Section 3.

If ν is regular and $\alpha>0$, then $\{x\colon \nu(K_x)\geqslant\alpha\}$ is compact, so that $\nu(K_x)$ is measurable in this case. The set $\{x\colon \nu(K_x)\geqslant\alpha\}$ is always *sequentially closed*, which is to say that every convergent sequence in this set converges to a member of this set (Johnson (1966), Theorems 4.1 and 4.2). A topological space is called *sequential* if every sequentially closed subset is closed (Franklin (1965), pp. 108–109). Hence, if X is sequential (for example, if X is first countable), then $\{x\colon \nu(K_x)\geqslant\alpha\}$ is closed.

A set A in X is called *$\aleph_0$-bounded* if the closure in A of each countable set in A is compact (Gulden, Fleischman and Weston (1970), p. 201). An $\aleph_0$-bounded set is clearly countably compact and hence sequentially closed. Although $\{x\colon \nu(K_x)\geqslant\alpha\}$ is not always compact, under suitable conditions it will be $\aleph_0$-bounded. It is not known if $\alpha>0$ implies $\{x\colon \nu(K_x)\geqslant\alpha\}$ is $\aleph_0$-bounded or even countably compact in general.

THEOREM 1. *Suppose K is a compact set in $X\times Y$ and that α is a positive number. If ν is purely atomic, then $\{x\colon \nu(K_x)\geqslant\alpha\}$ is $\aleph_0$-bounded.*

PROOF. Fix $\alpha>0$, and let $A=\{x\colon \nu(K_x)\geqslant\alpha\}$. Let C be a countable subset of A. Since the projection of K onto X (denoted $\mathrm{Pr}_X K$) is compact, we see that the closure of C in X is compact. It suffices to show that the closure of C is contained in A.

If A is nonempty, then $\nu(\mathrm{Pr}_Y K) \geqslant \alpha$. Since ν is purely atomic, we can find a nonempty, countable disjoint collection of ν-atoms in $\mathrm{Pr}_Y K$, say $\{B_n\}$ such that $\nu(\mathrm{Pr}_Y K) = \nu(\bigcup B_n)$. For each n, let

$$D_n = \bigcap \{K_x \cap B_n : x \in C \text{ and } \nu(K_x \cap B_n) = \nu(B_n)\}$$
$$- \bigcup \{K_x \cap B_n : x \in C \text{ and } \nu(K_x \cap B_n) = 0\}.$$

Then $\nu(D_n) = \nu(B_n)$, so that D_n is an atom for ν. Moreover, given $x \in C$, the set D_n is either contained in K_x or disjoint from K_x.

For each ν-atom D_n choose a corresponding point z_n in some discrete space, and let Z be the one-point compactification of Z_0, the set of z_n's. Let ν' be the smallest Borel measure on Z such that $\nu'(\{z_n\}) = \nu(D_n)$ for each n. Let

$$L = \{(x, z_n) \in C \times Z_0 : D_n \subset K_x\},$$

and let M be the closure in $X \times Z$ of L. If $x \in X$, notice that $z_n \in M_x$ if and only if $D_n \subset K_x$. Consequently, $\nu(K_x) \geqslant \nu'(M_x)$ for all $x \in X$. Now since $\nu'(M_x) \geqslant \alpha$ for all $x \in C$ and since ν' is regular, we have $\nu'(M_x) \geqslant \alpha$ for all x in the closure of C. Hence, A contains the closure of C, and we are done.

The space X is called *countably tight* if every limit point of a subset E is a limit point of a countable subset of E. If X is countably tight, then $\aleph_0$-bounded sets are clearly compact. However, $\aleph_0$-bounded sets may fail to be Borel sets, as the examples of Section 3 show.

SUMMARY. The function $\nu(K_x)$ is a measurable function in x for each compact set K in $X \times Y$ if any one of the following conditions holds:

(1) Each compact set K in $X \times Y$ is a member of the product σ-ring $B(X) \times B(Y)$.

