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# THE LAWS OF SOME METABELIAN VARIETIES

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**Abstract**

It is shown that if $m$, $n$ are relatively prime positive integers, then the variety consisting of those soluble groups of exponent $mn$ in which any subgroup of exponent $m$ or $n$ is abelian has a basis of two-variable laws.

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Since the paper of Higman (1959), it has been of interest to ask which varieties have a 2-variable basis for their laws. In this note, we show that certain metabelian varieties are defined by 2-variable laws.For unexplained results and notation on varieties of groups see Neumann (1967), while for other group-theoretical results see Gorenstein (1968).

THEOREM. *Let m and n be relatively prime positive integers. Then the following set of laws forms a basis for the laws of the variety* $\mathfrak{A}_m\mathfrak{A}_n \vee \mathfrak{A}_n\mathfrak{A}_m$:

(1) $x^{mn} = 1$.

(2) $[x^m, y^m]^m = 1$.

(3) $[x^n, y^n]^n = 1$.

(4) $[[x, y], [x^{-1}, y]] = 1$.

Let $\mathfrak{V}$ denote the variety defined by the laws (1)–(4), and let $\mathfrak{U}$ denote the variety $\mathfrak{A}_m\mathfrak{A}_n \vee \mathfrak{A}_n\mathfrak{A}_m$. We prove that $\mathfrak{U} = \mathfrak{V}$ in a series of lemmas. Note however that the laws (1)–(4) hold in $\mathfrak{A}_m\mathfrak{A}_n$ and in $\mathfrak{A}_n\mathfrak{A}_m$, so we have $\mathfrak{U} \leq \mathfrak{V}$.

LEMMA 1. (a) *Groups in* $\mathfrak{V}$ *of exponent dividing m or n are abelian.*

(b) *Finitely-generated soluble groups in* $\mathfrak{V}$ *are in* $\mathfrak{U}$.

(c) 2-*generator groups in* $\mathfrak{V}$ *are metabelian.*





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PROOF. The law (3) reduces to $[x, y] = 1$ in a group of exponent dividing $m$, as does the law (2) in a group of exponent dividing $n$. Hence (a) holds.

Let $G \in \mathfrak{V}$ be a finitely-generated soluble group. Then $G$ is finite. Now $F(G) = F_1 \times F_2$, where $F_1$ has exponent dividing $m$ and $F_2$ has exponent dividing $n$. Let $G_i = G/F_i$ for $i = 1, 2$. Then $G$ is a subgroup of $G_1 \times G_2$, and it suffices to show that $G_1$ and $G_2$ lie in $\mathfrak{U}$.

Now $F(G_1)$ has exponent dividing $n$, by law (1) and part (a). But if $g \in G_1$ has order dividing $n$ then again by law (1) $\langle g, F(G_1)\rangle$ has exponent dividing $n$, and so by (a) is abelian. Hence every element of $G_1$ of order dividing $n$ centralizes $F(G_1)$. But $\mathcal{C}_G(F(G_1)) \leqslant F(G_1)$ (Gorenstein (1968), Theorem 6.1.3), so $F(G_1)$ contains all the elements of $G_1$ of order dividing $n$. Hence $G_1/F(G_1)$ has exponent dividing $m$, and so by part (a) is abelian. Then $G_1 \in \mathfrak{A}_n\mathfrak{A}_m < \mathfrak{U}$. An exactly similar argument shows that $G_2 \in \mathfrak{A}_m\mathfrak{A}_n \leqslant \mathfrak{U}$. Hence $G \in \mathfrak{U}$.

By Theorem 2.1 of Higman (1959), (c) is a consequence of the law (4).

LEMMA 2. $[x^m, (x^n)^y] = 1$ *is a law of* $\mathfrak{V}$.

