Problems Associated with the Strength Assessment of Rotor Blades

By

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DR G S HISLOP *(Chairman of the Executive Council) occupying the Chair*

The CHAIRMAN, in introducing the Author, described the subject of his paper as one of paramount importance to all who were concerned in the design and operation of helicopters Mr ROGERS was an indentured aeronautical engineer apprentice at the Fairey Aviation Company from 1940 to 1946, during which time he obtained the Ordinary and Higher National Certificates in Mechanical and Aeronautical Engineering He was awarded an S B A C scholarship to the College of Aeronautics in 1946 and obtained a diploma with distinction in Aircraft Design

Returning to the Fairey Aviation Company in 1948 as a stressman, he worked on fixed wing aircraft from 1948 to 1950 and on rotary wing aircraft from 1950 onwards He was appointed Assistant Chief Stressman (Rotary Wing) in 1953 and was at present responsible for the strength assessment and airworthiness of all the Fairey Aviation Company's helicopter rotor blades and heads

MR V A B ROGERS

SUMMARY

This paper outlines some of the overall problems associated with the strength assessment of a Helicopter Rotor Blade

A picture of the dynamical behaviour of the aircraft and its rotor is presented From this all the necessary data for the strength assessment can be obtained (*e g* , Blade deflections, Shear Force and Bending moment diagrams)

The different dynamical conditions associated with a complete " Ground to Air " flight cycle are considered, and the problem of " Ground Resonance " is shown to be only a particular case of the " lag plane " forced oscillation problem

The calculation of the aerodynamic forcing loading, used in conjunction with the dynamical equations, is discussed, and an estimate of the necessary modification, to bring the calculated blade stress levels into line with those

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measured on the actual aircraft, is given

A brief outline of some typical fatigue problems is given and a comparison made between the estimated and actual life of a typical blade component

It is concluded that a theoretical strength assessment will prevent many a design pitfall, but that it is necessary to verify the results obtained by adequate flight and structural testing in order to produce a safe aircraft

INTRODUCTION

A rotary wing aircraft, whether on the ground or in the air, is basically a complex dynamical system, and consequently the behaviour of its rotor blades is dependent upon the behaviour of the system as a whole Thus the strength of a rotor blade can usually only be assessed in relation to a given aircraft configuration

It is the object of this paper, therefore, to build up a picture of the aircraft and blade motion for a typical " ground-flight " cycle, and to consider the associated strength assessment problem

For completeness, mention will also be made of the British Civil Airworthiness Requirements (B C A R) relevant to the cases considered

THE AIRCRAFT ON THE GROUND

Aircraft and Rotor Stationary

In this condition the rotor blade is usually only subjected to simple static cases, the loading arising from handling and picketing considerations

Consequently, all the normal stressing methods applicable to static aircraft structures will apply and need not be detailed here

The B C A R requirements relevant to this case specify arbitrary loads and deflections at the blade tip (B C A R , Sec G3-13)

Aircraft Moving and Rotor Stationary

In this condition the rotor blade experiences pitching and vertical acceleration due to " towing " considerations

Since the blade is an elastic body, a simple dynamical problem is presented The equations of motion for a typical system (Fig 1) are given in Appendix 1, and the solution obtained It is important to realise, however that the solution is very dependent upon the " nature " and " time of application " of the imposed accelerations, and it can be seen that the assumption of a maximum acceleration of un-specified duration (typical of undercarriage design cases) is unrealistic as it can give rise to an extremely large inertia loading (especially in pitch) along the blade

The real problem in this case therefore, is that of specifying the type of acceleration to be considered, and this is almost impossible to assess generally, as much depends upon the type of surface over which the aircraft is towed and the various towing technique employed

For practical purposes therefore, it is usually necessary to design the blade for a simple ultimate case based on a factored static blade weight, and then to investigate the towing accelerations that can be tolerated (For large rotors even a quite moderate ultimate factor of 2 5 constitutes a design case which is not easy to meet)

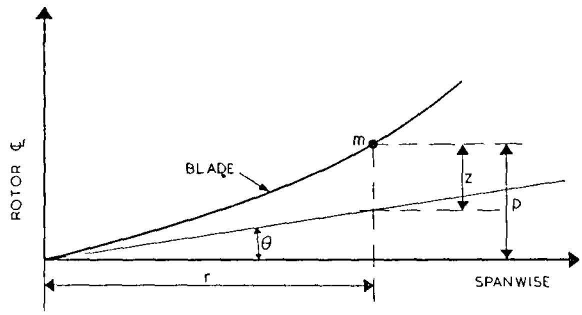
The B C A R requirements relevant to this case are given in B C A R Sec G3-5, and are supposed to give blade loads arising from towing considerations

The requirements are rather vague in the specification of the design loads since they are supposed to be determined from ultimate " maximum " undercarriage reactions, which are quoted for each set of wheels independently (*i e*, main and nose) , thus no indication of the required pitching and resultant vertical acceleration is given

If the pitching acceleration is ignored, a possible interpretation of the requirement is a vertical acceleration of 1 25 g with an ultimate factor 2 0 giving a static design ultimate factor of 2 5 as previously mentioned

It should be realised therefore, that the specification of maximum undercarriage acceleration in the present form does not define a design case for the rotor blade at all

Fig 1 Co-ordinate system for Appendix 1



Thus there is a need to specify a realistic towing case, with its associated acceleration (*i e*, type of acceleration and time of application) or alternatively a purely arbitrary ultimate factor on static blade weight

Aircraft Stationary, Rotor Moving

This is the condition of the aircraft running-up on the ground prior to take-off

We are now concerned with a more complicated dynamical system in which we must consider the dynamical properties of both the aircraft and its rotating rotor

The dynamical effects that usually give concern at this stage are those described under the heading of " ground resonance," but it should be realised that these effects are only part of a more general dynamical problem

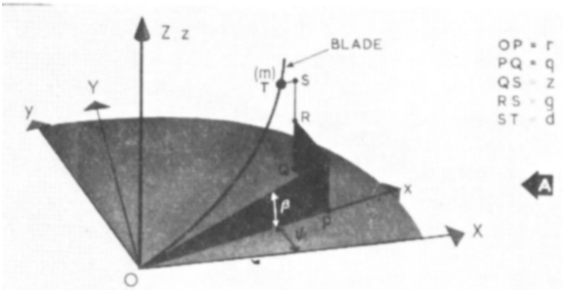
It is necessary therefore to consider this more general problem before proceeding further

The General Dynamical Problem

The problem to be solved at this stage is the determination of the motion of the elements in a three dimensional elastic system with certain space freedoms, rotating about an axis connected to a non-rotating " mass-spring " system, the elements being subjected to external periodic forces

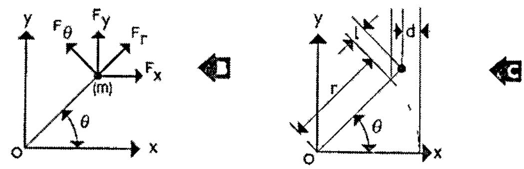
This represents the general case of a blade free to " flap " or " lag " and bend elastically while rotating about an axis, elastically restrained in the aircraft body, while the body is free in space, or elastically fixed to the ground