(2) ν is regular.

(3) Every sequentially closed set in X is a Borel set. This happens if X is a sequential space (for example, first countable space).

(4) ν is purely atomic and every $\aleph_0$-bounded set in X is a Borel set. The second condition holds if X is a countably tight space.

3. Examples of nonmeasurable cross section measure

In this section, we give the promised examples which show that the function $\nu(K_x)$ need not be measurable even though K is compact. We begin with some ideas that are used in Example 1; Theorem 2 and its Corollary allow us to say that certain sets are not Borel sets, and these results may be of independent interest.

By way of preliminaries, recall that a set Z in X is a *zero set* if it is the inverse image of $\{0\}$ under some real-valued continuous function on X. Every zero set is a closed G_δ, and the intersection of countably many zero sets is itself a zero set. In a completely regular space, every nonempty open set contains a zero set with nonempty interior.

Recall that a set in a topological space is *nowhere dense* if its closure has nonempty interior. The *weakly Borel* sets of the space are the smallest σ-field containing the class of open sets. If E is in the smallest σ-field containing the sequence of sets $\{E_n\}$, then

$$\bigcap\{E_n\colon x\in E_n\}\cap\bigcap\{X-E_n\colon x\notin E_n\}$$

is a subset of E for all $x\in E$.

THEOREM 2. *Suppose X is a completely regular space and E is a countably compact, weakly Borel set in X. Then E is nowhere dense or E contains a nonempty zero set.*

PROOF. Since E is a Borel set, there exists a sequence of open sets $\{U_n\}$ such that E is in the σ-algebra generated by the U_n's. We suppose that E is not nowhere dense and show that E contains a nonempty zero set in this case.

Since E is not nowhere dense, we may choose a nonempty open set V_0 contained in the closure of E. Choose a zero set Z_1 contained in V_0 such that Z_1 has nonempty interior. If interior$(Z_1)\cap U_1$ is nonempty, let $V_1=$ interior$(Z_1)\cap U_1$. Otherwise, let $V_1=$ interior(Z_1). We thus have a zero set Z_1 and a nonempty open set V_1 such that $V_1\subset Z_1\subset V_0$ and such that either $V_1\subset U_1$ or $V_1\subset X-U_1$. By induction, we can find zero sets Z_n and nonempty open sets V_n such that $V_n\subset Z_n\subset V_{n-1}$ for all n and such that for each n either $V_n\subset U_n$ or $V_n\subset X-U_n$.

We notice that each V_n is a nonempty open set in the closure of E. Hence, each $V_n\cap E$ is nonempty, and all the more each $Z_n\cap E$ is nonempty. Then since $Z_n\cap E$ is a decreasing sequence of nonempty closed sets in E and since E is countably compact, $\bigcap(Z_n\cap E)$ is nonempty. In other words, $(\bigcap V_n)\cap E=(\bigcap Z_n)\cap E$ is nonempty.

Choose x in $(\bigcap V_n)\cap E$. We observe that $x\in U_n$ if and only if $V_n\subset U_n$, and this means that $x\in X-U_n$ if and only if $V_n\subset X-U_n$. Since

$$\bigcap\{U_n\colon x\in U_n\}\cap\bigcap\{X-U_n\colon x\in X-U_n\}\subset E,$$

we have $\bigcap V_n\subset E$. Then since $\bigcap Z_n=\bigcap V_n$, we see that $\bigcap Z_n$ serves as the required nonempty zero set contained in E.

COROLLARY. *Suppose A is a dense, countably compact subset of a compact Hausdorff space. If A contains no nonempty zero set, then A is not a Borel set.*

Each of the following examples uses the fact that a nonempty G_δ and hence a nonempty zero set in a product space $\times X_i$ contains a nonempty set of the form $\times E_i$ where $E_i = X_i$ for all but countably many i's. This follows from the fact that if x is a member of an open set W, then x is in some basic open set $\times U_i$ contained in W, so that $U_i = X_i$ for all but finitely many i's (Kelley (1955), p. 90).

