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On the control theorem for the symplectic group

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Abstract. We build a theory of Λ-adic Siegel modular forms related to the Klingen parabolic subgroup of $\mathrm{GSp}(4)$. These correspond to families of cohomology classes of increasing levels whose Hecke eigenvalues enjoy strong congruence properties. In the spirit of Hida's theory, a control theorem to relate the family to finite-level members is proved for almost all primes p; in particular we show that the error term appearing in degree one cohomology is killed by the ordinary idempotent.

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0. Introduction

In this article we develop a theory of Λ-adic families of vector-valued Siegel modular forms. Such a family basically consists of a sequence of modular forms of varying weights, which are eigenforms for the Hecke algebra and whose Hecke eigenvalues enjoy strong congruence properties. The direction in the change of weight one considers corresponds to a choice of (a conjugacy class of) parabolic subgroups of $\mathrm{GSp}(4)$, and here we deal with the non-standard or Klingen parabolic subgroup. An analogous program has been carried out for the theory related to the Borel subgroup of $\mathrm{GSp}(4)$ in [TU1] and [TU2], and (by slightly different means) for the Siegel parabolic subgroup in [Tay1]. Our work builds on the independence of weight results proved in our previous paper [B2].

The general strategy of the construction is obviously a routine extension of the methods of [Hi1] and later works of Hida. However, these papers often encounter problems with the so-called control theorem. Our main contribution is to actually prove the vanishing of the ordinary part of one of the error terms by an explicit calculation. In the case of Borel-like congruence subgroups Hida has found a method to obtain exact control outside a finite set of primes – see [Hi3] Section 7. To complete our theory, we can employ a similar trick to show that the remaining error term is also non-ordinary, if we exclude a finite set of primes.

We obtain families as specialisations of elements in an 'infinite level' space of modular forms. Our approach is purely cohomological, so we can only really talk about systems of Hecke eigenvalues occurring on various cohomology groups, which may or may not come from genuine modular forms.

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The principal application of such families is to study the properties of the four-dimensional Galois representations attached to Siegel modular forms in the family. Here we show how to lift a given system of eigenvalues to a Λ-adic family, which is a prerequisite for such applications.

Let us describe our results in a little more detail. $\Gamma(r)$ is a certain congruence subgroup in $\mathrm{Sp}_4\mathbb{Z}$ described in Section 1, with level $p^r N$ for $p \nmid N$, whose reduction modulo p^r lies in the Klingen parabolic subgroup. Let $V_{m,n}$ be the unique irreducible representation of $\mathrm{Sp}_4(\mathbb{C})$ with highest weight (m,n) in the Weyl chamber corresponding to the standard Borel subgroup. Then there is a Hecke-equivariant embedding of holomorphic degree two Siegel modular forms for $\Gamma(r)$ and of weight (k_1,k_2) into $H^3(\Gamma(r), V_{k_1-3,k_2-3})$ (in analogy with the classical Eichler–Shimura isomorphism). We define a certain $\mathbb{Z}$-lattice $L_{\mathbf{k}} \subset V_{\mathbf{k}}$. Cohomology with coefficients in $L_{\mathbf{k}}$ does not correspond directly to modular forms with integer coefficients, but we have the same Hecke algebra acting on both. Let e be the projector onto the ordinary (with respect to a Hecke operator to be specified) subspace.

Define

$$\mathcal{W}^{\circ}_{\mathbf{k},r} = eH^3(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Z}/p^r\mathbb{Z}), \quad \mathcal{W}^{\circ}_{\mathbf{k}} = \varinjlim_{r} \mathcal{W}^{\circ}_{\mathbf{k},r}$$

and let $W^{\circ}_{\mathbf{k},r}$ and $W^{\circ}_{\mathbf{k}}$ be the corresponding Pontryagin dual spaces. In Section 1 we explain the action of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \cong \mathbb{Z}_p[[X]]$ and a twisted action of the Hecke operators on $\mathcal{W}^{\circ}_{\mathbf{k}}$. Then it follows from the main theorem of [B2] that $\mathcal{W}^{\circ}_{\mathbf{k}}$ is essentially independent of the second weight variable k_2.

To make the concept of infinite level a useful one, we require a 'control theorem' of the form $(\mathcal{W}^{\circ}_{\mathbf{k}})^{1+p^s\mathbb{Z}_p} = \mathcal{W}^{\circ}_{\mathbf{k},s}$. When attempting to prove this using a spectral sequence relating different level cohomology groups, one encounters several error terms. Let e_0 be the idempotent associated to level N. Under the assumption

$$p \nmid \# e_0 H^3(\Gamma(0), L_{\mathbf{k}})^{\mathrm{tor}}, \quad \text{for some } \mathbf{k} = (m,n) \text{ with } n \gg m \gg 0 \tag{H}$$

we can prove the control theorem as Theorem 2.3 and Corollary 2.7. The bulk of our work is in showing that $eH^1(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0$; by an application of the congruence subgroup property for Sp(4), this is reduced to a (slightly lengthy) calculation in the cohomology of finite groups. We then use the condition (H) to deduce that we also have $eH^2(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0$. We suggest a condition for this second error term to vanish for all p.

The control theorem (2.7) is the basis for studying the space $\mathcal{W}^{\circ}_{\mathbf{k}}$.

Now take $u = 1+p$, a topological generator of $1+p\mathbb{Z}_p$. For $a \in \mathbb{N}$ we have prime ideals $P_a = u - (1+p)^a$ in Λ. P_a is the kernel of the specialisation map from Λ to $\mathbb{Z}_p$ given by setting X to $(1+p)^a - 1$; these specialisation maps allow us to recover finite level spaces of modular forms from $\mathcal{W}^{\circ}_{\mathbf{k}}$.

Our main result is

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THEOREM A. *Assume hypothesis H. Define* $\mathbf{W} = W^{\circ}_{(m,n)}$. *Then* $\mathbf{W}$ *is finite and free over* Λ*; if* m *is a fixed even integer, the (twisted) Hecke module* $\mathbf{W}$ *is independent of* n. *Furthermore, we have specialisation maps* $\mathbf{W}/P_a\mathbf{W} \cong W^{\circ}_{(m,a),1}$.

A slightly more general version of Theorem A is proved as Theorem 5.4 in the text. As already remarked above, most of the proof follows the lines set out in [Hi1], apart from the vanishing of the degree one error term in the spectral sequence for the control theorem.

In [TU2] it is shown that the error terms in the control theorem vanish for complex coefficients, using results of Richard Taylor for the cohomology of the interior together with a theorem of Schwermer on the boundary cohomology. Hence the error terms can only be torsion, which gives us a weak version of Theorem 5.4 with finite kernel and cokernel. In the case where $\Gamma(r) \pmod{p^r}$ actually lies in the reduction of the Borel subgroup of $\mathrm{Sp}_4(\mathbb{Z})$, Tilouine and Urban use Hida's trick to obtain exact control provided p lies outside a finite set of primes. (See [TU3] for full details.)

On the other hand, in [Tay1], Taylor studied the Siegel parabolic subgroup, corresponding to parallel weight changes, where the weight ranges through values $(k_1 + \lambda, k_2 + \lambda)$ for fixed k_1 and k_2. He obtained p-adic families simply by multiplication by suitable scalar-valued Eisenstein series; ordinary eigenforms were then recovered using boundedness results for the ordinary part ([Tay1] Prop. 2.1; see [B2] Prop. 5.1).

On applying the going-down theorem from commutative algebra, we deduce from Theorem A that any given cohomological eigenform can be placed in a Λ-adic family.

THEOREM B. *Suppose* Θ *is a system of Hecke eigenvalues occurring on the group* $W^{\circ}_{\mathbf{k},1}$. *Then there exists a local ring* $\mathcal{I}$ *finite over* Λ*, a system of Hecke eigenvalues* $\widehat{\Theta}$ *on the universal space* $\mathbf{W}$ *and valued in* $\mathcal{I}$*, and an ideal* $\wp_n$ *of* $\mathcal{I}$ *lying above* P_n*, such that*

$$\widehat{\Theta} \bmod \wp_n = \Theta.$$

Furthermore, if $(m, n') = (m, n) + \lambda(0, p-1)$ *is another weight, and* $\wp_{n'}$ *is an ideal of* $\mathcal{I}$ *lying above* $P_{n'}$ *with* $\mathcal{I}/\wp_{n'} \cong \mathbb{Z}_p$*, then*

$$\widehat{\Theta} \bmod \wp_{n'} \equiv \Theta \pmod{p}.$$

Theorem B will be proved as Theorem 5.6 at the end of this paper.

The layout of the paper is as follows. In Section 1 we review the cohomological setup of Siegel modular forms and the relevant theorems proved in [B1] and [B2]. We also define the spaces $\mathcal{W}_{\mathbf{k},r}$ etc. discussed above together with their Iwasawa and Hecke actions. In Section 2 we prove the control theorem that will allow us to recover finite level spaces from $\mathbf{W}$, assuming that the H^1 error term is zero.

Section 3 is devoted to this vanishing result. In Section 4 we show that the H^2 error is zero provided p does not lie in the exceptional set of primes. Finally, in Section 5 we deduce the main theorems; they are simply algebraic consequences of the control Theorem 2.7.

1. Notation and definitions

We begin by summarising very briefly our notation for vector-valued Siegel modular forms of degree 2. For more details, see Section 1 of [B2] or the exhaustive discussion in [God].