A completely general solution for the above problem has not yet been



GENERAL THREE DIMENSIONAL BLADE DISPLACEMENT

Fig 2 Co-ordinate system for Appendix 2



CO-ORDINATE SYSTEM FOR APPENDIX 2

FIG 2

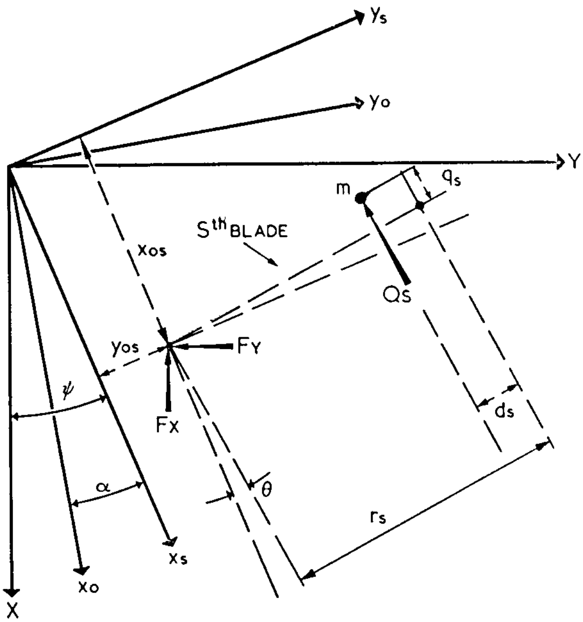


Fig 3 Co-ordinate system for Appendix 3

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attempted, but if the " flapping " and " lag " plane motions are assumed to be independent (suitable allowance being made for the " cross terms " that exist as a result of the combined motion, *e g*, " inertia forces," etc) a completely general solution can be obtained for the simplified " two dimensional " system (Fig 3)

An outline of the formal development of this two dimensional problem is given in Appendix 3

De-Coupling the Flap and Lag Plane Motion

In most physical systems, the elastic displacements are usually small enough to justify the assumption of rectilinear motion, but when an elastic displacement is superimposed upon a free body motion (*i e*, blade freely flapping and bending) the accelerations resulting from the assumptions must be carefully considered

A system considering " three dimensional " displacement of a mass rotating about a fixed axis (*e g*, free body motion with rectilinear elastic displacements superimposed, see Fig 2) is considered in Appendix 2

The results obtained show the presence of a " Coriolis " inertia term in the lag plane equation resulting from the flapping plane free body motion, and thus justifies (to the first order of small quantities) the introduction of the " Coriolis " inertia force as an external applied force together with the aerodynamic applied load in a de-coupled two dimensional lag plane system

Radial accelerations associated with this three dimensional system are also obtained, but generally these are small in magnitude when compared with the centrifugal force

It is essential, however, to include these radial terms in any balance of forces in " fixed directions " (*e g*, at the rotor head) as neglect of these radial terms will produce constant forces which do not exist (It is neglect of these terms which led to error in Mr Payne's lecture to this Association, Ref 1, and in other of his published articles)

The General Two Dimensional Dynamical Problems

The equations of motion and methods of solution are outlined in Appendix 3, but it is instructive to see how the different physically observed phenomena (*e g*, ground resonance) are related to the analytical solution

The equations of motion for the system yield a set of simultaneous differential equations, the solution of which is in the form of " a complementary function " (*i e*, solution for zero external applied force) and a " particular integral " (*i e*, solution associated with the particular external applied force)

The complementary function gives information about the natural vibration of the system (*e g*, modes and frequencies) whereas the particular integral gives the amplification effects (*e g*, displacements) associated with applied forcing loads (*e g*, oscillating blade loads)

It is the complementary function which tells us about " ground resonance " since for a given combination of dynamical parameters a range of rotor r p m exists over which the blade natural frequencies become complex (see Fig 5a and Fig 5b) and this means that the displacements become divergent over this range

The complementary function also defines the positions of the normal blade resonance conditions (*e g*, forcing frequency coincident with natural frequency), but it should be noted that displacements associated with the latter die down when the forcing load is removed whereas those associated with the former do not

Once we are satisfied that the " complementary function " is not going to give trouble (*i e* , aircraft operating away from resonance regions and unstable ranges) we must then consider the " particular integral " which assesses the effect of the forcing loads

During the " run up " of the rotor from zero to flight r p m , the frequency of the forcing loads is continually changing but provided the run up is sufficiently fast, the steady-state particular integral solution does not

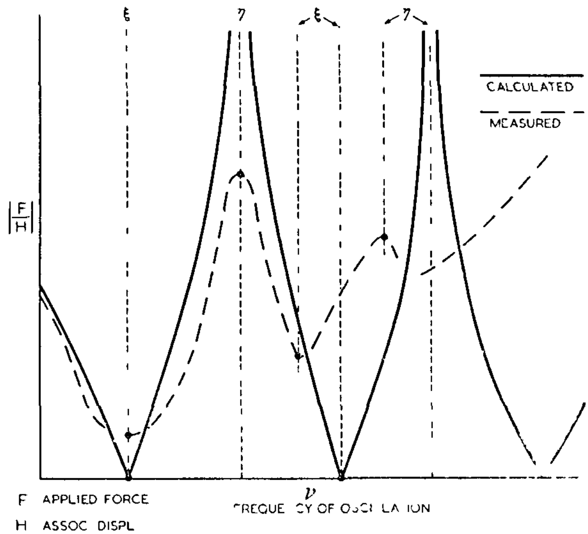

Fig 4 Calculated and Measured Impedance Plots (App 3)

have time to develop and thus it is only necessary to consider the particular integral effects at take-off or other specified steady r p m This procedure is justified since it is usually impossible to assess the blade loading during this transitional period and experience has shown that provided the rotor r p m does not dwell at resonant conditions, no undue amplification need occur (This, however, should always be checked experimentally since the actual loads resulting are very dependent upon the starting procedure)

Finally, as the problem of forced oscillation on the ground is only a particular case of the general flight conditions, this condition will be covered by the considerations of the next section

The Aircraft in the Air

Dynamical Considerations in the Air

When airborne, the considerations of Appendix 3, still apply, but in this case, however, emphasis is removed from the contribution of the undercarriage to the dynamical characteristics and transferred to any other flexible support for the rotor (*e g* , say flexibly mounted pylon) and also to the fact that the aircraft body is a mass which for small oscillations is unrestrained in space

Thus the general problem remains but with a change of dynamical constants It is therefore possible to have free air resonance (analogous to ground resonance) as well as the usual problem of forced oscillation It should also be noted that the amplification factors associated with the forced-oscillation can be considerably modified by the inclusion of the body effects (*i e* , the assumption of an axis of rotation rigidly fixed in space can give misleading results)

Before discussing the forced oscillation problem in detail, however, it is necessary to determine the stressing cases and then to assess the magnitude of the forcing loads