PROOF. First we show that $[x^m, (x^n)^y] = 1$ is a law of $\mathfrak{U}$. In other words, we must show that it is a law in $\mathfrak{A}_m\mathfrak{A}_n$ and in $\mathfrak{A}_n\mathfrak{A}_m$. In $\mathfrak{A}_m\mathfrak{A}_n$, a commutator $c$ has order $m$, and so since $(m, n) = 1$, $c$ is an $n$th power. Also $n$th powers commute. So $[x, y, z^n] = 1$ is a law of $\mathfrak{A}_m\mathfrak{A}_n$. But $[x^m, (x^n)^y] = [x^m, y^{-1}, x^n]^y$, so $[x^m, (x^n)^y] = 1$ is a law of $\mathfrak{A}_m\mathfrak{A}_n$.

Similarly $[z^m, [x, y]] = 1$ is a law of $\mathfrak{A}_n\mathfrak{A}_m$. But $[x^m, (x^n)^y] = [x^m, [x^n, y]]$, so $[x^m, (x^n)^y] = 1$ is a law of $\mathfrak{A}_n\mathfrak{A}_m$. Hence $[x^m, (x^n)^y] = 1$ is a law of $\mathfrak{U}$.

But now suppose $G \in \mathfrak{V}$ does not satisfy $[x^m, (x^n)^y] = 1$. Then $G$ contains elements $g$, $h$ with $[g^m, (g^n)^h] \neq 1$. But by Lemma 1 (b) and (c) $\langle g, h\rangle \in \mathfrak{U}$. Hence $[g^m, (g^n)^h] = 1$, a contradiction. Hence $[x^m, (x^n)^y] = 1$ is a law of $\mathfrak{V}$.

LEMMA 3. $\mathfrak{V}$ *contains no non-abelian simple group*.

PROOF. Suppose $G \in \mathfrak{V}$, $G$ a non-abelian simple group. Then we deduce some properties of $G$.

(i) If $g \in G$, then $g^m = 1$ or $g^n = 1$.

For if $g \in G$ with $g^m \neq 1$ and $g^n \neq 1$, then by Lemma 2, $\mathcal{C}_G(g^m)$ contains all conjugates of $g^n$. But $G$ is simple, so $G$ is generated by the conjugates of $g^n$. Then $g^m$ is central in $G$, which is absurd, since $G$ is a non-abelian simple group.

(ii) Suppose that $g$, $h$ are non-commuting elements of $G$ of the same order, and let $H = \langle g, h\rangle$. Then $H$ is a Frobenius group, with $\langle g\rangle$ and $\langle h\rangle$ as Frobenius complements. In particular, there is an integer $\alpha$ with $\langle g\rangle = \langle g^\alpha\rangle$ and $g^{-\alpha}h \in H'$.

Let $g$, $h$ have order $\mu$. By (i), we may assume for definiteness, that $\mu$ divides $m$. Then by Lemma 1 (c), $H$ is metabelian, and so finite. Again $g$ and $h$ are $n$th powers. But $H'$ is generated by elements $[a, b]$ with a conjugate of $g$ and $b$ a conjugate of $h$. Then by law (3), these elements have order dividing $n$. Since $H'$ is abelian, it follows that $H'$ has exponent dividing $n$. Now by (i), $H/H'$ acts regularly on $H'$. Hence $H$ is a Frobenius group. Since $H/H'$ is abelian, it is cyclic (Gorenstein (1968), Theorem 5.3.14(ii). See also Theorem 10.3.1).

Now by (i), $H' \cap \langle g \rangle = H' \cap \langle h \rangle = 1$, as $(\mu, |H'|) = 1$. But $H/H' = \langle gH', hH' \rangle$, so $H/H'$ has exponent exactly $\mu$. Since $H/H'$ is cyclic, we have $|H : H'| = \mu$, whence $H = H'\langle g \rangle = H'\langle h \rangle$. In other words, $\langle g \rangle$ and $\langle h \rangle$ are Frobenius complements. Then $\langle g \rangle$ and $\langle h \rangle$ are conjugate in $H$ (Gorenstein (1968), Theorem 6.2.1(ii)). Choose $a \in H$ with $\langle h^a \rangle = \langle g \rangle$, say $h^a = g^\alpha$. Then $g^{-\alpha}h = (h^a)^{-1}h = [a, h] \in H'$ as required.

(iii) $G$ contains a non-cyclic abelian subgroup.