As usual, we let an ordinal denote the set of all its predecessors. For example, ω_1 denotes the set of all ordinals less than the first uncountable ordinal ω_1, and ω_1+1 denotes the set of all ordinals less than or equal to ω_1.

EXAMPLE 1. Let I be an uncountable index set, and let $X=(\omega_1+1)^I$. That is, X is the space of all functions from I into ω_1+1, where ω_1+1 has the order topology and X has the product topology. Let $Y=\omega_1+1$ with the order topology, and let ν be the (nonregular) Borel measure on Y such that $\nu(E)$ is 1 if E contains a closed, unbounded subset of ω_1 and $\nu(E)$ is 0 otherwise (Halmos (1950), Exercise 52.10). Let $K=\{(x,y)\in X\times Y\colon x_i\leqslant y \text{ for all } i\in I\}$. In other words,

$$K=\{(x,y)\in X\times Y\colon \sup x_i\leqslant y\}.$$

If $(p,q)\notin K$, then there is an index j such that $q<p_j$. If $U=\{x\in X\colon x_j>q\}$ and $V=\{y\in Y\colon y<q+1\}$, then $U\times V$ is a neighbourhood of (p,q) which is disjoint from K. Hence, K is closed and thus compact in $X\times Y$.

If $x\in X$, then $K_x=\{y\in Y\colon \sup x_i\leqslant y\}$. Hence $\nu(K_x)=1$ if $\sup x_i<\omega_1$, and $\nu(K_x)=0$ otherwise. Let $A=\{x\colon \nu(K_x)\geqslant 1\}$. That is, $A=\{x\colon \sup x_i<\omega_1\}$. We know that A is $\aleph_0$-bounded and hence countably compact by Theorem 1, and it is easy to see that A is dense in X. Each nonempty zero set in X contains a point x such that $x_i=\omega_1$ for some $i\in I$. Hence, A contains no nonempty zero set, so that A is *not* a Borel set in view of the Corollary to Theorem 2. Hence, $\nu(K_x)$ is not measurable as a function in x.

EXAMPLE 2. Let $Y=\omega_1+1$, with the order topology. Let X be 2^{ω_1}, the space of all functions from ω_1 into the discrete space of two elements, where the topology is the product topology. Let μ be Haar measure on X; that is, μ is a translation invariant measure on the Borel sets of X such that $0<\mu(X)<\infty$. If E is a Borel set in Y, let $\nu(E)$ be 1 if E contains a closed, unbounded subset of ω_1 and $\nu(E)$ be 0 otherwise (Halmos (1950), Exercise 52.10).

Let $K=\{(x,y)\in X\times Y\colon x_\alpha=1 \text{ if } \alpha\geqslant y\}$. If $(p,q)\notin K$, then there exists a coordinate β such that $p_\beta=0$ and $\beta\geqslant q$. If $U=\{x\in X\colon x_\beta=0\}$ and

$$V=\{y\in Y\colon y<\beta+1\},$$

then U and V are open neighbourhoods of p and q, respectively, and $U\times V$ is disjoint from K. Thus, K is closed and hence compact in $X\times Y$.

We show that $\nu(K_x)$ is not even almost everywhere measurable with respect to μ by showing that the set $A = \{x \in X: \nu(K_x) = 1\}$ fails to be in the domain of completion of μ. We know that