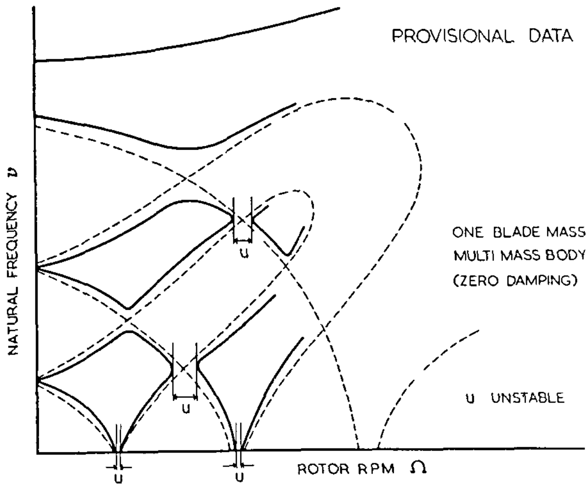

Fig 5a Natural Frequency Plot for a Two Bladed Rotor System

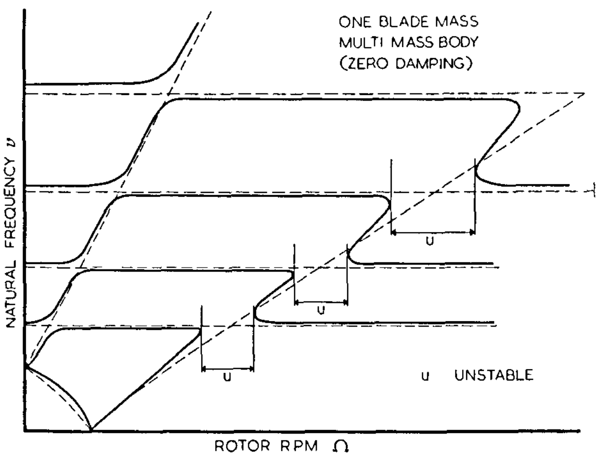

Fig 5b Natural Frequency Plot for a Four Bladed Rotor System

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The Flight Envelope

The flight envelope for a rotary wing aircraft is mentioned in B C A R Sec G3-2, but it is not defined since its formulation is very dependent upon the type of aircraft concerned

For fixed wing aircraft, a flight envelope is defined by the requirements, and provided the strength is assessed at suitable points on it, the whole envelope can then be considered as having been covered , also the aerodynamic loading is reasonably well defined and the calculated stresses are usually quite representative of those actually encountered in practice

The specification of a helicopter flight envelope is, however, more complex since there are more variables to consider than ' g ' and speed

It is useful, however, to consider a flight envelope of rectangular* form with parameters similar to those used for " fixed wing " aircraft but with the independent variables (*e g* , rotor speed, flap angle, cyclic pitch, etc) specified at certain points on it (The type of rotor head decides which variables are dependent and independent)

Having defined the flight envelope it is not always obvious which points on it constitute design cases (*e g* , for the Rotodyne a high ' g ' hovering case with rotor overspeed, can be just as critical as an off-loaded high speed case) Thus it is necessary to perform some preliminary investigations before the design cases can be specified, since these must be kept to a minimum if the subsequent analysis is not to become prohibitive

The Aerodynamic Loading

The spanwise distribution of aerodynamic loading associated with a given design case can be derived by the application of normal aerodynamic theories and will not be detailed here However, it should be realised that it is only possible to deal with a few of the lower order harmonics of forcing load (*i e* , first and second order) if the resulting analysis is not to become prohibitive, since the complexity of the loading equations increases considerably as the harmonic order is increased Even then the estimation of the lower order harmonics of forcing load usually involves much tedious calculation, since the loading equations usually fill pages rather than lines

The loading resulting from such equations is then only approximate since the induced velocity distribution, to which it partly owes its origin, is not well defined by existing theories

It is true that theories exist which determine induced velocity distributions with sufficient accuracy for performance calculations, but there is no manageable theory, to my knowledge, which accurately defines the magnitude of the induced velocity at every point in the rotor disc

The presence of oscillating loads occurring during hovering flight, as shown by strain gauge results from the Fairey Gyrodyne, Ultra Light and Rotodyne wind tunnel model (Fig 6), indicates the presence of an induced velocity distribution which is not even constant with time

It is obvious therefore, that the calculation of the lower orders of harmonic load, assuming a constant uniform induced velocity distribution, can only be approximate It would therefore be pointless to attempt to calculate higher harmonic orders, where the accuracy of the result is very dependent upon the accuracy of the induced velocity distribution

*A rectangular flight envelope is now defined for rotary wing aircraft in A P 970 Vol 3

Although the magnitude of the higher harmonic of loading decreases as the harmonic order increases, the effects of the higher harmonic loading cannot be ignored, as certain of the harmonic orders of loading have their effects amplified by the nearness of their frequencies to the overtones of the blade natural frequency

It is true that the actual harmonic which is being amplified can be altered by changing the blade's dynamical characteristics, but unless one of the higher harmonics is almost coincident with a blade natural frequency the overall effect is just to replace one amplified state by another

The effect of the higher harmonics of loading is generally to produce an additional level of stress approximately equal to that calculated from the first and second order of harmonic loading alone This indicates that an empirical factor of two, at least, should be applied to the calculated first and second harmonic orders of loading, if a realistic estimate of the flight stress levels is to be made

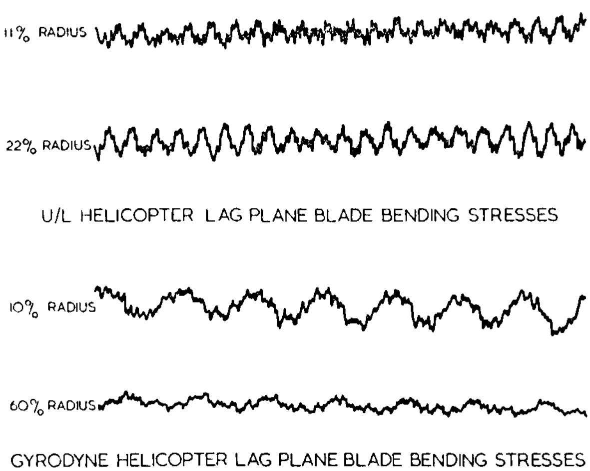

Fig 6 Typical Strain Gauge Results for a Hovering Flight Condition

A factor of this magnitude is indicated by published flight test results of American experiments (Ref 2), and is also confirmed by flight strain gauge measurements on the Fairey Gyrodyne, Ultra Light and Rotodyne wind tunnel model

Even after introducing the above factor, there is still a tendency for the calculated loading to produce stress levels which under-estimate the actual measured stress levels at outboard spanwise blade stations

Thus without considering the dynamical effects which are to be covered in a later section, much still remains to be done from the aerodynamic point of view if a true picture of the blade loading is to be obtained by calculation

The " Free Body Motion " Inertia Loading

In the aerodynamic calculations of the previous section the rotor blade is usually considered as a rigid body, and thus depending upon the flight case and the independent variables considered (*i e*, flap angle of applied

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cyclic pitch) a condition of rigid body equilibrium is obtained The resulting rigid body motion introduces various inertia forces, one of which, the "Coriolis Force," is well known It has been mentioned previously, and verified in Appendix 2, that in the decoupled lag plane analysis of blade motion, this inertia force can be added to the aerodynamic forces, obtained in the previous section, to give resulting external applied load for the forced oscillation problem

The Problem of Forced Oscillation

Having now defined the loading system, we can apply it directly to the dynamical equations formulated in Appendix 3, and from them obtain the bending moment, shear force, and deflection diagrams associated with the different harmonics of loading (see Appendix 3, Section 4 8, 4 9)

