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# INVERSE MULTIPARAMETER EIGENVALUE PROBLEMS FOR MATRICES II

*by* PATRICK J. BROWNE* and B. D. SLEEMAN

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## 1. Introduction

This is a sequel to our previous paper [4] where we initiated a study of inverse eigenvalue problems for matrices in the multiparameter setting. The one parameter version of the problem under consideration asks for conditions on a given $n \times n$ symmetric matrix $A$ and on $n$ given real numbers $s_1 \leqq s_2 \leqq \cdots \leqq s_n$ under which a diagonal matrix $V$ can be found so that $A+V$ has $s_1, \ldots, s_n$ as its eigenvalues. Our motivation for this problem and our method of attack on it in [4] comes chiefly from the work of Hadeler [5] in which sufficient conditions were given for existence of the desired diagonal $V$. Hadeler's approach in [5] relied heavily on the Brouwer fixed point theorem and this was also our main tool in [4]. Subsequently, using properties of topological degree, Hadeler [6] gave somewhat different conditions for the existence of the diagonal $V$. It is our desire here to follow this lead and to use degree theory to give some results extending those in [6] to the multiparameter case.

In Section 2 we study the inverse eigenvalue problem for one equation with two spectral parameters and in Section 3 we apply these results to linked systems of such equations and to the quadratic eigenvalue problem thus paralleling our earlier work [4].

## 2. One equation with two parameters

In this section we are given $n \times n$ symmetric matrices $A$, $B$, $C$ where, without loss of generality, we assume that the leading diagonal elements of $A$ namely $a_{ii}=0$, $1 \leqq i \leqq n$. For each $(\lambda, \mu) \in \mathbb{R}^2$ the matrix

$$W(\lambda, \mu) = A + \lambda B + \mu C$$

is also symmetric and we list its eigenvalues as

$$\rho_1(A; \lambda, \mu) \leqq \cdots \leqq \rho_n(A; \lambda, \mu).$$

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We are interested in the eigencurves given by

$$Z_i(A)=\{(\lambda,\mu)\in\mathbb{R}^2 \,|\, \rho_i(A;\lambda,\mu)=0\}.$$

There is no *a priori* guarantee that the sets $Z_i(A)$ are nonempty but various fairly weak conditions preventing $Z_i(A)=\varnothing$ have been discussed in [**2**]. It will be enough for us here to assume that at least one of $B$, $C$ is positive (or negative) definite.

As in [**4**], we use the cone $\hat{C}\subset\mathbb{R}^2$ given by

$$\hat{C}=\{(\lambda,\mu)\,|\,\lambda(Bx,x)+\mu(Cx,x)\leqq 0,\ \forall x\in\mathbb{R}^n\}$$

and we assume

**Hypothesis 2.1.** The points $(s_i, t_i)$ are $\hat{C}$-ordered; i.e.

$$(s_j,t_j)-(s_i,t_i)\in\hat{C} \text{ whenever } j\geqq i.$$

We put

$$g_i^j=\sum_{\substack{k=1\\k\neq i}}^{n}|a_{ik}+s_jb_{ik}+t_jc_{ik}|$$

and make the further

**Hypothesis 2.2.**

$$(s_j-s_i)b_{ii}+(t_j-t_i)c_{ii}<-g_i^j-g_j^j,$$

$$(s_k-s_j)b_{kk}+(t_k-t_j)c_{kk}<-g_k^j-g_j^j,$$

$$1\leqq i<j<k\leqq n.$$

Note that Hypothesis 2.1 ensures that the left-hand sides of these two inequalities are, in fact, negative. Now select $\eta>0$ and consider the open bounded region $E\subset\mathbb{R}^n$ given by

$$E=\{(v_1,\dots,v_n)\,|\,v_1+s_1b_{11}+t_1c_{11}>-\eta,\ v_n+s_nb_{nn}+t_nc_{nn}<\eta,$$

$$v_i+s_jb_{ii}+t_jc_{ii}+g_i^j<v_j+s_jb_{jj}+t_jc_{jj}-g_j^j-\varepsilon,$$

$$v_j+s_jb_{jj}+t_jc_{jj}+g_j^j<v_k+s_jb_{kk}+t_jc_{kk}-g_k^j-\varepsilon,$$

$$1\leqq i<j<k\leqq n\}.$$

It is easy to check that the point

$$x=(-s_1b_{11}-t_1c_{11},\dots,-s_nb_{nn}-t_nc_{nn})$$

belongs to $E$.