Throughout the paper, we fix a prime p, and an integer N prime to p. The power of p occurring in the level, r, is usually positive, but at one point we will consider level prime-to-p. We define a congruence subgroup $\Gamma(r)$ of level Np^r as follows

- for $q \nmid Np$, let $U_q = \mathrm{GSp}_4(\mathbb{Z}_q)$;
- for $q \mid N$, we allow $U_q \subset \mathrm{GSp}_4(\mathbb{Z}_q)$ to be any subgroup such that $\nu(U_q) = \mathbb{Z}_q^*$, $\mathrm{diag}(p^r, p^{2r}, p^r, 1) \in U_q$, and whenever $\gamma \in \mathrm{GSp}_4(\mathbb{Z}_q)$ is congruent to I_4 modulo N, then $\gamma \in U_q$ (i.e. basically a congruence subgroup of level N);
- for $q = p$,

$$U_p = \left\{ g \in \mathrm{GSp}_4(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^r},\quad p^{2r} | g_{42} \right\},$$

(here g_{42} denotes the $(4,2)$ matrix entry of g).

Let $U = \prod U_l$ as an open compact subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$. Then define $\Gamma(r) = U \cap \mathrm{Sp}_4(\mathbb{Z})$. In [B2], we similarly define a semigroup $\Delta(r)$ and construct a commutative Hecke algebra $\mathbb{T}_{(m,n),r}$ as the $\mathbb{Z}$-module generated by double cosets $\Gamma(r)g\Gamma(r)$ with $g \in \Delta(r)$, acting on weight (m,n) modular forms. We do not repeat the definition of $\Delta(r)$ here; the important point is that the only Hecke operator at p is $R_{p^r} = [\Gamma(r)\,\mathrm{diag}(p^r, p^{2r}, p^r, 1)\,\Gamma(r)]$.

Let $\mathcal{Z} = \{Z \in M_2(\mathbb{C}) : Z^T = Z, \mathrm{Im}(Z) > 0\}$ be the Siegel upper half-space of degree 2, which comes equipped with the usual action of $\mathrm{GSp}_4^+(\mathbb{Q})$. For $n \geqslant m \geqslant 0$, let $X = \mathrm{Sym}^{n-m}\,\mathbb{C}^2 \otimes \det^{m+n}$ be an $(n-m+1)$-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$. Then the space of Siegel cusp forms of weight (m,n) consists of holomorphic functions $F\colon \mathcal{Z} \to X$ satisfying

$$f(\gamma Z) = (CZ + D).f(Z) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(r),\quad Z \in \mathcal{Z},$$

together with a cuspidal condition. We denote it by $S_{m,n}(\Gamma(r))$; it is equipped with an action of $\mathbb{T}_{(m,n),r}$. Denote by e the projector onto the subspace spanned by forms on which R_p acts as a p-adic unit.

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Pick the standard Borel subgroup $B \subset \mathrm{Sp}(4)$; this fixes a Weyl chamber $n \geqslant m \geqslant 0$ in $\mathrm{Hom}_{\mathrm{alg}}(T(\mathbb{C}), \mathbb{C}^*) \cong \mathbb{Z}^2$ (T being the torus of diagonal matrices). Recall from the representation theory of Lie groups (see eg. [Hmph]) that there is a unique irreducible representation $V_{m,n}$ of $\mathrm{Sp}_4(\mathbb{C})$ with highest weight (m, n) in the given Weyl chamber. $V_{m,n}$ breaks up as a direct sum of weight spaces, $V_{m,n} = \oplus_{x,y} V_{m,n}^{x,y}$, so that $\mathrm{diag}(\alpha, \beta, 1/\alpha, 1/\beta)$ acts as $\alpha^x \beta^y$ on $V_{m,n}^{x,y}$. We give $V_{m,n}$ an action of $\mathrm{GSp}_4(\mathbb{C})$ by letting the centre λI_4 act as λ^n. By a standard construction, we can form the group cohomology $H^*(\Gamma(r), V_{m,n})$ equipped with an action of Hecke operators (and hence also an action of the idempotent e). There is a $\mathbb{T}_{(m,n),r}$-equivariant embedding

$$S_{m,n}(\Gamma(r)) \hookrightarrow H^3(\Gamma(r), V_{m-3,n-3}),$$

(see e.g. [Tay1] Section 2.3), and for the rest of this paper we shall be working with cohomological systems of eigenvalues. These may not always correspond to a modular form, but the above embedding allows us, for example, to obtain boundedness results for spaces of genuine modular forms.

In [B2] we showed how to construct an admissible $\mathbb{Z}$-lattice $L_{\mathbf{k}} \subset V_{\mathbf{k}}$ using the action of the universal enveloping algebra of the Lie algebra $\mathfrak{sp}_4$ on $V_{\mathbf{k}}$. Then the main theorem of [B2] states that

THEOREM 1.1. *For* $0 \leqslant i \leqslant 6$, *we have an isomorphism of* $\mathbb{T}_{(m,n),r}$*-modules*

$$\begin{aligned} &eH^i(\Gamma(r), L_{m,n} \otimes \mathbb{Z}/p^r\mathbb{Z}) \otimes \chi_r^{-n} \\ &\quad \cong eH^i(\Gamma(r), L_{m,n+1} \otimes \mathbb{Z}/p^r\mathbb{Z}) \otimes \chi_r^{-n-1}. \end{aligned}$$

Here χ_r is the character of $\mathrm{GSp}_4(\mathbb{Z})$ given by sending a matrix to its bottom right-hand entry modulo p^r.

Now we define

$$\mathcal{W}_{\mathbf{k},r} := \lim_{\substack{\longrightarrow \\ t}} H^3(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Z}/p^t\mathbb{Z}).$$

(We will later also consider this limit in degree two, but this is the group we are really interested in.) Then we have $\mathcal{W}_{\mathbf{k},r} = H^3(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$; this can be seen from the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/p^t\mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r, "p^t"] \arrow[d, equal] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/p^{t+1}\mathbb{Z} \arrow[r, hook] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r, "p^{t+1}"] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

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on taking derived functors $H^*(\Gamma(r), L_{\mathbf{k}} \otimes -)$, as follows (where we have abbreviated $H^*(\Gamma(r), L_{\mathbf{k}} \otimes A)$ to $H^*(A)$)

$$\begin{tikzcd}
H^2(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "p^t"] \arrow[d, equal] & H^2(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d, "p"] & H^3(\mathbb{Z}/p^t\mathbb{Z}) \arrow[r] \arrow[d] & H^3(\mathbb{Q}_p/\mathbb{Z}_p)[p^t] \arrow[r] \arrow[d] & 0 \\
H^2(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "p^{t+1}"] & H^2(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d, "p"] & H^3(\mathbb{Z}/p^{t+1}\mathbb{Z}) \arrow[r] \arrow[d] & H^3(\mathbb{Q}_p/\mathbb{Z}_p)[p^{t+1}] \arrow[r] \arrow[d] & 0 \\
 & H^2(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] & H^3(\mathbb{Z}/p^{t+2}\mathbb{Z}) \arrow[r] \arrow[d] & H^3(\mathbb{Q}_p/\mathbb{Z}_p)[p^{t+2}] \arrow[r] \arrow[d] & 0 \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & \varinjlim_t H^3(\mathbb{Z}/p^t\mathbb{Z}) \arrow[r] & H^3(\mathbb{Q}_p/\mathbb{Z}_p). &
\end{tikzcd}$$

Diagram-chasing gives the injectivity and surjectivity.

Let $\mathcal{W}^\circ_{\mathbf{k},r} = e\mathcal{W}_{\mathbf{k},r}$ be the ordinary component of $\mathcal{W}_{\mathbf{k},r}$. Define the direct limit

$$\mathcal{W}^\circ_{\mathbf{k}} = \varinjlim_r \mathcal{W}^\circ_{\mathbf{k},r} = \varinjlim_r eH^3(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)),$$

where the transition maps are simply restriction morphisms of cohomology groups. If we also define $\mathcal{W}'_{\mathbf{k},r} = eH^3(\Gamma(r), L_{\mathbf{k}}(\mathbb{Z}/p^r\mathbb{Z}))$, we have

$$\mathcal{W}^\circ_{\mathbf{k}} = \varinjlim_r \varinjlim_t eH^3(\Gamma(r), L_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})) = \varinjlim_r \mathcal{W}'_{\mathbf{k},r},$$

(now the transition maps are restriction composed with embedding $\mathbb{Z}/p^r\mathbb{Z}$ into $\mathbb{Z}/p^s\mathbb{Z}$).

We have Pontryagin dual modules

$$W^\circ_{\mathbf{k},r} = \mathrm{Hom}(\mathcal{W}^\circ_{\mathbf{k},r}, \mathbb{Q}_p/\mathbb{Z}_p)$$

which behave well with respect to limits

$$W^\circ_{\mathbf{k}} = \varprojlim_r W^\circ_{\mathbf{k},r} \cong \mathrm{Hom}(\mathcal{W}^\circ_{\mathbf{k}}, \mathbb{Q}_p/\mathbb{Z}_p).$$

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We now discuss giving these objects a Λ-module structure. Define a new congruence subgroup $\Gamma_0(r)$ adelically as above, with the same conditions as before at $q \mid N$ and $q \nmid Np$, but replacing the condition at p by

$$U_p = \left\{ g \in \mathrm{GSp}_4(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{p^r}, \quad p^{2r} | g_{42} \right\}.$$

Also, for $r \geqslant s \geqslant 1$, let $\Phi_r^s = \Gamma(s) \cap \Gamma_0(r)$. So we have $\Gamma(r) \triangleleft \Phi_r^s \subset \Gamma(s)$ and

$$\Phi_r^s / \Gamma(r) \cong G_s / G_r \cong \mathbb{Z}/p^{r-s}\mathbb{Z},$$

where we have written $G_n := (1 + p^n \mathbb{Z}_p)^*$.