With the above data available, the problem of strength assessment then reverts to the normal stressing problems associated with any fixed wing aircraft and consequently need not be dealt with here

It is necessary, however, to give special attention to the problems of fatigue, and these will be considered in a later section

It should be realised, however, that other methods of solution exist for determining blade bending moment diagrams, shear force, and deflection diagrams, and many such methods are summarized by A Flax (Ref 3)

The advantage of the matrix approach outlined in this paper (Appendix 3), however, is that the solution to the forced oscillation problem has been obtained as the particular integral of the equations formulated for the "General Dynamical Problem," thus showing the true relationship that exists between the forced oscillation problem, and that of "Ground" or "Air" Resonance Also, it is possible to assess the effects of the aircraft body characteristics on the blade bending moments, natural frequencies, etc

To investigate such an overall problem in detail (*i e*, the numerical solution for a problem with a multi-mass distribution along the blade, together with a set of body masses), it is necessary to enlist the aid of a digital computer (Reference has already been made in Appendix 3 to a digital computer programme for the simple natural frequency problem)

It is essential, however, to choose a digital computer capable of dealing with at least twenty significant figures, since past experience has shown that it is possible for a computer to lose so many of the significant figures during the calculation, that it has actually 'lost' the problem being solved

Another important point on this aspect is, that although the physical data, associated with the problem, can be as approximate as one wishes (*e g*, choice of mass distribution, geometric characteristics, etc), it is essential that all the derived data prepared for submission to the digital computer (*i e*, stiffness matrices, etc) is exact

Without the aid of a digital computer, however, much can be learned of the blade behaviour by setting up simplified problems capable of solution on desk calculating machines

It is possible to investigate the behaviour of a blade comprising a multi-mass system with the rotor hub fixed against translation and rotation (rotating axes), and then to modify the solution by considering the effects of hub translation and rotation obtained from a single blade mass, and multi-mass body system

The results obtained by such a process, although not exact, give an indication of the correctness or otherwise of the assumptions generally used in assessing blade behaviour This is very necessary, since in all the reports on this subject I have seen to date, no account has been taken of the effect of body freedoms on blade bending moments, etc , and in the case of a flexibly mounted rotor system this effect can be quite significant

It is possible to present the equations developed in Appendix 3, in many ways in order to give the solutions of particular problems associated with any given aircraft (*e g* , the effect of " Jet cut " on the Rotodyne blade and variations of the problems indicated above), but it is not possible to enlarge further on these points within the scope of this lecture

Other Dynamical Systems

The dynamical system described in Appendix 3, is usually sufficient to deal with most rotor configurations, except for problems of dynamic stability, etc , associated with a " See-Saw " two bladed, tilting head rotor assembly In this case a straight forward application of " Euler's Dynamical Equations of Motion " as developed in most standard dynamical text books can give very interesting results

Whatever the rotor configuration being considered, however, it is essential that its dynamical characteristics are understood Failure to assess fundamental dynamical effects in the early stages of design, can seriously affect the strength and performance of the resulting aircraft

Data for Blade Stressing

The information obtained from the foregoing considerations is usually sufficient to enable a realistic strength assessment to be made, and finally a provisional blade life to be assessed

Problems associated with the fatigue aspect in the calculation of blade life, are described in the next section

Blade Life

To enable a rotary wing aircraft to become an economical transport vehicle, it is necessary to achieve a blade life of 1,000 hours at least For most helicopters this implies over 10^7 cycles of stress reversals at first harmonic (This is an optimistic assumption)

For steel rotor blades, therefore, if a life of 1,000 hours can be achieved, an infinite fatigue life should be obtained Conversely, if a life of 1,000 hours is required the blade should be designed for infinite life

The Fatigue Problem

It is not possible within the scope of this lecture to give a detailed description of all the problems that arise in connection with the fatigue life assessment of a rotor blade as methods of " fatigue life assessment " could easily be the subject of a lecture in its own right

Mention will be made, however, of some of the important factors which influence the calculation of the fatigue life, but it should be realised that the considerations which follow do not in any way constitute a complete analysis of the problem

Types of Fatigue Problems

Generally speaking fatigue problems fall into two categories (*a*) Those associated with oscillating stresses occurring in the body of a component (*e g*, normal fatigue) and (*b*) those associated with oscillating stresses, as above, together with a relative movement between the two adjacent parts, resulting in " fretting " (*i e*, fretting fatigue)

There are, of course, many other aspects of fatigue such as corrosion fatigue, rate of crack propagation, etc, but since much has already been written on the subject, they will not be considered here

Normal Fatigue Problems

The problems occurring in this category are usually those of determining the fatigue concentration factor for a component, and the fatigue allowable for the material

It is well known that the fatigue concentration factors are usually less than the equivalent elastic stress concentration factors, and much useful data on this subject is given in Ref 6

Care should always be taken, however, in applying fatigue concentration factors to redundant structures, since often the " calculated " stress at a given point results from assumptions which can usually be justified only on arguments for " Ultimate Case " stressing (*i e*, considerations in which plastic deformation is allowed to re-distribute some of the loading typical of fixed wing stressing) This does not give a true indication of the actual stress occurring at the point considered It usually results in the stress being underestimated and premature failure occurring

It is also necessary to consider the material " fatigue allowable " very carefully, since, although this is usually quite consistent for a given material specification, the allowable achieved for a given component is usually very dependent upon the state of its surface fibres

The fatigue allowable for T 60 Tube for example, is ±19 tons/sq in (25 3% Ult) in the " as drawn heat-treated " condition, whereas the removal of 0 015 ins of the internal and external " decarburized " fibres increases the fatigue allowable to ±27 5 tons/sq in (36 7% Ult)

Tests have also shown that some pickling techniques, used in conjunction with cadmium plating, can reduce the fatigue allowable of D T D 331 (S99B) by 33% (although this is not true for some of the lower grade steels)

The above are examples of reductions in the fatigue strength due to weak surface fibres On the other hand the fatigue allowable can be improved by " fine " surface finishes or the more practical process of " Capri-honing " or " Vapour blasting " For example, a specimen in D T D 331, with a 16 micro ins finish (000016 inches), has a fatigue allowable of ±22 3 tons/sq in (notched) and ±34 0 tons/sq in (unnotched), whereas a similar specimen with a basic surface finish of 54 micro ins prior to " Capri-honing " had in the final capri-honed condition a fatigue allowable of 27 5 tons/sq in (notched) and 42 7 tons/sq in (unnotched)

Consequently, bearing all the above factors in mind it is then possible to estimate the fatigue life for a given component

The " Fretting " Fatigue Problem

The most well known problem in this category is that of the normal lug

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There is considerable experimental evidence now available to show that the infinite life allowable for steel lugs is only 4 5% of the ultimate tensile stress A typical S/N curve for steel lugs is given in Fig 7

Similar effects are also obtained for sheet "lap joints," but these will be discussed in detail in the next section

Another example of fretting fatigue, which is not so obvious, is shown in Fig 8 In this case a simple oscillating tension stress was applied to the bolt, and as a result of fretting on its shear face, failed across its maximum