For $v \in E$, $V$ will denote the diagonal matrix $V = \operatorname{diag}(v_1, \ldots, v_n)$. We also use $\hat{B}, \hat{C}$ to denote the diagonal matrices

$$\hat{B} = \operatorname{diag}(b_{11}, \ldots, b_{nn}), \quad \hat{C} = \operatorname{diag}(c_{11}, \ldots, c_{nn})$$

and $B^{\#}, C^{\#}$ for $B - \hat{B}$, $C - \hat{C}$ respectively. Now consider the mapping $F_\theta$, $0 \leqq \theta \leqq 1$, $F_\theta : E \to \mathbb{R}^n$ given by

$$F_\theta(v) = (\rho_1(\theta(A + s_1 B^{\#} + t_1 C^{\#}) + V + s_1 \hat{B} + t_1 \hat{C}),$$

$$\rho_n(\theta(A + s_n B^* + t_n C^*) + V + s_n \hat{B} + t_n \hat{C})).$$

Our problem of finding a diagonal matrix $V$ so that $(s_i, t_i) \in Z_i(A + V)$, $1 \leqq i \leqq n$, is equivalent to finding a point $v$ so that $F_1(v) = 0$.

Note that for $v \in E$,

$$v_i + s_j b_{ii} + t_j c_{ii} < v_j + s_j b_{jj} + t_j c_{jj} < v_k + s_j b_{kk} + t_j c_{kk},$$

$$1 \leqq i < j < k \leqq n.$$

Thus it follows that

$$F_0(v) = v + x$$

and accordingly $F_0(v) = 0$ has a unique solution, viz. $v = -x$. Moreover, in terms of the topological degree we see that

$$d(F_0, E, 0) = 1.$$

It is clear that $F_0$ and $F_1$ are homotopy equivalent. To use the homotopy invariance of topological degree we need to show that for each $\theta \in [0, 1]$ we have $0 \notin F_\theta(\partial E)$. Suppose then that $v \in \partial E$ and $F_\theta(v) = 0$. Should $v \in \partial E$, because $v_1 + s_{11} b_{11} + t_1 c_{11} = -\eta$, we can argue that $W(A + V; s_1, t_1)$ is positive semi-definite (since zero is its smallest eigenvalue) and thus its diagonal entries must be non-negative. Hence $v_1 + s_1 b_{11} + t_1 c_{11} \geqq 0$—a contradiction. In like fashion we can dismiss the case $v_n + s_n b_{nn} + t_n c_{nn} = \eta$. We next note that the matrix $\theta(A + s_j B^{\#} + t_j C^{\#}) + V + s_j \hat{B} + t_j \hat{C}$ has diagonal entries $v_i + s_j b_{ii} + t_j c_{ii}$, $1 \leqq i \leqq n$, which are the centres of the Gerschgorin circles for this matrix. The radii of the circles are $\theta g_i^j$, $1 \leqq i \leqq n$, respectively. From the relations defining $E$ we see that the circles corresponding to $i = 1, \ldots, j-1$ are all disjoint from the $j$th circle which in turn is disjoint from the circles corresponding to $i = j+1, \ldots, n$. We use the theorems of Hadamard and Gerschgorin (see [1, Theorems 6.2.1, 6.2.2, p. 231]) to infer that the $j$th circle contains $\rho_j(\theta(A + s_j B^{\#} + t_j C^{\#}) + V + s_j \hat{B} + t_j \hat{C})$ and thus if $F_\theta(v) = 0$ we must have $|v_j + s_j b_{jj} + t_j c_{jj}| \leqq \theta g_j^j$. This observation is now sufficient to complete the proof that $F_\theta(v) \neq 0$ for $v \in \partial E$.

The upshot of these remarks is

**Theorem 1.** *Suppose Hypothesis 2.2 holds. Then there is a diagonal* $V =$

$\operatorname{diag}(v_1, \ldots, v_n)$ *such that*

$$(s_i, t_i) \in Z_i(A+V)$$

*and*

$$|v_i + s_i b_{ii} + t_i c_{ii}| \leqq g_i^i, \qquad 1 \leqq i \leqq n.$$

**Theorem 2.** *The conclusion of Theorem 1 holds if equality is permitted in the two inequalities of Hypothesis 2.2.*

**Proof.** If $g_j^j \neq 0$ for each $1 \leqq j \leqq n$ then the argument above shows that for each $\theta \in [0, 1)$ we have a solution $v^\theta$ of $F_\theta(v) = 0$. We select a sequence $\theta_k \to 1$ with corresponding solutions $v^k$. A suitable subsequence of $v^k$ must converge and the limit will be a solution of $F_1(v) = 0$. Whenever $g_j^j = 0$ it is easy to see that it is necessary to use $v_j = -s_j b_{jj} - t_j c_{jj}$.

We should point out that while we have considered here an equation with exactly two parameters, similar arguments can be presented for eigenvalue problems of the form $(A + \lambda_1 B_1 + \cdots + \lambda_n B_n)x = 0$.