Also $\Gamma(r) \triangleleft \Gamma_0(r)$ and

$$\Gamma_0(r)/\Gamma(r) \cong (\mathbb{Z}/p^r\mathbb{Z})^* \quad \text{via}$$
$$[M] \mapsto M_{4,4} \pmod{p^r}.$$

Then we have an action of $(\mathbb{Z}/p^r\mathbb{Z})^*$ on $\mathcal{W}_{\mathbf{k},r}$ by letting $d \in (\mathbb{Z}/p^r\mathbb{Z})^*$ act as σ_d, where σ_d is a representative of d in $\Gamma_0(r)$. Hence there is an action of

$$\varprojlim_r (\mathbb{Z}/p^r\mathbb{Z})^* \cong \mathbb{Z}_p^* \cong (\mathbb{Z}/p\mathbb{Z})^* \times (1 + p\mathbb{Z}_p)^*$$

on $\mathcal{W}_{\mathbf{k}}^{\circ}$. Regarding $G_1 = (1 + p\mathbb{Z}_p)^*$ as a subgroup of $\mathbb{Z}_p^*$, and letting $\mathbb{Z}_p$ act simply by multiplication on the coefficients, we see that $\mathcal{W}_{\mathbf{k}}^{\circ}$ becomes a continuous module under the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[G_1]]$.

Pick a topological generator u of G_1, for example $u = 1 + p$. Then Λ is isomorphic to the one-variable Iwasawa algebra $\mathbb{Z}_p[[X]]$ via $u \leftrightarrow 1 + X$.

There are specialisation maps

$$s_a : \mathbb{Z}_p[[X]] \cong \mathbb{Z}_p[[G_1]] \to \mathbb{Z}_p$$
$$1 + X \leftrightarrow u \mapsto (1+p)^a \quad (a \in \mathbb{N}).$$

The kernel of s_a is the height one prime ideal of Λ generated by

$$P_a = u - (1+p)^a \in \Lambda;$$

more generally, let

$$P_{a,s} = u^{p^s} - (1+p)^{ap^s}.$$

Let $g_K = \mathrm{diag}(K, K^2, K, 1)$. We will be referring to the following decomposition of the Hecke operator R_p (see Corollary 2.2.6 of [B1]; note the condition $g_p \in U_q$ in the definition).

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LEMMA 1.2. *Suppose* $r, s > 0$. *Then we have a coset decomposition*

$$[\Gamma(r)g_{p^s}\Gamma(r)] = \coprod \alpha_u\Gamma(r),$$

where

$$\alpha_u = \begin{pmatrix} 1 & 0 & 0 & z \\ t & 1 & z & w \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} g_{p^s}.$$

Here z, t *run through residue classes modulo* p^s *and* w *runs through residue classes modulo* p^{2s} *and the representatives are all chosen to be congruent to zero modulo* N. $\square$

It follows from this that the Hecke operators are compatible with restriction maps between spaces of different levels, so we obtain operators $T = \varprojlim_r T^{(r)}$ acting on $\mathcal{W}_{\mathbf{k}}$, by the universal property of inverse limits. We define $\mathbb{T}_{\mathbf{k}}$ to be the Λ-subalgebra of the endomorphism ring of $\mathcal{W}_{\mathbf{k}}$ generated by all these operators. It will be seen as a consequence of Corollary 2.7 that $\mathcal{W}_{\mathbf{k},s} \hookrightarrow \mathcal{W}_{\mathbf{k},r}$ for $r \geqslant s$, so $\mathcal{W}_{\mathbf{k},r} \hookrightarrow \mathcal{W}_{\mathbf{k}}$ and $\mathbb{T}_{\mathbf{k}} \twoheadrightarrow \mathbb{T}_{\mathbf{k},r}$.

We write $\mathcal{W}_{\mathbf{k}}(a)$ for the Λ-module obtained from $\mathcal{W}_{\mathbf{k}}$ by twisting the Hecke action by the ath power of $\varprojlim_r \chi_r$, and the Λ-action by the ath power of the tautological character $G_1 \to \mathbb{Z}_p^*$. Notice that at finite level p^r, this induces the character χ_r^a of Theorem 1.1 (the action of $\Gamma_0(r)$ on lattices $L_{m,n}$ and $L_{m,n+1}$ differs by precisely this character). This twisting is essential to our theory, since it removes differences in the action on different weights. Indeed, Theorem 1.1 now says that

$$\mathcal{W}'_{(m,n_1),r}(-n_1) \cong \mathcal{W}'_{(m,n_2),r}(-n_2)$$

as Λ-modules and as Hecke modules.

We check that the isomorphism of Theorem 1.1 is compatible with restriction between different levels. Thus on taking direct limits, we can conclude

COROLLARY 1.3. *We have* $\mathcal{W}^\circ_{(m,n_1)}(-n_1) \cong \mathcal{W}^\circ_{(m,n_2)}(-n_2)$ *as modules for the Hecke algebra*.

There is a Λ-action on the dual $W_{\mathbf{k}}$ in the usual way

$$(uf)(\phi) = f(u\phi) \quad (u \in G_1,\ \phi \in \mathcal{W}_{\mathbf{k}}, f \in W_{\mathbf{k}}).$$

Twisted objects have twisted duals: the dual of $\mathcal{W}^\circ_{\mathbf{k},r}(a)$ is $W^\circ_{\mathbf{k},r}(a)$, and that of $\mathcal{W}^\circ_{\mathbf{k}}(a)$ is $W^\circ_{\mathbf{k}}(a)$. A result analogous to Corollary 1.3 holds for the dual spaces $W^\circ_{\mathbf{k}}(-n)$.

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We can therefore define an object **W** $:= W^{\circ}_{m,n}(-n)$ independent of n. Our aim is to determine hypotheses under which **W** is a finite free Λ-module, and to recover the spaces $\mathcal{W}^{\circ}_{\mathbf{k},s}$ from $W^{\circ}_{(m,n)}$ for any finite level p^s.

2. Control theorem

The following identity of double cosets is analogous to Lemma 4.3(i) in [Hi1]. We omit the proof, which is a straightforward matrix calculation (see Lemma 3.2.1 in [B1]).

LEMMA 2.1. *Let* $r \geqslant s \geqslant 1$. *Then* $\Phi^s_r\, g_p^{r-s}\Phi^s_r = \Gamma(s) g_p^{r-s}\Phi^s_r$. □

To recover finite levels from our direct limit, we first need to compare $\mathcal{W}^{\circ}_{\mathbf{k},r}$ and $\mathcal{W}^{\circ}_{\mathbf{k},s}$ for all $r \geqslant s$. The first step is given by the next elementary lemma on Hecke operators, which is modelled on [Hi1] Lemma 4.3(ii).

LEMMA 2.2. *Let* $r \geqslant s \geqslant 1$ *and* $0 \leqslant q \leqslant 6$. *Then restriction of cocycles induces an isomorphism of Hecke modules*

$$eH^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = eH^q(\Phi^s_r, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)).$$

Proof. Consider the diagram

$$\begin{tikzcd}
H^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) \arrow[r, "\text{res}"] \arrow[d, "R_p^{r-s}"] & H^q(\Phi^s_r, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) \arrow[d, "R_p^{r-s}"] \arrow[dl] \\
H^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) \arrow[r, "\text{res}"] & H^q(\Phi^s_r, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)),
\end{tikzcd}$$

where the diagonal arrow is given by the Hecke operator $[\Gamma(s)g_p^{r-s}\Phi^s_r]$, and the horizontal maps are restrictions of cocycles.

In the notation of Lemma 1.2 above, we have

$$R_p^{r-s} = [\Gamma(s)g_p^{r-s}\Gamma(s)] = \coprod_u \alpha_u\Gamma(s).$$

On the other hand, by Lemma 2.1, we have that

$$\begin{aligned}
[\Gamma(s)g_p^{r-s}\Phi^s_r] &= [\Phi^s_r g_p^{r-s}\Phi^s_r] \\
&= \coprod \alpha_u\Phi^s_r
\end{aligned}$$

for the same representatives α_u. (This follows from Lemma 1.2, as $\Gamma(s) \cap g_p^{s-r}\Phi^s_r g_p^{r-s} \subset \Phi^s_r$.)

Let $[\phi] \in H^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))$. Then for $(g_1, \ldots, g_q) \in \Gamma(s)^q$, choose $v_1 = v_1(u)$ such that $\alpha_u^{-1}g_1\alpha_{v_1} \in \Phi^s_r$, choose $v_2 = v_2(u)$ such that $\alpha_{v_1}^{-1}g_2\alpha_{v_2} \in \Phi^s_r$, etc,

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up to $v_q = v_q(u)$. We have

$$\begin{aligned}
&((\text{res}\,\phi)|[\Phi_r^s g_p^{r-s}\Gamma(s)])(g_1,\dots,g_3)\\
&\quad = \sum_u \alpha_u(\text{res}\,\phi)(\alpha_u^{-1}g_1\alpha_{v_1},\dots,\alpha_{v_{q-1}}^{-1}g_q\alpha_{v_q})\\
&\quad = \sum_u \alpha_u\phi(\alpha_u^{-1}g_1\alpha_{v_1},\dots,\alpha_{v_{q-1}}^{-1}g_q\alpha_{v_q})\\
&\quad = (\phi|R_p^{r-s})(g_1,\dots,g_3),
\end{aligned}$$

i.e. the left-hand triangle commutes. Similarly the right-hand triangle commutes.

Then on the ordinary components, the vertical maps are isomorphisms. Hence so are the horizontal maps. □

THEOREM 2.3. *Suppose that* $eH^i(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p) = 0$ *for all* $i < q$, *and for all* $r \geqslant 1$. *Then for* $s \geqslant 1$ *and* $G_s = (1+p^s\mathbb{Z}_p)^*$

$$\left(\varinjlim_r eH^q(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))\right)^{G_s} = eH^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)).$$

Proof. Note that it suffices to prove that for $r \geqslant s \geqslant 1$

$$eH^q(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))^{G_s} = eH^q(\Phi_r^s, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)).$$

Indeed, combining this statement with Lemma 2.2 gives us $eH^q(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))^{G_s} = eH^q(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))$, for $r \geqslant s$. On taking the limit over r, we obtain the theorem.