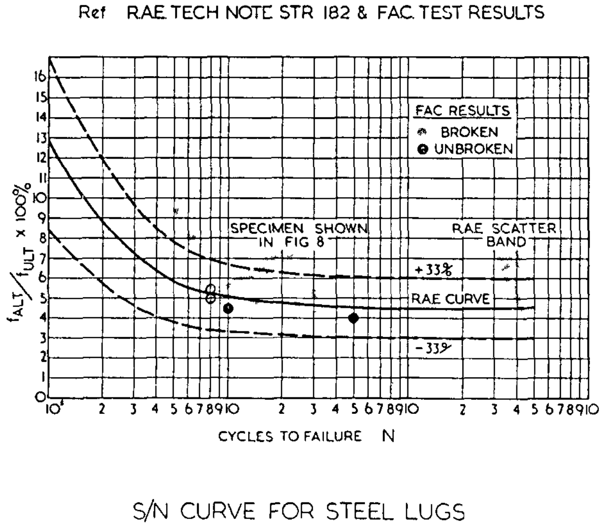

Fig 7 Lug S/N Curves

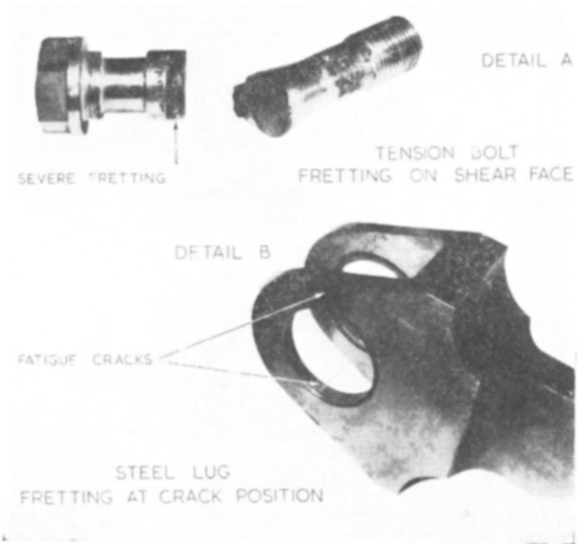

Fig 8 Typical Example of Fretting Failures (Lugs-Bolts)

diameter (no geometric stress concentration factor) at approximately 3 5% of its ultimate tensile stress after 10^6 cycles

The fretting failure of the tension bolt, referred to above, only occurred after the design preload had been reduced to zero No fretting occurred during the previous tests when preloads of 50% and 25% of its ultimate tension strength were applied Also, the specimen was unbroken after 2×10^6 cycles, in each case, for an oscillating stress approximately equal to the failing stress in the un-preloaded case above

Thus, much can be done to improve the fatigue life of such components (*e g*, preloading or clamping, etc), but it should be noted that unless the preloading is faithfully applied (either in the workshop or in the field) premature fatigue failure can occur

The Problem of Loaded Holes in a Plain Sheet

The problem to be considered here is that of a plain sheet subjected to an oscillating stress across its width, together with an additional local oscillating stress due to an oscillating load applied via a rivet or bolt hole (Fig 9)

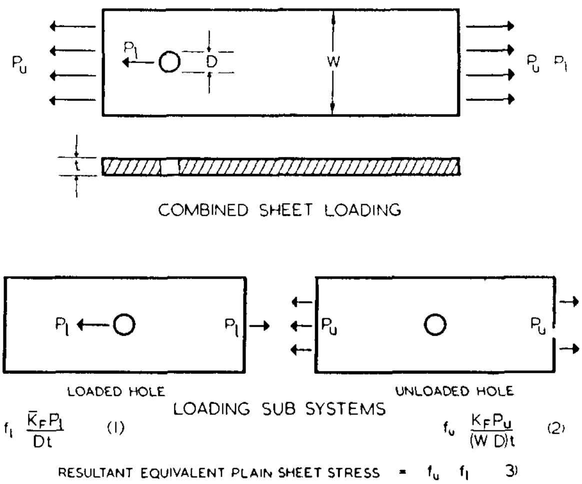

Fig 9 Load Diagram for Appendix 4

In order to obtain the life of a component loaded in this manner, it is convenient to split the loading into two systems which can be called the " loaded hole " system, and the " unloaded hole " system (see Fig 9) The stresses resulting from the two systems are added together, after the application of the respective fatigue concentration factors (K_F) The life is then estimated from an " unnotched " S/N curve

The normal definition of K_F (equivalent plain sheet stress divided by " notched " sheet stress) is quite satisfactory for the " unloaded hole " system However, it is not really suitable for the " loaded hole " system, since the local stress in the vicinity of the loaded hole is approximately constant irrespective of sheet width (*i e*, width greater than five times hole diameter) Thus, using the normal definition for K_F results in a K_F which varies with sheet width

If, however, the " notched sheet stress " is replaced by P/Dt, P (applied

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load), D (hole dia), and t (sheet thickness), a value for K_F (say $\overline{K_F}$) will be obtained, which is independent of sheet width

A provisional graph of $\overline{K_F}$ (for " loaded holes "), obtained from coupon specimens, is given in Fig 11 Also, an example of this procedure, applied to actual components, is given in Appendix 4, and the results obtained are compared with actual test data for the components analysed

It is not suggested that the above procedure is exact, but it does enable an estimate to be obtained of the importance of the local stress occurring in the vicinity of the loaded hole, since quite small oscillating rivet loads can considerably reduce the life of a given sheet

Note Fig 11 is based on coupon test data in which D (hole dia) was constant (*i e*, 0 125 ins) Consequently the test data to date strictly

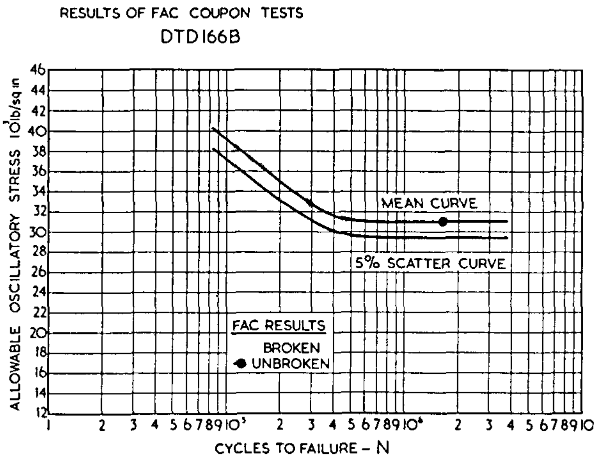

Fig 10 S/N Curve, Un-notched D T D 166B

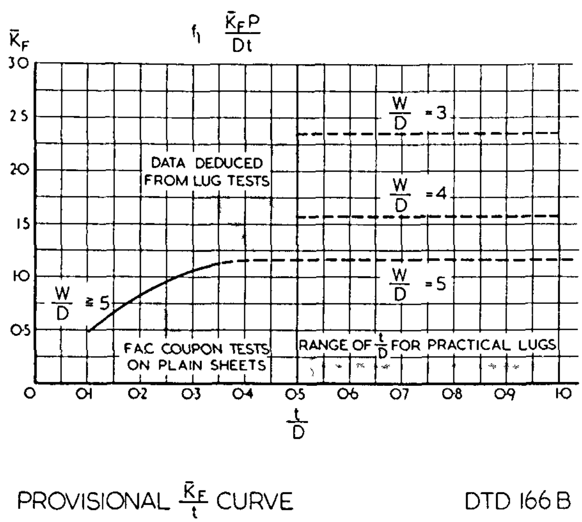

Fig 11 Plot of $\overline{K_F}$ against t/D for a Loaded Hole Condition

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only gives the variation of $\overline{K}_F$ with t for constant D