### 3. Linked systems and quadratic eigenvalue problems

Firstly suppose we are given Hermitean matrices $A_1$, $B_1$, $C_1$ of size $n_1 \times n_1$, and $A_2$, $B_2$, $C_2$ of size $n_2 \times n_2$. Consider the $2 \times 2$ multiparameter eigenvalue problem

$$(A_1 + \lambda_1 B_1 + \lambda_2 C_1)x_1 = 0, \quad x_1 \neq 0, \quad x_1 \in \mathbb{R}^{n_1},$$

$$(A_2 + \lambda_1 B_2 + \lambda_2 C_2)x_2 = 0, \quad x_2 \neq 0, \quad x_2 \in \mathbb{R}^{n_2}.$$

An eigenvalue is a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ for which this problem can be solved. A customary hypothesis which we shall adopt to ensure the existence of eigenvalues is "right definiteness":

$$RD\text{: for all } x_1 \neq 0,\ x_2 \neq 0,$$

$$\det \begin{vmatrix} (B_1 x_1, x_1) & (C_1 x_1, x_1) \\ (B_2 x_2, x_2) & (C_2 x_2, x_2) \end{vmatrix} > 0.$$

There is now no loss in assuming that say both $B_1$ and $B_2$ are positive definite. Under $RD$ there are $n_1 n_2$ eigenvalues $\lambda = (\lambda_1, \lambda_2)$ which can be indexed systematically as $\lambda^{(i,j)}$, $1 \leqq i \leqq n_1$, $1 \leqq j \leqq n_2$, in the sense that

$$W_k(\lambda^{(i,j)}) = A_k + \lambda_1^{(i,j)} B_k + \lambda_2^{(i,j)} C_k, \qquad k = 1, 2,$$

has 0 as its $i$th (respectively $j$th) eigenvalue for $k = 1$ (respectively 2). The cone $\hat{C}$ for this situation is

$$\hat{C}=\{(\lambda_1,\lambda_2)\in\mathbb{R}^2 \mid \lambda_1(B_ix_i,x_i)+\lambda_2(C_ix_i,x_i)\leqq 0 \quad \forall x_i\neq 0, i=1,2\}.$$

The recent survey paper [3] provides an overview of the (direct) theory of multiparameter eigenvalue problems.

As before we may assume $A_1$, $A_2$ have zero leading diagonals.

**Theorem 3.** *Suppose we are given points*

$$s^{(i,j)}=(s_1^{(i,j)},s_2^{(i,j)})\in\mathbb{R}^2 \qquad 1\leqq i\leqq n_1, \quad 1\leqq j\leqq n_2,$$

*with*

$$s^{(1,1)},\ldots,s^{(n_1,n_1)},s^{(n_1,n_1+1)},\ldots,s^{(n_1,n_2)}$$

*ordered by $\hat{C}$—here we have assumed that $n_1\leqq n_2$. If $s^{(1,1)},\ldots,s^{(n_1,n_1)}$ satisfy Hypothesis 2.2 with respect to $A_1$, $B_1$, $C_1$ and $s^{(1,1)},\ldots,s^{(n_1,n_1)}$, $s^{(n_1,n_1+1)},\ldots,s^{(n_2,n_2)}$ satisfy Hypothesis 2.2 with respect to $A_2$, $B_2$, $C_2$, then diagonal matrices $D_1$, $D_2$ of sizes $n_1\times n_1$ and $n_2\times n_2$ respectively can be found so that*

$$s^{(i,i)}\in Z_i(A_1+D_1)\cap Z_1(A_2+D_2), \qquad 1\leqq i\leqq n_1$$

$$s^{(n_1,i)}\in Z_i(A_2+D_2), \qquad n_1+1\leqq i\leqq n_2.$$

This is parallel to our earlier result [4, Theorem 4.1].

As a further application of our main result we consider the quadratic eigenvalue problem

$$(A+\lambda B+\lambda^2C)x=0, \quad x\neq 0,$$

where $A$, $B$, $C$ are given $n\times n$ symmetric matrices. We can assume that either $B$ or $C$ is positive definite and we ask for conditions under which a diagonal $D$ can be found so that the problem with $A+D$ in place of $A$ has given numbers $s_1,\ldots,s_n$ as eigenvalues.

**Theorem 4.** *Suppose $(s_i,s_i^2)$, $1\leqq i\leqq n$, are $\hat{C}$-ordered and satisfy Hypothesis 2.2 with respect to $A$, $B$, $C$. Then a diagonal $D$ can be found so that the quadratic eigenvalue problem $(A+D+\lambda B+\lambda^2C)x=0$ has $\lambda=s_1,\ldots,s_n$, as eigenvalues.*

The above results answer but a few of the many open questions in inverse eigenvalue theory in the multiparameter setting. Our earlier discussion ([4], Section 6) gave a brief outline of other interesting possibilities.

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Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4

Department of Mathematical Sciences
University of Dundee
Dundee DD1 4H4