To prove the above statement, we consider the Serre–Hochschild spectral sequence

$$H^i(\Phi_r^s/\Gamma(r), H^j(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))) \Rightarrow H^{i+j}(\Phi_r^s, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)).$$

By the argument preceeding Theorem 9.1 in [Hi2], the differential maps in the spectral sequence are compatible with our Hecke operators. A Hecke operator $[\Phi_r^s g\Phi_r^s]$ acts on $H^i(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))$ as $[\Gamma(r)g\Gamma(r)]$, compatibly with the action of $\Phi_r^s/\Gamma(r)$. Thus it induces an endomorphism of the cohomology groups at stage $E_2^{i,j}$, in other words the $H^i(\Phi_r^s/\Gamma(r), eH^j(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)))$, and we get

$$\begin{aligned}
&H^i(\Phi_r^s/\Gamma(r), eH^j(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)))\\
&\quad \Rightarrow eH^{i+j}(\Phi_r^s, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)).
\end{aligned}$$

Then it is a consequence of the hypotheses that the spectral sequence degenerates to give $eH^q(\Phi_r^s, L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) \cong eH^q(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))^{\Phi_r^s/\Gamma(r)}$. On the other hand, it is easy to see that $\Phi_r^s/\Gamma(r) \cong (1+p^s\mathbb{Z}_p)^*/(1+p^r\mathbb{Z}_p)^* = G_s/G_r$, and the theorem follows. □

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We will now demonstrate the vanishing conditions of the theorem as far as possible.

LEMMA 2.4. $eH^0(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = 0$.

Proof. To examine the effect of e on $H^0(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = \varinjlim_{t} L_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})^{\Gamma(r)}$, we use the coset decomposition of R_p given in Lemma 1.2. This gives that for $m \in L_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})$,

$$m \,|\, R_p = \sum_{z,w,t} \begin{pmatrix} 1 & 0 & 0 & z \\ t & 1 & z & w \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} g_p m.$$

Now on all but the $(*, -n)$ weight spaces in $V_{m,n}$, g_p itself acts by multiplication by a positive power of p. On the other hand, when restricted to $(L_{\mathbf{k}} \cap \oplus_x V_{x,-n}) \otimes \mathbb{Z}/p^t\mathbb{Z}$, each of the above coset representatives acts merely as the identity (cf. Section 3 in [B2]: one can factorise them into $\exp W_\alpha$, where the W_α are elements of a Chevalley basis for $\mathfrak{sp}_4$, whose action is easily described). Thus, for $m \in V_{m,n}^{*,-n}$, $m \,|\, R_p = p^4 m + m'$ with $m' \in \oplus_{y>-n} V^{x,y}$, and m has been killed after at most t applications of R_p. □

PROPOSITION 2.5. $eH^1(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = 0$.

This result will be proved in the next section. Here we continue with the proof of our control theorem.

PROPOSITION 2.6. *Assume the condition* (H) *holds for* p. *Then* $eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = 0$.

Proposition 2.6 will be proved in Section 4. Note that its proof will in fact rely on Theorem 2.3 with $q = 2$, i.e. using Lemma 2.4 and Proposition 2.5.

Remark. One might expect the conclusion of Proposition 2.6 to hold for all primes, if the projector e is replaced by something stronger: let $\mathfrak{m}_\theta \lhd \mathbb{T}_{\mathbf{k},r}$ be a maximal ideal corresponding to a system of eigenvalues $\theta\colon T_{\mathbf{k},r} \to \mathbb{Q}_p$ (i.e. $\mathfrak{m}_\theta = \mathrm{Ker}(\theta)$) occurring on the ordinary component $eH^3(\Gamma(r), L_{\mathbf{k}}(\mathbb{Z}_p))$. Let $\overline{\theta}$ be the mod p reduction of θ. Weissauer has proved (see [W]) that there is an associated Galois representation $\rho_\theta\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$ which is unramified outside p, and such that the characteristic polynomial of $\rho_\theta(\mathrm{Frob}_q)$ is equal to $\overline{\theta}(Q_q(X))$, where $Q_q(X)$ is the relevant Hecke polynomial.

Suppose θ is such that ρ_θ is *irreducible*. Then by analogy with the GL_2 case, it is reasonable to assume that θ does not occur on $eH^i(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))$ for degrees $i \neq 3$. This would mean that the localisation $eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))_{\mathfrak{m}_\theta}$ vanishes.

This is certainly true for the torsion-free case: [Tay2], Proposition 2 says that any representation coming from $H^2(\Gamma\backslash\mathcal{Z}_2, \mathcal{V}_{m,n}(\mathbb{Q}_p))$ can only have irreducible

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constituents of dimension at most two – for any subrepresentation, we can obtain another one as a suitable twist by the cyclotomic character. However, it is not clear how to prove an analogous result for torsion coefficients.

As the Hecke algebra $\mathcal{H} = \mathbb{T}_{\mathbf{k},r}$ is semilocal, there is a projector $e_{\mathfrak{m}_\theta}$ such that

$$e_{\mathfrak{m}_\theta}\,\mathcal{H} = e_{\mathfrak{m}_\theta}\left(\bigoplus_{\mathfrak{m}\lhd\mathcal{H}} \mathcal{H}_{\mathfrak{m}}\right) = \mathcal{H}_{\mathfrak{m}_\theta}.$$

Now $H^2(\mathbb{Z}_p)_{\mathfrak{m}_\theta} = (\mathcal{H})_{\mathfrak{m}_\theta} \otimes_{\mathcal{H}} H^2(\mathbb{Z}_p)$, and to require that θ does not occur in degree two is the same as demanding $e_{\mathfrak{m}_\theta} H^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = 0$. This is clearly weaker than assuming $e\,H^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) = 0$: e projects onto the sum of all localisations at maximal ideals which do not contain R_p, whereas $e_{\mathfrak{m}_\theta}$ projects onto just one of these (certainly $R_p \notin \mathfrak{m}_\theta$). So $ee_{\mathfrak{m}_\theta} = e_{\mathfrak{m}_\theta} e = e_{\mathfrak{m}_\theta}$.

We can now deduce our key result on infinite level cohomology groups.

COROLLARY 2.7. *Suppose the condition (H) holds. Then for $s \geqslant 1$ and $G_s = (1 + p^s\mathbb{Z}_p)^*$,*

$$(\mathcal{W}_{\mathbf{k}}^\circ)^{G_s} = eH^3(\Gamma(s), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)). \qquad \square$$

3. Proof of Proposition 2.5

The plan of attack is as follows: first we apply the congruence subgroup property to reduce the proposition to a statement about the cohomology of finite groups. We then break this up into pieces on which the action of R_p is sufficiently simple to prove our result; first we break off a prime-to-p part, then we split the remainder into Levi-like and unipotent components.

It suffices to prove that $eH^1(\Gamma(r), L_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})) = 0$ for all finite t. Now $\alpha_u = \gamma_u.g_p$ (see Lemma 1.2) and g_p will act as a factor of p^{n+y} on a weight space $V^{x,y}$ in L. So without loss of generality, we can replace the coefficients L by $L' = \oplus_x(V^{x,-n} \cap L)$ where α_u acts trivially. L' is an SL_2-module, but modulo p^t it may cease to be irreducible. For the rest of this section, abbreviate $L'_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})$ to L and $\Gamma(r)$ to Γ. The action of R_p on a 1-cocycle ϕ in $H^1(\Gamma, L)$ is then given by

$$(\phi\,|\,R_p)(\gamma) = \sum_u \phi(\alpha_u^{-1}\gamma\alpha_v),$$

where for each u, v is chosen so that $\alpha_u^{-1}\gamma\alpha_v \in \Gamma$. The action of Γ on L factors modulo p^t ([B2] Lemma 3.2), and also (at least for $r \geqslant t$) through the projection

$$\begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \pmod{p^r} \mapsto \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \pmod{p^r}.$$

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We first show

LEMMA 3.1. *Let* Γ_M *be the principal congruence subgroup of level* M. *Then*

$$H^1(\Gamma, L) = \bigcup_{p^r N | M \in \mathbb{N}} \inf_{\Gamma/\Gamma_M}^{\Gamma} H^1(\Gamma/\Gamma_M, L^{\Gamma_M}).$$

Proof. Consider a class $[\phi] \in H^1(\Gamma, L)$ and pick a representative cocycle $\phi \in [\phi]$. Let $\Omega \subset \Gamma$ be the kernel of ϕ, i.e. the set $\{g \in \Gamma : \phi(g) = 0\}$. Ω has finite index in Γ because L is finite. Now we appeal to the congruence subgroup property for the symplectic group – see Satz 10, Corollary 3 in [Menn], or [BMS] for a more general treatment. Thus there is an integer M such that $\Gamma_M \subset \Omega$ is a principal congruence subgroup. Then ϕ induces

$$\overline{\phi} : \Gamma/\Gamma_M \to L.$$

Clearly $\inf(\overline{\phi}) = \phi$ and so $[\inf(\overline{\phi})] = [\phi]$, even though Ω depends on the choice of ϕ. Also as $\Gamma_M \subset \Gamma$ we have $p^r N \mid M$, by minimality of the level $p^r N$. □

The Hecke action on the mod M cohomology groups will be defined compatibly with inflation. Hence the Hida idempotent e associated to $[\Gamma g_p \Gamma]$ will be compatible with inflation, too.

Suppose $M = p^c K$, $K = \prod_{l \neq p} l^{c_l}$, where $c \geqslant r$ and $N \mid K$. Write $\Gamma_n' = \Gamma_n \cap \Gamma$ for $n \in \mathbb{N}$. Then

$$\Gamma/\Gamma_M \cong \Gamma/\Gamma_{p^c}' \times \Gamma/\Gamma_K'$$

by the Chinese remainder theorem, using Lemma 3.3.2 of [A]: given $([g], [h])$ on the right, lift each to $\mathrm{Sp}_4(\mathbb{Z})$ and form $X = ga + hb \in M_4(\mathbb{Z})$ where $a, b \in M_4(\mathbb{Z})$, and $a \equiv 1\ (p^c), a \equiv 0\ (K), b \equiv 0\ (p^c), b \equiv 1\ (K)$. Then $X \bmod M \in \mathrm{Sp}_4(\mathbb{Z}/M\mathbb{Z})$ has a lift $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$ satisfying $\gamma \equiv g \pmod{p^c}$ and $\gamma \equiv h \pmod{K}$, so $\gamma \in U_q$ for all $q \mid N$ and $\gamma \in U_p$, i.e. $\gamma \in \Gamma$.