The curve has been plotted against t/D provisionally, as there is evidence from other miscellaneous tests, that it is of this form (Tests are at present proceeding to establish the final form of this curve)

Scatter

It is obvious from the example considered in Appendix 4, that the life calculated for a given component is very dependent upon the magnitude of the scatter factor applied to the test data

Since we are generally concerned with that part of the S/N curve where $N > 10^7$, it is obvious that a factor on stress is much more important than a factor on life For steel components a factor on life is generally meaningless for $N > 10^7$, whereas a small percentage on stress would considerably reduce calculated life

The choice of scatter factor (for use in deciding a safe life) is very dependent upon the number of specimens tested and other general experience with regard to similar components

The choice of an unrealistically large scatter factor involves a considerable weight penalty Alternatively, the reduction to the min safe value, involves considerable expense in testing Consequently the choice of a scatter factor is a compromise, the actual value chosen being dependent on the nature of the component considered

Fatigue Substantiation—Conclusion

The calculated blade life, based on calculated stress obtained by the methods outlined in this paper, can only be considered as very preliminary evidence in relation to fatigue substantiation

However, provided care has been taken during all of the stages in the strength assessment, the blades and rotor system should be quite safe for ground running and preliminary flying to a restricted flight envelope

It is essential, however, that all the calculated stresses are checked as early as possible by flight strain gauging, and the calculated life by destruction testing of respresentative components Only by these means is it possible to obtain the actual data required for the fatigue substantiation

The virtue of the strength assessment calculations as outlined in this paper, however, is that it has enabled many design pitfalls to be avoided (*e g* , nearness to resonance, rotor operating in an unstable r p m range, etc), and also that it has prepared a basis on which the flight test results can be considered

Pure theory will not produce a safe aircraft, neither will flight test results alone, but an intelligent combination of the two will bring out the best of both worlds, and consequently, safe aircraft

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Appendix 1

The Problem of a Stationary Rotor Blade Subjected to an Applied Acceleration

A solution is given for two different types of pitching acceleration applied to a blade idealised to a " one mass spring " system

Notation

- θ applied pitching displacement
- p total displacement ($p = z + r\theta$)
- z elastic displacement
- r spanwise position of blade mass
- m blade mass
- K blade spring stiffness
- w $(K/m)^{\frac{1}{2}}$
- t time
- A magnitude of applied acceleration
- e time of applied acceleration
- C rA/w^2
- N amplification factor

Basic equation of motion

With reference to fig 1, basic equation of motion becomes

$$m(\ddot{z} + r\ddot{\theta}) = - Kz \qquad (1)$$

Applied Acceleration

Condition 1 (Rectangular distribution)

$$0 = o \qquad t \leqslant o$$
$$0 = A \qquad o \leqslant t \leqslant e \qquad \text{i e } 0 = \tfrac{1}{2}At^2$$
$$0 = o \qquad t \geqslant e$$

Condition 2 (Triangular distribution)

$$0 = o \qquad t \leqslant o$$
$$0 = A\left(1 - \frac{t}{e}\right) \qquad o \leqslant t \leqslant e \qquad \text{i e } 0 = A\left(\frac{t^2}{2} - \frac{t^3}{6e}\right)$$
$$0 = o \qquad t \geqslant e$$

Solution

Solving equation (1) in the normal way and applying the above boundary conditions gives the following solution

Condition 1

$$\left.\begin{aligned} p &= z + \tfrac{1}{2}rAt^2 \qquad (2) \\ z &= C(\cos wt - 1) \qquad (3) \end{aligned}\right\} \quad o \leqslant t \leqslant e$$

$$\left.\begin{aligned} p &= z + rAe\left(t - \frac{e}{2}\right) \qquad (4) \\ z &= -2C \sin \frac{we}{2} \sin w\left(t - \frac{e}{2}\right) \qquad (5) \end{aligned}\right\} \quad t \geqslant e$$

Condition 2

$$\left.\begin{aligned} p &= z + \tfrac{1}{2}rAt^2\left(1 - \frac{t}{3e}\right) \qquad (6) \\ z &= C\left[\cos wt - \frac{\sin wt}{we} - \left(1 - \frac{t}{e}\right)\right] \qquad (7) \end{aligned}\right\} \quad o \leqslant t \leqslant e$$

$$\left.\begin{aligned} p &= z + \tfrac{1}{2}rAe\left(t - \frac{e}{3}\right) \qquad (8) \\ z &= C\left[(\cos we - 1)\frac{\sin wt}{we} + \left(1 - \frac{\sin we}{we}\right)\cos wt\right] \qquad (9) \end{aligned}\right\} \quad t \geqslant e$$

Note The displacement associated with the inertia force resulting from an instantaneous acceleration (i e , neglecting dynamical effects) is

$$z_s \text{ (say)} = \frac{-mAr}{K} \equiv -C$$

Discussion of Solution

The overall solution obtained is in the form of a rigid body displacement with an elastic displacement superimposed The stresses induced in the blade however are only a function of the elastic displacement, and thus only this part of the displacement need be considered

The dynamical effects can then easily be seen by introducing an amplification factor N defined as $| z/z_s |$

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Amplification factors

Condition 1

From Equation (5)

$$N_1 = 2 \sin\left(\frac{we}{2}\right) \quad \sin w\left(t - \frac{e}{2}\right)$$

This is a maximum when $t = \frac{e}{2} + \frac{\pi}{2}(1 + 4n)$ [n = integer]

$$\text{i e ,} \ N_{1(max)} = 2 \sin \frac{we}{2} \quad \text{for any e}$$

Condition 2

From Equation (9)

$$N_2 = (\cos we - 1)\frac{\sin wt}{we} + \left(1 - \frac{\sin we}{we}\right)\cos wt$$

For certain t and small e, N_2 (max) ——→ we/2
„ „ t „ some e, N_2 (max) ——→ 1 26 approx
„ „ t „ large e, N_2 (max) ——→ 1

Thus condition 1 gives a maximum amplification factor of 2 0 and condition 2 gives a maximum amplification factor of 1 26 but it should be noted that in each case the actual amplification factor is very dependent on e (the time of application of the acceleration) and the resulting amplification can be made as small as we please by suitable choice of e (i e , a small amplification only will be obtained if the acceleration is removed after a very short time)

Appendix 2

The Equations of Motion for a " Three Dimensional " Mass-Spring System with Elastic and Space Freedoms

1 *Introduction*

The equations of motion are obtained for the above system to provide data for the discussion on the de-coupling of the flap and lag plane motion

A " one " mass-spring system is considered in detail (for simplicity) but the more general solution is stated