Let $p$ be the largest prime dividing the exponent of $G$. Then $G$ is generated by elements of order $p$. Hence $G$ contains a pair $g$, $h$ of non-commuting elements of order $p$. Let $H = \langle g, h \rangle$. Then by (ii) $H$ is a Frobenius group, with $H'$ abelian. Let $C$ be a complement to $H'$ in $H$, and let $q$ be a prime dividing $|H'|$. Then $|C| = p$, and $C$ acts regularly on the abelian group $O_q(H')$. Since $q < p$, $O_q(H')$ must be non-cyclic.

(iv) $G$ does not exist.

By (iii), $G$ contains a non-cyclic abelian subgroup, so for some prime $p$, $G$ contains the non-cyclic group of order $p^2$. Hence choose $g, h \in G$ such that $\langle g, h \rangle$ is non-cyclic of order $p^2$. Suppose for definiteness that $p$ divides $m$.

Let $A = \mathcal{C}_G(g)$. By (i) $A$ has exponent dividing $m$, so by Lemma 1 (a) $A$ is abelian. If $a \in A^\#$ then again $\mathcal{C}_G(a)$ has exponent dividing $m$, and is abelian. Also $A \leqslant \mathcal{C}_G(a)$. So $\mathcal{C}_G(a)$ centralizes $g$. Then $\mathcal{C}_G(a) \leqslant A$. Hence we have $A = \mathcal{C}_G(a)$ for any $a \in A^\#$. In particular $A = \mathcal{C}_G(g^{-1}h)$.

Now as $G$ is simple, $G$ is generated by elements of order $p$ (for example, the conjugates of $g$). Then there is an element $k$ of order $p$ in $G - A$, as $A$ is abelian but $G$ is not. Let $H_1 = \langle g, k \rangle$ and $H_2 = \langle h, k \rangle$. Then by (ii) there are integers $\alpha$, $\beta$ with $1 \leqslant \alpha, \beta < p$ such that $g^{-\alpha}k \in H_1'$ and $h^{-\beta}k \in H_2'$. Replacing $g$ by $g^\alpha$ and $h$ by $h^\beta$, we suppose that $g^{-1}k \in H_1'$ and $h^{-1}k \in H_2'$. But $H_1'$ and $H_2'$ have exponent dividing $n$, so $(g^{-1}k)^n = (h^{-1}k)^n = 1$. Then $g^{-1}k$, $h^{-1}k$ are $m$th powers, and so by law (2), $[g^{-1}k, k^{-1}h]^m = 1$. But since $[g, h] = 1$, $[g^{-1}k, k^{-1}h] = [k, g^{-1}h]$. As $k^m = (g^{-1}h)^m = 1$, $[g^{-1}k, k^{-1}h]^n = [k, g^{-1}h]^n = 1$, by law (3). Since $(m, n) = 1$, we have $[k, g^{-1}h] = 1$. Then $k \in \mathcal{C}_G(g^{-1}h) = A$, contradicting the choice of $k$.

LEMMA 4. $\mathfrak{U} = \mathfrak{B}$.

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PROOF. Suppose $\mathfrak{U} \neq \mathfrak{V}$. Then as $\mathfrak{U} < \mathfrak{V}$, there is a law of $\mathfrak{U}$ which is not a law of $\mathfrak{V}$. Hence there is a finitely-generated group $G$ with $G \in \mathfrak{V} - \mathfrak{U}$. By Lemma 3 and Lemma 1 (b), all finite groups in $\mathfrak{V}$ are in $\mathfrak{U}$. Hence $G$ is infinite. We show first that $G''$ is perfect. Since $G/G'''$ is finitely-generated and soluble, we have by Lemma 1 (b) that $G/G''' \in \mathfrak{U}$. But all groups in $\mathfrak{U}$ are metabelian. Hence $G'' = G'''$ as required.

Now $G/G''$ is finite, while $G$ is finitely-generated. Then $G''$ is finitely-generated. Now by Zorn's Lemma, $G''$ has a maximal normal subgroup $N$. Then $G''/N$ is a simple group, which is non-abelian since $G''$ is perfect. But $G''/N \in \mathfrak{V}$, contradicting Lemma 3 and completing the proof.

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