Now if $g \equiv I \pmod{p^c}$ then g acts trivially on L^{Γ_M}, and so we have a decomposition of cohomology groups

$$H^1(\Gamma/\Gamma_M, L^{\Gamma_M}) \cong H^1(\Gamma/\Gamma_{p^c}', L^{\Gamma_{p^c}}) \times H^1(\Gamma/\Gamma_K', L^{\Gamma_K}),$$
$$\phi \mapsto (\phi|_{\Gamma/\Gamma_{p^c}'}, \phi|_{\Gamma/\Gamma_K'}),$$
$$\psi \leftarrow\!\shortmid (\psi_1, \psi_2),$$

where $\psi(\gamma) = \psi_1(\gamma \bmod p^c) + (\gamma \bmod p^c, I).\psi_2(\gamma \bmod K)$.

So consider $H^1(\Gamma/\Gamma_K', L^{\Gamma_K})$. We will show this is not ordinary. Choose a k such that $p^k \equiv 1 \pmod{K}$ and pick representatives $\alpha_u = \gamma_u g_p^k$ for R_p^k such that $\gamma_u \equiv I_4 \pmod{K}$ for each u. We define the action of R_p^k on cocycles $\phi \in Z^1(\Gamma/\Gamma_K', L)$ by

$$(\phi \mid R_p^k)(\overline{\gamma}) = \sum \alpha_u \phi(\overline{\gamma}) \quad (\gamma \in \Gamma),$$

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where the bar denotes reduction modulo K. This is manifestly independent of the choice of lift γ of an element of Γ/Γ'_K.

Furthermore, we must check that the action commutes with inflation from Γ/Γ'_K to Γ. So suppose we are given $\gamma \in \Gamma$; let $v = v(u)$ be the unique index of the coset representatives satisfying $\alpha_u^{-1}\gamma\alpha_v \in \Gamma$. Then

$$\begin{aligned}((\inf\phi) \mid R_p^k)(\gamma) &= \sum_u \alpha_u(\inf\phi)(\alpha_u^{-1}\gamma\alpha_v)\\ &= \sum_u \alpha_u\phi(\overline{\alpha_u^{-1}\gamma\alpha_v})\\ &= \sum_u \alpha_u\phi(\overline{\gamma})\\ &= (\inf(\phi \mid R_p^k))(\gamma).\end{aligned}$$

The third equality above can be seen by considering all the matrices modulo K; we see that with our choices for the coset representatives and k we have $\alpha_u^{-1}\gamma\alpha_v \equiv \gamma \pmod{l}$, and ϕ takes the same value on both.

But then

$$(\phi \mid R_p^k)(\overline{\gamma}) = \sum_u (\gamma_u g_{p^k}\phi(\overline{\gamma})).$$

Again, g_{p^k} will act as a power of p unless $\phi(\overline{\gamma}) \in V^{?,-n}$, in which case γ_u acts as the identity. So as in the proof of Lemma 2.4, repeated application of R_p will eventually multiply by p^t, and so e kills ϕ.

We now turn to $H^1(\Gamma/\Gamma'_{p^c}, L^{\Gamma'_{p^c}})$. This group is not preserved by R_p; instead $R_p\colon H^1(\Gamma/\Gamma'_{p^c}, L^{\Gamma'_{p^c}}) \to H^1(\Gamma/\Gamma'_{p^{c+2}}, L^{\Gamma'_{p^{c+2}}})$: if ψ is a cocycle then $\psi \mid R_p \in Z^1(\Gamma/\Gamma'_{p^{c+2}}, L)$ is given by

$$(\psi \mid R_p)(\gamma \bmod p^{c+2}) = \sum \alpha_u\psi(\alpha_u^{-1}\gamma\alpha_v \bmod p^c).$$

One can see that the right-hand side is well-defined independently of the choice of representative for γ modulo p^{c+2} (the problem is it does depend on γ modulo p^c). The R_p-action is automatically compatible with inflation

$$\begin{tikzcd} \psi \arrow[r, "R_p"] \arrow[d, "\inf^{\Gamma}_{\Gamma/\Gamma'_{p^c}}"] & \psi \mid R_p \arrow[d, "\inf^{\Gamma}_{\Gamma/\Gamma'_{p^{c+2}}}"] \\ (\inf\psi\colon \Gamma \to L) \arrow[r, "R_p"] & (\inf\psi) \mid R_p \end{tikzcd}$$

and for $\gamma \in \Gamma$

$$\begin{aligned}((\inf \psi) \mid R_p)(\gamma) &= \sum_u \alpha_u (\inf \psi)(\alpha_u^{-1} \gamma \alpha_v) \\ &= \sum_u \alpha_u \psi(\alpha_u^{-1} \gamma \alpha_v \bmod p^c) \\ &= (\psi \mid R_p)(\gamma \bmod p^{c+2}) \\ &= (\inf(\psi \mid R_p))(\gamma).\end{aligned}$$

Recall that the Klingen parabolic subgroup of $\mathrm{Sp}(4)$ is the semidirect product of its normal subgroup U, which consists of matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ with $A = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)$, and $D = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, and a Levi component isomorphic to $\mathrm{GL}_2 \times \mathbb{G}_m$. Motivated by this, we define

$$S_r = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}/p^r\mathbb{Z}) : \gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & e & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & e' \end{pmatrix} \right\}$$

and consider the projection map $\Gamma/\Gamma'_{p^c} \xrightarrow{\pi} S_r$, given by picking out the corresponding entries from a matrix and reducing modulo p^r. This is well-defined as $c \geqslant r$, and it is a homomorphism. Its image is $\widetilde{S}$ (say), which is independent of c, and its kernel is

$$U_c = \left\{ \gamma \in \Gamma/\Gamma'_{p^c} : \gamma \equiv \begin{pmatrix} 1 & 0 & 0 & * \\ * & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^r} \right\}.$$

(Again this is a good definition as $c \geqslant r$).

The idea here is that the action of U_c on L is nearly trivial, whereas on $H^1(\widetilde{S}, L)$ we can define a convenient action of R_p. We have the inflation-restriction sequence arising from $0 \to U_c \hookrightarrow \Gamma/\Gamma'_{p^c} \xrightarrow{\pi} \widetilde{S} \to 0$

$$0 \to H^1(\widetilde{S}, L^{U_c}) \xrightarrow{\inf} H^1(\Gamma/\Gamma'_{p^c}, L) \to H^1(U_c, L),$$

which allows us to consider the 'unipotent' and 'Levi-like' parts separately.

We define the Hecke action on $H^1(\widetilde{S}, L)$ by

$$(\phi \mid R_p)(\gamma) = \sum_u \alpha_u \phi(\gamma) \quad (\phi \in Z^1(\widetilde{S}, L),\ \gamma \in \widetilde{S}).$$

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This is also compatible with inflation: $\inf_{\tilde{S}}^{\Gamma/\Gamma'_{p^c}}(\phi)$ lies in $Z^1(\Gamma/\Gamma'_{p^c}, L)$ and $(\inf_{\tilde{S}}^{\Gamma/\Gamma'_{p^c}}\phi)\,|\,R_p$ lies in $Z^1(\Gamma/\Gamma'_{p^{c+2}}, L)$; let g be a lift of γ to $\Gamma/\Gamma'_{p^{c+2}}$ and $\widehat{g}$ a lift of g to Γ. Then

$$\begin{aligned}((\inf\phi)\,|\,R_p)(g) &= \sum_u \alpha_u(\inf\phi)(\alpha_u^{-1}\widehat{g}\alpha_v \bmod p^c)\\ &= \sum_u \alpha_u\phi(\pi(\alpha_u^{-1}\widehat{g}\alpha_v \bmod p^c))\\ &= \sum_u \alpha_u\phi(\pi(\widehat{g} \bmod p^c))\\ &= \sum_u \alpha_u\phi(\pi(g))\\ &= (\inf(\phi\,|\,R_p))\,(g).\end{aligned}$$

Again, the third equality follows from explicit computation.

So we get

$$\phi\,|\,R_p = \left(\sum \alpha_u\right)\phi = p^{?}\,\phi$$

and as before e annihilates all cocycles.

The remaining term is $H^1(U_c, L)$. The Hecke action on the image of restriction in $H^1(U_c, L)$ is inherited from that on $H^1(\Gamma/\Gamma_{p^{c+2}}, L)$.

By definition of U_c, any cocycle $\phi \in Z^1(U_c, L)$ is a homomorphism modulo p^r, i.e. for $x, y \in U_c$

$$\phi(xy) = \phi(x) + \phi(y) + p^r(\lambda X_{2,0}^k + \mu X_{-2,0}^l)\phi(y)$$

for some $\lambda, \mu, k, l \in \mathbb{Z}$, where $X_{2,0}$ and $X_{-2,0}$ are the two non-diagonal elements of the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$. Assume for now that we know $\phi\,|\,R_p^r = 0$ if ϕ is a homomorphism. Then in general

$$(\phi\,|\,R_p^r)(\gamma) = p^r \sum_{?\in U_u} (\lambda_? X_{2,0}^{k_?} + \mu_? X_{-2,0}^{l_?})\;\phi(?)\quad(\gamma \in U_{c+2r})$$

and $(\phi\,|\,R_p^{rt})(\gamma) = 0$ in $L_{\mathbf{k}}(\mathbb{Z}/p^t\mathbb{Z})$.

It remains to verify the above assumption. So take $\phi \in \mathrm{Hom}(U_c, L)$ and $\gamma \in \Gamma$ a lift of $\overline{\gamma} \in U_{c+2r}$. Since L is just an abelian group, ϕ must vanish on the commutator subgroup of U_c.