2 *Notation*

r	spanwise reference position of blade mass
m	blade mass
x, y, z	co-ords of blade mass (rotating ref frame)
e, q, g	elastic displacement along x, y, z
d	fore-shortening of blade along x
K_x, K_y, K_z	blade " elastic " stiffness coeffs along x, y, z
a, b, c	blade " effective " stiffness coeffs along x, y, z
F_x, F_y, F_z	applied external forces
β	blade root slope ($\simeq$ rigid blade flap angle)
ψ	blade azimuth angle
Ω	blade angular velocity

3 *Derivation of "effective" stiffness coefficients*

The physical elastic displacements are radial ($\simeq$ spanwise) and spanwise bending (see fig 2b)

Thus $F_r = K_r l$ $\quad F_\theta = \frac{K_\theta \theta}{r} \simeq \frac{K_\theta y}{r^2}$

Since we are considering small displacements and large radial force (i e $ae \simeq mr\Omega^2$ to be shown later) we have

$$F_x = ae = F_r \cos \theta - F_\theta \sin \theta \simeq F_r$$

$$F_y = by = F_\theta \cos \theta + F_r \sin \theta \simeq F_\theta + F_x \frac{y}{r}$$

Hence $b = \frac{F_\theta}{y} + \frac{F_x}{r} = \frac{K_\theta}{r^2} + \frac{ae}{r}$

Now $K_y = K_\theta / r^2$ (equivalent linear bending stiffness)

Thus $b = K_y + \frac{ae}{r}$ $\quad$ (e g $K_y + m\Omega^2$)

Similarly for flapping plane, allowing for flapping freedom

$$c = K_z + \frac{ae}{r} \quad \frac{z}{g} \left(\text{e g} \quad K_z + \frac{m\Omega^2 z}{g} \right)$$

4 *Derivation of blade fore-shortening "d"*

From fig 2c (two dimensional example)

$$\begin{aligned} d &= r(1 - \cos \theta) - l \cos \theta \\ &= r\theta^2/2 - e \\ &= y^2/2r - e \end{aligned}$$

Thus generally $d = \frac{1}{2r} (y^2 + z^2) - e$

5 *Derivation of equations of motion*

Co-ordinates of mass point m

With reference to fig 2a, we have

$$x = r - d, \; y, \; z = r \sin \beta + g$$

Velocity components of m with respect to x, y, z

Velocity components are

$$\dot{x} - y\dot{\psi}, \; \dot{y} + x\dot{\psi}, \; \dot{z}$$

Thus kinetic energy

$$T = \frac{m}{2} \left\{ \dot{x}^2 - 2y\dot{x}\dot{\psi} + y^2\dot{\psi}^2 + \dot{y}^2 + 2x\dot{y}\dot{\psi} + \dot{z}^2 + x^2\dot{\psi}^2 \right\}$$

Potential energy

$$V = \tfrac{1}{2} \left\{ ae^2 + by^2 + cg^2 \right\}$$

Sub in Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q$$

Where $q = x, y$ or z $\quad Q = F_{x, y \text{ or } z}$ (Note $F_x = o$)

Equations of motion become

$$\left.\begin{array}{ll} x\ dir^n & m(x - 2y\psi - y\psi - x\psi^2) + ae = F_x = 0 \\ y\ dir^n & m(y + 2x\psi + x\psi - y\psi^2) + by = F_y \\ z\ dir^n & mz + cg = F_z \end{array}\right\} \quad (1)$$

If we assume

(a) β small i e $\sin\beta = \beta \quad \cos\beta = 1$

(b) $\psi = 0$ i e $\psi = \Omega$ (constant angular velocity)

(c) $x - 2y\psi$ small compared with $x\psi^2$

(d) $d = \frac{1}{r}(yy + zz) - e \simeq r\beta\beta$

since the elastic displacements and their derivatives are small compared with the free body displacement,
then equation (1) becomes

$x\ dir^n$

$$-mx\Omega^2 + ae = 0$$

$$ae = mx\Omega^2 \simeq mr\Omega^2 \quad (2)$$

$y\ dir^n$

$$m(y - 2d\Omega - y\Omega^2) + (K_y + m\Omega^2)y = F_y$$

$$\left.\begin{array}{r} my + [K_y + (m-m)\Omega^2]y = F_y + 2mr\Omega\beta\beta \\ my + K_y y = F_y + 2mr\Omega\beta\beta \end{array}\right\} \quad (3)$$

$z\ dir^n$

$$mz + \left(K_z + \frac{m\Omega^2{}_z}{g}\right)g = F_z$$

$$mz + (K_z g/z + m\Omega^2)z = F_z \quad (4)$$

Note When more than one mass is considered the term (m—m) above, equation 3, is not zero (see below)

6 *Statement of General Solution*

The general solution for equations 2, 3 and 4 for "N" mass points on one blade defined by the co-ords (r_1, , r_N) become

*x dir*ⁿ Normal centrifugal force equations

*y dir*ⁿ *and z dir*ⁿ Written in formal matrix notation (considering the Coriolis acceleration $2\ mr\Omega\beta\beta$ as an external applied force in conjunction with F_y) equations become

$$[m]\left\{q\right\} + \left([S] + [C]\Omega^2\right)\left\{q\right\} = \left\{Q\right\} \quad (5)$$

or less formally, matrix notation implied

$$mq + (S + C\Omega^2)q = Q \quad (6)$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \\ q_N \end{bmatrix} \quad Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \\ Q_N \end{bmatrix} \quad \begin{array}{l} S = a^{-1} \quad (\text{y dir}_n \quad \text{lag}) \\ \\ S = a^{-1}\left(I - \dfrac{rr'a^{-1}}{r'a^{-}r'}\right) \quad (\text{z dir}_n \quad \text{flap}) \end{array}$$

$$\begin{array}{lll} C = R^{-1}F - m & (\text{y dir}_n & \text{lag}) \\ C = R^{-1}F & (\text{z dir}_n & \text{Flap}) \end{array}$$

$$F = \begin{bmatrix} F_1, & F_2, & F_3 & , F_N \\ -\sum_{i=2}^{n} F_i, & F_2, & F_3 & , F_N \\ 0, & -\sum_{i=3}^{n} F_i, & F_3 & , F_N \\ \\ 0, & 0, & 0 & , F_N \end{bmatrix} \quad F_N = m_N r_N$$

$$R = \begin{bmatrix} r_1, & r_2, & & , r_N \\ 0, & r_2 - r_1, & & , r_N - r_1 \\ \\ \\ 0, & & , & , r_N - r_{N-1} \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11}, & a_{12}, & & , a_{1N} \\ \\ \\ a_{n1}, & & & , a_{NN} \end{bmatrix}$$ Flexibility matrix for blade with "built-in" root

$$m = \begin{bmatrix} m_1, & 0, & 0, & , 0 \\ 0, & m_2, & 0, & \\ \\ \\ 0, & & & , m_N \end{bmatrix}$$ mass matrix

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \\ \\ r_N \end{bmatrix}$$

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Appendix 3

The General " Two Dimensional " Dynamical Problem

1 *Introduction*

The " Two dimensional " dynamical problem considered in this appendix is that associated with " in-plane " oscillations (e g , oscillations in the plane perpendicular to the axis of rotation) as this has a direct bearing on the problem of Ground Resonance and general lag plane motion   Similar results can also be obtained for flapping plane motion