$$K_x = \{y \in Y: x_\alpha = 1 \text{ if } \alpha \geqslant y\}$$

and that K_x is nonempty since $\omega_1 \in K_x$. Let $y(x)$ be the smallest member of K_x. Then $K_x = \{y \in Y: y(x) \leqslant y\}$, so that $\nu(K_x) = 1$ if and only if $y(x) < \omega_1$. In other words, $x \in A$ if and only if there exists an element β in ω_1 such that $x_\alpha = 1$ whenever $\alpha \geqslant \beta$. If we think of the members of X as increasingly directed nets with values in $\{0, 1\}$, then A consists of those such nets which converge to 1. Now given any countable collection of ω_1, there exists an element of ω_1 which is greater than all elements of that countable collection (Kelly (1955), Theorem 0.23). It is thus easy to see that every nonempty zero set in X contains members that are eventually 1 and members which are not eventually 1. If A were in the domain of completion of μ, then A or its complement would contain a Baire set of positive μ-measure since Haar measure is completion regular (Halmos (1950), Theorem 64.I). Then that Baire set would contain a zero set of positive μ-measure. Since neither A nor its complement contains a nonempty zero set, we see that A fails to be almost everywhere measurable.

Incidentally, we sometimes think of nonmeasurable sets as contrived, unnatural sets. But such a 'natural' subset of X as the subset of convergent nets is a nonmeasurable set. Indeed, if G is any nontrivial compact group and X is the collection of functions from the ordered space ω_1 into G, then the subset of eventually constant functions is not almost everywhere measurable with respect to Haar measure on X.

EXAMPLE 3. Let $X = Y = 2^I$, the space of all functions from an uncountable set I into the discrete space of two elements, with the product topology, and let μ be Haar measure on X. Let m be any nonzero measure on a σ-algebra of subsets of I such that countable subsets have measure zero. Let S be the collection of all y in Y such that $y = 1$ almost everywhere with respect to m. Let us say that a subset F of S is cofinal in S if given y in S, there exists z in F such that $z \leqslant y$. Define a Borel measure ν on Y by $\nu(E) = 1$ if E contains a closed, cofinal subset F of S and $\nu(E) = 0$ otherwise (see Johnson (1969), p. 98).

Let $K = \{(x, y) \in X \times Y: y \leqslant x\}$. If $(p, q) \notin K$, then there exists a coordinate β such that $q_\beta > p_\beta$. Necessarily, $q_\beta = 1$ and $p_\beta = 0$. Letting $U = \{x \in X: x_\beta = 0\}$ and $V = \{y \in Y: y_\beta = 1\}$, we see that $U \times V$ is a neighbourhood of (p, q) which is disjoint from K. Hence, K is closed and thus compact in $X \times Y$.

Since $K_x = \{y \in Y: y \leqslant x\}$, it is clear that $\nu(K_x) = 1$ if $x \in S$ and $\nu(K_x) = 0$ if $x \notin S$. Hence, $\{x \in X: \nu(K_x) = 1\}$ is precisely the set S. Clearly, neither S nor its complement contains a nonempty zero set, so that by the reasoning used in Example 2,

S is not in the domain of completion of μ. Hence, $\nu(K_x)$ is not almost everywhere measurable with respect to μ.

QUESTION. If μ is a purely atomic (nonregular) Borel measure on X, is the function $\nu(K_x)$ almost everywhere measurable with respect to μ? We do not know.

References

S. K. Berberian (1965), *Measure and integration* (Macmillan, New York).
S. P. Franklin (1965), 'Spaces in which sequences suffice', *Fund. Math.* **57**, 107–115.
S. L. Gulden, W. M. Fleischman and J. H. Weston (1970), 'Linearly ordered topological spaces', *Proc. Amer. Math. Soc.* **24**, 197–203.
P. R. Halmos (1950), *Measure theory* (Van Nostrand, New York).
R. A. Johnson (1966), 'On product measures and Fubini's theorem in locally compact spaces', *Trans. Amer. Math. Soc.* **123**, 112–129.
R. A. Johnson (1969), 'Some types of Borel measures', *Proc. Amer. Math. Soc.* **22**, 94–99.
J. L. Kelley (1955), *General topology* (Van Nostrand, New York).

Department of Mathematics
Washington State University
Pullman, Washington 99164
U.S.A.