We have

$$\begin{aligned}(\phi\,|\,R_p^r)(\overline{\gamma}) &= \sum \phi(\overline{\alpha_u^{-1}\gamma\alpha_v})\\ &= \phi\left(\prod(\overline{\alpha_u^{-1}\gamma\alpha_v})\right).\end{aligned}$$

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Using the fact that $\sum_{0\leqslant z<p^r} z$ is divisible by p^r, etc., we can compute this product as

$$A =: \prod_u (\overline{\alpha_u^{-1}\gamma\alpha_v}) \equiv \begin{pmatrix} 1 & 0 & 0 & p^r* \\ p^r* & 1 & p^r* & * \\ 0 & 0 & 1 & p^r* \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^{2r}}.$$

Now ϕ is trivial modulo p^c; so we wish to show that A differs from a commutator only by the identity mod p^c.

Firstly, the commutator subgroup is quite large

$$\left[\begin{pmatrix} 1 & p^r\delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p^r\delta & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & p^{2r}\delta^2 & p^r\delta \\ 0 & 1 & p^r\delta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & p^r\delta & 1 & 0 \\ p^r\delta & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^r\delta & 1 & 0 & 0 \\ p^{2r}\delta^2 & 0 & 1 & -p^r\delta \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & 1 & 0 & 0 \\ 0 & 0 & 1 & -\delta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & 1 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and in each case we can choose a δ to ensure that these commutators satisfy some congruence condition modulo the auxiliary level N of Γ, and so lie in U_c. So modulo commutators, $A = I + p^{2r}A_1$. But the Lie algebra $\mathfrak{sp}_4$ is simple; so if $I + p^{2r}A_1 \in \mathrm{Sp}_4(\mathbb{Z})$ then A_1 mod p^r is a sum of commutators $\sum[a_i, b_i]$ in $\mathfrak{sp}_4$.

Therefore

$$\begin{aligned} \prod I + p^{2r}A_1 &\equiv I + p^{2r}\sum[a_i, b_i] \pmod{p^{3r}} \\ &\equiv \prod[I + p^r a_i, I + p^r b_i] \pmod{p^{3r}} \end{aligned}$$

i.e. $(I+p^{2r}A_1)(I+p^{3r}A_2)$ is a product of commutators. Repeating this process, we obtain that $A(I + p^c A_c)$ is nothing but a product of commutators. Thus $\phi(A) = 0$, hence $\phi \mid R_p^r = 0$.

This completes the proof of the proposition.

4. Proof of Proposition 2.6

The argument runs along the same lines as Section 7 of [Hi3]: we will show that the error term $eH^2(\Gamma(r), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is isomorphic to the p-torsion in a fixed finite

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group, so that it is zero for almost all p. The following lemma neatly encapsulates the calculations on cocycles that are required

LEMMA 4.1 ([Tay1] Lemma 1.1). *Let Δ be a semi-group; $\Gamma_1 \supset \Gamma_2$ subgroups of Δ; $g \in \Delta$ with $[\Gamma_1 : \Gamma_1 \cap g\Gamma_2 g^{-1}] < \infty$; M_1 (resp. M_2) a module for $\langle \Gamma_1, g\rangle$ (resp. $\langle \Gamma_2, g\rangle$); and $j\colon M_1 \to M_2$ a $\langle \Gamma_2, g\rangle$ morphism such that $j\colon gM_1 \xrightarrow{\sim} gM_2$. Suppose*

(1) *there exist elements $\gamma_i \in \Gamma_1$ such that $\Gamma_1 g \Gamma_1 = \coprod \gamma_i g \Gamma_1$ and $\Gamma_1 g \Gamma_2 = \coprod \gamma_i g \Gamma_2$, and*
(2) *there exist elements $\delta_i \in \Gamma_1$ with $\Gamma_1 g \Gamma_2 = (\Gamma_2 g \Gamma_2) \amalg (\coprod \Gamma_2 \delta_i g \Gamma_2)$, and such that $j\delta_i g M_1 = 0$.*

Then there is a map $I\colon H^q(\Gamma_2, M_2) \to H^q(\Gamma_1, M_1)$ such that $I \circ j_* = [\Gamma_1 g \Gamma_1]$ and $j_* \circ I = [\Gamma_2 g \Gamma_2]$.

We will apply the lemma with $\Delta = \Delta(r)$ as in Section 1, $\Gamma_1 = \Gamma_0(0)$ (i.e. no conditions at p, only at N), $\Gamma_2 = \Gamma_0(1)$ (level Np), $M_1 = L_{\mathbf{k}} \otimes \mathbb{Z}/p\mathbb{Z}$, j the projection from V to $V' = \oplus_x V^{x,-n}$, $M_2 = j(L)$ and $g = \mathrm{diag}(p, p^2, p, 1)$. In [B2] Section 4 we have already seen that j satisfies the conditions of the lemma, although there we worked with $\Gamma_1 = \Gamma_2$. It remains to verify the conditions on double cosets.

(1) is easy, using Lemma 1.2 and the fact that $\Gamma_1 \cap g\Gamma_1 g^{-1} \subset \Gamma_1 \cap g\Gamma_2 g^{-1}$, because $g^{-1}\Gamma_1 g \cap \Gamma_1 \subset \Gamma_2$.

Condition (2) is equivalent to the existence of δ_i such that $j\delta_i g M_1 = 0$ and

$$\Gamma_1 = \Gamma_2 \left(\Gamma_1 \cap g\Gamma_2 g^{-1}\right) \amalg \left(\coprod \Gamma_2 \delta_i \left(\Gamma_1 \cap g\Gamma_2 g^{-1}\right)\right).$$

Consider the following representatives for elements of the Weyl group

$$w_1 = I_4 \qquad w_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$w_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and for $i = 1$ to 4, let ω_i be any matrix satisfying $\omega_i \equiv w_i \pmod p$ and $\omega_i \equiv I_4 \pmod N$.

Then in fact $\Gamma_1 \cap g\Gamma_2 g^{-1} = g\Gamma_2 g^{-1} = \omega_4 \Gamma_2 \omega_4^{-1}$; on the other hand we obtain from the Iwasawa decomposition that

$$\Gamma_1 \cong \coprod_{i=1}^{4} \Gamma_2 \, \omega_i \, B(\mathbb{Z}).$$

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We claim that $\delta_i = \omega_i$ for $i = 2,3,4$ will work. Indeed, given $\alpha \in \Gamma_1$, we can write $\alpha\omega_4 = \gamma\omega_i b$ with $\gamma \in \Gamma_2$, $b \in B(\mathbb{Z})$, and thus $\alpha = \gamma.\,\omega_i\omega_4^{-1}.\,\omega_4 b\,\omega_4^{-1}$. (Here $\omega_i\,\omega_4^{-1}$ differs only by an element of Γ_2 from one of the ω's.) Moreover, the effect of elements of the Weyl group on weight spaces is simply given by a symmetry of the weight diagram; since $gM_1 \subset V'$, we have, for example, $\omega_2 gM_1 \subset \oplus_{y=-m}^{m} V^{-n,y}$, which is killed by j provided the highest weight satisfies $n > m \geqslant 0$. Thus we have satisfied the conditions of Lemma 4.1.

Let e_0 be the idempotent associated to $[\Gamma_1 g\Gamma_1]$ at level N, and e the one associated to $[\Gamma_2 g\Gamma_2]$ at level Np. Then j_* induces an isomorphism on the ordinary components. But we have already seen in [B2], Section 4 that $eH^q(\Gamma_0(1), L'_{\mathbf{k}} \otimes \mathbb{Z}/p\mathbb{Z}) \cong eH^q(\Gamma_0(1), L_{\mathbf{k}} \otimes \mathbb{Z}/p\mathbb{Z})$.

COROLLARY 4.2. *If* $\mathbf{k} = (m,n)$ *with* $n > m$, *the restriction map composed with the idempotent* e *gives an isomorphism*

$$e_0 H^q(\Gamma_0(0), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p) \cong eH^q(\Gamma_0(1), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Proof. Lemma 4.1 together with [B2] Section 4 implies that the top arrow in the following commutative diagram is an isomorphism

$$\begin{array}{ccc} e_0 H^q(\Gamma_0(0), L_{\mathbf{k}} \otimes \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & eH^q(\Gamma_0(1), L_{\mathbf{k}} \otimes \mathbb{Z}/p\mathbb{Z}) \\ \downarrow & & \downarrow \\ e_0 H^q(\Gamma_0(0), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p] & \longrightarrow & eH^q(\Gamma_0(1), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p]. \end{array}$$

Here $[p]$ denotes the part killed by p. The restriction map at the bottom is an injection (because $[\Gamma_0(0) : \Gamma_0(1)]$ is prime to p, so the kernel in the inflation-restriction sequence vanishes). But the vertical maps are surjections, as can be seen from the long exact sequence associated to the sequence

$$0 \to \mathbb{Z}_p/p\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Therefore the bottom map is also a surjection, and the result follows by Nakayama's lemma. $\square$

We proceed with the proof of Proposition 2.6.

Let $\mathcal{V}^{\circ}_{\mathbf{k},r} = eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))$ and $\mathcal{V}^{\circ}_{\mathbf{k}} = \varinjlim_{r} \mathcal{V}^{\circ}_{\mathbf{k},r}$. We give this a twisted Iwasawa and Hecke action in the same way as for $\mathcal{W}^{\circ}_{\mathbf{k}}$ in Section 1. By Theorem 2.3 and using Lemma 2.4 and Proposition 2.5, we have that

$$(\mathcal{V}^{\circ}_{\mathbf{k}})^{G_s} = eH^2(\Gamma(s), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Recall that $P_{a,s} = u^{p^s} - (1+p)^{ap^s}$.

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LEMMA 4.3. $\mathcal{V}^\circ_{\mathbf{k}}(-n)\,[P_{n,s}] = \mathcal{V}^\circ_{\mathbf{k},s+1}$.

Proof. $\mathcal{V}^\circ_{\mathbf{k}}(-n)[P_{a,s}] = \varinjlim_{r} \mathcal{V}^\circ_{\mathbf{k},r}(-n)[P_{a,s}]$. On setting $a = n$ we get

$$\begin{aligned}
&\mathcal{V}^\circ_{\mathbf{k},r}(-n)\,[P_{n,s}]\\
&\quad = \{\phi \in eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p))(-n) : (u^{p^s} - (1+p)^{np^s})(\phi) = 0\}\\
&\quad = \{\phi \in eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) : u^{p^s}(1+p)^{np^s}(\phi) - (1+p)^{np^s}\phi = 0\}\\
&\quad = \{\phi \in eH^2(\Gamma(r), L_{\mathbf{k}}(\mathbb{Q}_p/\mathbb{Z}_p)) : u^{p^s}\phi = \phi\}\\
&\quad = \mathcal{V}^{\circ\, u^{p^s}}_{\mathbf{k},r}\\
&\quad = \mathcal{V}^{\circ\, G_{s+1}}_{\mathbf{k},r},
\end{aligned}$$

because u is a generator for G_1, and the G_1-action is continuous. But we have just seen that $\mathcal{V}^{\circ\, G_s}_{\mathbf{k},r} = \mathcal{V}^\circ_{\mathbf{k},s}$.

After taking limits over r again, we obtain the result. □

We now show that $\mathcal{V}^\circ_{\mathbf{k}}(-n) = 0$ for almost all p. By Corollary 4.2, and using the fact that $[\Gamma_0(1) : \Gamma(1)] = p-1$ to pass from $\Gamma(1)$ to $\Gamma_0(1)$, we see $\mathcal{V}^\circ_{\mathbf{k},1} \cong e_0 H^2(\Gamma_0(0), L_{\mathbf{k}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$.

Consider the exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$, which yields (abbreviating $H^q(\Gamma_0(0), L_{\mathbf{k}} \otimes A)$ to $H^q(A)$)

$$H^2(\mathbb{Q}_p) \to H^2(\mathbb{Q}_p/\mathbb{Z}_p) \to H^3(\mathbb{Z}_p) \to H^3(\mathbb{Q}_p).$$

Now the cohomology with coefficients in $\mathbb{Q}_p$ vanishes: this follows from the vanishing theorems proved by Schwermer for the boundary cohomology ([Schw2] Section 4.5) under the condition on the weight that $n \gg m \gg 0$, and by R. Taylor for the cohomology of the interior (calculation in Section 1 of [Tay2]), under the condition that $n > m > 0$.

Therefore we have that $\mathcal{V}^\circ_{\mathbf{k},1}$ injects into the p-torsion part of $H^3(\Gamma_0(0), L_{\mathbf{k}} \otimes \mathbb{Z}_p)$. Since $\mathbb{Z}_p$ is flat over $\mathbb{Z}$, this is the same as $H^3(\Gamma_0(0), L_{\mathbf{k}})\,[p^\infty]$.

But $H^3(\Gamma_0(0), L_{\mathbf{k}})^{\mathrm{tor}}$ is a finitely generated torsion abelian group, and hence is finite. So provided p does not divide its order, the p-torsion vanishes and by Lemma 4.3 $\mathcal{V}^\circ_{\mathbf{k}}(-n)[P_a] = 0$ for all $a \gg 0$.

By Nakayama's lemma this is enough to show that $\mathcal{V}^\circ_{\mathbf{k}}(-n) = 0$, and hence, again by the control theorem for H^2 (Lemma 4.3), we obtain

$$eH^2(\Gamma(r), L_{m,n} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0 \;\; \forall n \gg m \gg 0 \;\text{ for almost all }\; p.$$

The values of p that are excluded are precisely the ones not satisfying (H).

Now consider another weight $\mathbf{k}' = (m, n')$ with $n' \gg m \gg 0$. Then the above argument will show that $eH^2(\Gamma_0(1), L_{\mathbf{k}'} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ injects into a finite p-group,

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which this time is not a priori zero. However, its p-torsion also vanishes: indeed, by Theorem 2.5 (vanishing of eH^1), we have

$$eH^2(\Gamma_0(1), L_{\mathbf{k}'} \otimes \mathbb{Z}/p\mathbb{Z}) \cong eH^2(\Gamma_0(1), L_{\mathbf{k}'} \otimes \mathbb{Q}_p/\mathbb{Z}_p)\,[p].$$

The left-hand side is independent of the weight by Theorem 1.1, so is zero by the result already proved for weight $\mathbf{k}$. This implies that we also have $\mathcal{V}^\circ_{\mathbf{k}',1} = 0$.

5. Λ-adic families of Siegel modular forms

Recall that we have defined $\mathbf{W} := W^\circ_{m,n}(-n)$, and that Λ is isomorphic to the one-variable Iwasawa algebra $\mathbb{Z}_p[[X]]$ via $u \leftrightarrow 1+X$. We also have specialisation maps $s_a\colon \Lambda \to \mathbb{Z}_p$ with kernels $P_a = u-(1+p)^a \in \Lambda$.

Let $\mathbf{k} = (m,n)$. Then the quotient $W^\circ_{\mathbf{k}}(-n)/P_a W^\circ_{\mathbf{k}}(-n)$ is dual to $\mathcal{W}^\circ_{\mathbf{k}}(-n)\,[P_a]$, i.e. elements of $\mathcal{W}^\circ_{\mathbf{k}}(-n)$ annihilated by P_a. The following key lemma is a consequence of the Control Theorem.

LEMMA 5.1. *Assume Condition* (H). *Then* $\mathcal{W}^\circ_{\mathbf{k}}(-n)\,[P_{n,s}] = \mathcal{W}^\circ_{\mathbf{k},s}$ *and dually*

$$W^\circ_{\mathbf{k}}(-n)/P_{n,s}W^\circ_{\mathbf{k}}(-n) = W^\circ_{\mathbf{k},s}.$$

Proof. The proof is the same as that of Lemma 4.3. □

On the other hand, we saw in Section 1 that $W^\circ_{m,n_1}(-n_1) \cong W^\circ_{m,n_2}(-n_2)$ as Λ-modules. Therefore

$$\begin{aligned} W^\circ_{m,n_1}(-n_1)/P_{n_2}W^\circ_{m,n_1}(-n_1) &\cong W^\circ_{m,n_2}(-n_2)/P_{n_2}W^\circ_{m,n_2}(-n_2) \\ &= W^\circ_{(m,n_2),1}. \end{aligned}$$

Thus we can recover an entire family of weights from our universal object $\mathbf{W}$. We can now analyse the Λ-module structure of $\mathbf{W}$.

LEMMA 5.2. *Assume the condition* (H). *Then* $\mathcal{W}^\circ_{\mathbf{k},r}$ *is* p*-divisible*.

Proof. We have a short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

This gives rise to the sequence

$$eH^3(\mathbb{Z}_p)^{\mathrm{tf}} \to eH^3(\mathbb{Q}_p) \to eH^3(\mathbb{Q}_p/\mathbb{Z}_p) \to eH^4(\mathbb{Z}_p)^{\mathrm{tor}}.$$

(Again we have written $H^*(A)$ for $H^*(\Gamma(r), L_{\mathbf{k}} \otimes A)$). By Pontryagin duality, the hypothesis (H) implies $eH^4(\mathbb{Z}_p)^{\mathrm{tor}} = 0$ and we have

$$\mathcal{W}^\circ_{\mathbf{k},r} \subset eH^3(\mathbb{Q}_p)/eH^3(\mathbb{Z}_p),$$

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so we are done. □

Dually, this says that $W^{\circ}_{\mathbf{k},r}$ is p-torsion free, and hence is free of finite rank over $\mathbb{Z}_p$ ($\mathbb{Z}_p$ being a principal ideal domain). We now resort to the following criterion.

LEMMA 5.3. *Let M be a continuous compact Λ-module and $\mathfrak{p}_i$ an infinite collection of height one primes, such that $M/\mathfrak{p}_i$ is a finite free $\mathbb{Z}_p$-module for each i. Then M is a finite free Λ-module.*

Proof. This is a standard application of Nakayama's lemma, and can be found e.g. as [Hi1] Lemma 6.3. □

We have seen that $\mathbf{W}/P_a\mathbf{W}$ is finite and free over $\mathbb{Z}_p$. Thus we can apply Lemma 5.3 to deduce that $\mathbf{W}$ is finite and free over Λ.

Let us summarise our results.

THEOREM 5.4. *Assume hypothesis (H). Define $\mathbf{W} = W^{\circ}_{m,n}(-n)$, a twisted Λ-module. Then $\mathbf{W}$ is finite and free over Λ; if m is a fixed even integer, $\mathbf{W}$ is independent of n. Furthermore, if $P_{a,s} = u^{p^s} - (1+p)^{ap^s} \in \Lambda$, we have specialisation maps $\mathbf{W}/P_{a,r}\mathbf{W} \cong W^{\circ}_{(m,a),r}$.*

Now recall the definitions of the Hecke algebras $\mathbb{T}_{\mathbf{k},r}$ and $\mathbb{T}_{\mathbf{k}}$ from Section 1 as endomorphisms generated by Hecke operators. We have seen that in fact, $\mathbb{T} := \mathbb{T}_{\mathbf{k}}$ is independent of the second weight parameter. As $\mathbf{W}$ is finite and free we know that $\mathbb{T} \hookrightarrow \Lambda^{a^2}$ is finite and torsion-free, though not necessarily free (as Λ is not a PID). We have, by restriction of operators

$$\mathbb{T}/P_{\mathbf{k},r}\mathbb{T} \twoheadrightarrow \mathbb{T}_{\mathbf{k},r}.$$

We are interested in systems of eigenvalues in

$$\mathcal{S}^{\circ} := \mathrm{Hom}_{\mathrm{alg}}(\mathbb{T}, \Lambda),$$

where we are demanding ring homomorphisms. (Note that given a complete set of Hecke eigenvalues at finite level, there may be more than one Siegel modular form in the corresponding eigenspace).

So suppose we are given a system of eigenvalues occurring on $\mathcal{W}^{\circ}_{\mathbf{k},1}$

$$\Theta \colon \mathbb{T}_{\mathbf{k},1} \twoheadrightarrow \mathcal{O},$$

with values in the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$. We aim to lift Θ to an element of $\mathcal{S}^{\circ}$.

We have a composite map

$$\mathbb{T} \xrightarrow{s_n} \mathbb{T}/P_n\mathbb{T} \twoheadrightarrow \mathbb{T}_{(m,n),1} \xrightarrow{\Theta} \mathcal{O},$$

with kernel $\wp_n \lhd \mathbb{T}$, say. Then $\wp_n$ is a prime ideal, as $\mathcal{O}$ is an integral domain, and $(P_n) \subset \wp_n$. Also $\wp_n \cap \Lambda$ is a prime ideal of Λ containing P_n but not (p), so by height considerations $\wp_n \cap \Lambda = P_n\Lambda$.

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We now appeal to the following version of the Going-down Theorem.

LEMMA 5.5. *Let A be an integrally closed domain and B a ring containing A and integral over A. Suppose that 0 is the only element of A which is a divisor of zero in B. Let $\mathfrak{p}, \mathfrak{q}$ be two prime ideals of A such that $\mathfrak{q} \supset \mathfrak{p}$, and $\mathfrak{Q}$ a prime ideal of B lying above $\mathfrak{q}$. Then there exists a prime ideal $\mathfrak{P}$ of B lying above $\mathfrak{p}$ and such that $\mathfrak{Q} \supset \mathfrak{P}$.*

Proof. See [Bour] V 2.4. I am grateful to Jacques Tilouine for pointing out this reference to me. □

Let $A = \Lambda$, $B = \mathbb{T}$, $\mathfrak{q} = (P_n)$, $\mathfrak{Q} = \wp_n$ and $\mathfrak{p} = 0$. Then the Going-down Theorem clearly applies. Λ is integrally closed because it is a UFD ([Bour] VII 3.9), and the second hypothesis is just the torsion-freeness of $\mathbb{T}$.

Therefore there exists a prime ideal $\wp_n^* \subset \wp_n$ with $\wp_n^* \cap \Lambda = (0)$

$$\begin{array}{ccc} \Lambda & \hookrightarrow & \mathbb{T} \\ (P_n) & \subset & \wp_n \\ \bigcup & & \bigcup \\ (0) & \subset & \wp_n^*. \end{array}$$

Now $\mathcal{I} = \mathbb{T}/\wp_n^*$ is finite over Λ ($\Lambda \cap \wp_n^* = 0$) and so is a semilocal ring ([Bour] II 3.5), i.e. it has only finitely many maximal ideals and is the direct sum of its localisations: $\mathcal{I} = \oplus \mathcal{I}_{\mathfrak{m}}$. But $\mathcal{I}$ is also an integral domain; hence it must be a local ring. Denote its maximal ideal by $\mathfrak{m}_{\mathcal{I}}$. Note that $\mathcal{I}$ is not necessarily integrally closed.

Denote by $\widehat{\Theta}$ the natural surjection

$$\widehat{\Theta}: \mathbb{T} \twoheadrightarrow \mathbb{T}/\wp_n^* = \mathcal{I}.$$

Let $\widetilde{\wp}_n$ be the image of $\wp_n$ in $\mathcal{I}$. The image of P_n in $\mathcal{I}$ lies in $\widetilde{\wp}_n$.

THEOREM 5.6. *Suppose $\Theta: \mathbb{T}_{\mathbf{k},1} \to \mathcal{O}$ is a system of eigenvalues occurring on the group $W_{\mathbf{k},1}$, where $\mathbf{k} = (m, n)$. Then there exists a local ring $\mathcal{I}$ finite over Λ, a system of eigenvalues $\widehat{\Theta}: \mathbb{T} \to \mathcal{I}$ on the universal space $\mathbf{W}$, and an ideal $\widetilde{\wp}_n$ of $\mathcal{I}$ lying above P_n, such that*

$$\widehat{\Theta} \bmod \widetilde{\wp}_n = \Theta.$$

Furthermore, if $\mathbf{k}' = \mathbf{k} + \lambda(0, p-1) = (m, n')$ is another weight, and $\widetilde{\wp}_{n'}$ is an ideal of $\mathcal{I}$ lying above $P_{n'}$ with $\mathcal{I}/\widetilde{\wp}_{n'} \cong \mathbb{Z}_p$, then

$$\widehat{\Theta} \bmod \widetilde{\wp}_{n'} \equiv \Theta \pmod{p}.$$

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Proof. Let I and $\widetilde{\wp}_n$ be as before. Let $p_\Theta = \mathrm{Ker}(\Theta)$ and let $\widetilde{p}_n$ be the kernel of the map $\mathbb{T} \to \mathbb{T}_{\mathbf{k},1}$. Then for $T \in \mathbb{T}_{\mathbf{k},1}$ with lift $\widetilde{T}$ to $\mathbb{T}$

$$\begin{aligned}\Theta(T) = T \bmod p_\Theta &= \widetilde{T} \bmod \widetilde{p}_n \bmod p_\Theta \\ &= \widetilde{T} \bmod \wp_n \\ &= \widehat{\Theta}(\widetilde{T}) \bmod \widetilde{\wp}_n.\end{aligned}$$

If we denote maps by their kernels, the picture looks like this

$$\begin{tikzcd} & \mathbb{T}/P_n\mathbb{T} \arrow[dr] & \\ \mathbb{T} \arrow[ur, "P_n"] \arrow[rr, "\tilde{p}_n"] \arrow[d, "\wp_n^*"] \arrow[drr, "\wp_n"] & & \mathbb{T}_{\mathbf{k},1} \arrow[d, "p_\Theta"] \\ \mathcal{I} \arrow[rr, "\tilde{\wp}_n"] & & \mathcal{O}. \end{tikzcd}$$

For the second part, note that both $(p, \widetilde{\wp}_n)$ and $(p, \widetilde{\wp}_{n'})$ generate the unique maximal ideal $\mathfrak{m}_\mathcal{I}$ of $\mathcal{I}$. Thus the images of $\widehat{\Theta} \bmod \wp_n$ and $\widehat{\Theta} \bmod \wp_{n'}$ in $\mathbb{F}_p$ are both just the reductions of $\widehat{\Theta}$ modulo $\mathfrak{m}_\mathcal{I}$. □

Remark. It is perhaps illuminating to note that in the case where $\mathcal{I} = \Lambda$, we can see the second part of the theorem explicitly, as reduction mod $\wp_n$ is then just evaluation of power series at $X = (1+p)^n - 1$. Then if $\mathbf{k} = \mathbf{k}' + p^a(0, p-1)$, we see that $\widehat{\Theta} \bmod P_n \cong \widehat{\Theta} \bmod P_{n'} \pmod{p^{a+1}}$.

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References

[A] Andrianov, A. N.: Quadratic forms and Hecke operators, *Grundlehren d. Math. Wiss.* 286, Springer (1987).

[BMS] Bass, H., Milnor, J. and Serre, J.-P.: Solution of the congruence subgroup problem for SL_n $(n \geqslant 3)$ and Sp_{2n} $(n \geqslant 2)$, *Publ. Mathematiques* IHES 33 (1967) 59–137.

[Bour] Bourbaki, N.: Commutative algebra, *Elements of Mathematics*, Hermann (1972).

[B1] Buecker, K.: Congruences between Siegel modular forms, Ph.D. Thesis, Cambridge (1995).

[B2] Buecker, K.: Congruences between Siegel modular forms on the level of group cohomology, *Ann. Inst. Fourier* 46 No. 4 (1996) 877–897.

[God] Godement, R.: Exposés 5–10, in: Fonctions automorphes, Séminaire Henri Cartan 1957/58 Vol. 1, Paris (1958).

[Hi1] Hida, H.: Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, *Invent. Math*. 85 (1986) 545–613.

[Hi2] Hida, H.: On p-adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math*. 128 (1988) 295–384.

[Hi3] Hida, H.: Control theorems of p-nearly ordinary cohomology groups for $\mathrm{SL}(n)$, *Bull. Soc. Math*. France 123 No. 3 (1995) 425–475.

[Hmph] Humphreys, J. E.: Introduction to Lie Algebras and Representation Theory, *Graduate Texts in Mathematics* 9, Springer (1972).

[Menn] Mennicke, J.: Zur Theorie der Siegelschen Modulgruppe, *Math. Annalen* 159 (1965) 115–129.

[Schw1] Schwermer, J.: *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, LNM 988, Springer (1983).

[Schw2] Schwermer, J.: On arithmetic quotents of the Siegel upper half space of degree two, *Comp. Math*. 58 (1986) 233–258.

[Tay1] Taylor, R. L.: Ph.D. thesis, Princeton 1988.

[Tay2] Taylor, R. L.: On the l-adic cohomology of Siegel Threefolds, *Invent. Math*. 114 (1993) 289–310.

[TU1] Tilouine, J. and Urban, E.: *Nearly ordinary cohomology for* Sp_4, preprint.

[TU2] Tilouine, J. and Urban, E.: Familles p-adique a trois variables de formes de Siegel et de representations galoisiennes, *C.R. Acad. Sci*. Paris 321 No. 1 (1995) 5–10.

[TU3] Tilouine, J. and Urban, E.: Several variable p-adic families of Siegel-Hilbert cusp eigenforms and their Galois representations, *Prépublications Mathématiques de l'Université Paris-Nord* (1997).

[W] Weissauer, R.: *A special case of the Fundamental Lemma: the case of* $\mathrm{GSp}(4)$, preprint.

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