In order to simplify the analysis of this problem it is convenient to break the basic system into two sub-systems, and to replace the internal forces occuring in the basic system at the break, by external forces in the two sub-systems

Each sub-system is then analysed separately and a relationship obtained between the external forces acting at the break and the associated displacement (such a relationship defines the " Impedance " of the sub-systems)

The solution of the overall problem is then obtained by considering compatibility relationships between the forces and displacements at the break in the basic system

The sub-systems considered in this appendix are, (a) The aircraft body (b) The rotor, the basic system being broken into two sub-systems at the rotor centre line

2 *Notation*

r	Spanwise ref position of blade mass
m_s	Blade mass on Sth Blade
x, y, z	Co-ordinates of blade mass (rotating ref frame)
q	Elastic displacement of blade
d	Fore-shortening of the blade along x
H_s	$x_{0s} + iy_{0s}$, Hub displacement with respect to Sth axis
Suffix S	Denotes Sth Blade
θ	Body rotation of complete system
ψ	Blade azimuth angle
Ω	Blade angular velocity
I	Body impedance $K\xi/\eta$ (Operator)
F_x, F_y	Force applied by body to Rotor in direction X, Y
Q	Force applied to blade in direction q
S	Elastic stiffness matrix
C	Centrifugal stiffness matrix for unit Ω
N	Number of masses per blade
j	Number of blades
D	Operator d/dt
Π	Product

NOTE Other notation is defined as required

3 *The aircraft Body Impedance*

The equations of motion for the body can be obtained by the usual Lagrangian method, and the impedance at the junction of the rotor head to the aircraft body subsequently derived

It is not practical to develop a general solution for the body impedance, since the specification of the dynamical characteristics for a " general aircraft body " would be extremely complex

The final form of the body impedance however is stated, and a method of obtaining the impedance parameters from a vibration test is given

NOTE In deriving the body impedance it is convenient to assume that a mass equal to the total rotor mass is situated at the rotor centre line (See section 4 below) This mass should be included in all body impedance calculations or tests

3 1 *General Form of Body Impedance*

The equations of motion for the body (at the break) yield the following

$$F = IH$$

Where H = displacement

I = impedance = $K\xi/\eta$

$\xi = \prod_{i=1}^{p} (\xi_i^2 + D^2)$, Polynomial in D^2 (ξ_i^2 root of $\xi = 0$)

$\eta = \prod_{i=1}^{p-1} (\eta_i^2 + D^2)$, Polynomial in D^2 (η_i^2 root of $\eta = 0$)

K = constant

p = number of degrees of body freedom

3 2 *Impedance Parameters Obtained from Vibration Tests*

F (applied forcing load) = $F_0 \cos wt$
H (resulting displacement) = $H_0 \cos wt$

Then $$\frac{F_0}{H_0} = K \frac{\prod_{i=1}^{p} (\xi_i^2 - w^2)}{\prod_{i=1}^{p-1} (\eta_i^2 - w^2)}$$

Thus if $\left|\frac{F_0}{H_0}\right|$ is plotted against w (see Fig 4) the values of ξ_i and η_i can be obtained

NOTE $$K = \frac{\prod_{i=1}^{p-1} \eta_i^2}{\prod_{i=1}^{p} \xi_i^2} \qquad \left|\frac{F_0}{H_0}\right| \ w = 0$$

4 0 *Equations of Motion for the Rotor*

4 1 *Co-ords of Mass Point m_s (i e, mass on Sth Blade)*

$$z_s = H_s + e^{i\theta} (ir_s - id_s - q_s) \tag{1}$$

NOTE

(a) For simplicity only one mass point on each blade will be considered (General solution for " N " mass points however will be stated)

(b) The suffix " s " will be dropped while considering one blade only, but will be re-introduced when combined blade effects are considered

(c) We assume H, θ, q, are first order terms and d second order All terms above second order in the subsequent analysis will be neglected

4 2 *Kinetic Energy of Mass Point m*

$$T = \tfrac{1}{2}m\left\{\dot{\bar{z}}\dot{z} + i\Omega(\bar{z}\dot{z} - \dot{\bar{z}}z) + \Omega^2\bar{z}z\right\} \tag{2}$$

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Substituting for z from (1) gives

$$T = (A) + (B) + (C)$$

Where

$$(A) = \tfrac{1}{2}m\left\{\overline{H}H + i\Omega\left(\overline{H}H - \overline{H}H\right) + \Omega^2\overline{H}H\right\}$$

$$(B) = \tfrac{1}{2}m\left\{q^2 + \Omega^2\left(r^2 + q^2 - 2rd\right) + 2\Omega rq\right\}$$

$$(C) = \tfrac{1}{2}m\Bigg[\left\{-\left(\theta r+q\right)\left(H + \overline{H}\right) + \left(\theta r\right)^2 + 2\theta rq\right\} +$$

$$i\Omega\left\{-\left(\theta r + q\right)\left(H - \overline{H}\right) - \left(\theta r + q\right)\left(\overline{H} - H\right) + ir\left(\overline{H} + H\right) - 2i\theta r^2\right\} +$$

$$\Omega^2\left\{-\left(\theta r + q\right)\left(\overline{H} + H\right) + ir\left(\overline{H} - H\right) + \left(\theta r + q\right)^2\right\}\Bigg]$$

NOTE

Bracket (A) is the Kinetic energy of a mass " m " situated at the rotor hub expressed in co-ords defined with respect to the rotating ref frame

Bracket (B) is the Kinetic energy of the mass "m" situated at point r for a blade whose root is fixed against translation and rotation (i e it can be shown to be identical with the K E in Appendix 2 when $z = 0$)

Bracket (C) is the Kinetic energy of the mass point " m " due to the translation and rotation of the blade root

4 3 *Total Kinetic Energy of Complete System*

We must now re-introduce suffix " s " in order to differentiate between the different blades

$$\text{Thus} \quad T = \sum_{s=0}^{j-1} T_s = \sum_{s=0}^{j-1} \left\{(A)_s + (B)_s + (C)_s\right\} \tag{4}$$

Since we have j blades ($s = 0, 1 \quad , j-1$) each separated from its neighbour by the angle $2\pi/j$, H_s and H_0 are related by the equation

$$H_s = H_0 e^{-\alpha i} \quad (s = 0, \quad , j-1) \tag{5}$$

$$\text{Where } \alpha = \frac{2\pi s}{j}$$

For convenience in representing products of the form $H_s q_s$ we define

$$\left.\begin{aligned} (j)^{\frac{1}{2}} R &= \sum_{s=0}^{j-1} q_s e^{\alpha i} \\ (j)^{\frac{1}{2}} \overline{R} &= \sum_{s=0}^{j-1} q_s e^{-\alpha i} \end{aligned}\right\} \tag{6}$$

NOTE $\quad \sum_{s=0}^{j-1} H_s = j H_0$, since $\sum_{s=0}^{j-1} e^{-\alpha i} = 1$

Substituting the terms (5) and (6) in (4) we have

$$T = -\tfrac{1}{2}m(j)^{\frac{1}{2}}\left\{R\overline{H} + \overline{R}H + \Omega^2\left(\overline{H}R + H\overline{R}\right) + i\Omega\left(\overline{H}R - H\overline{R} - \overline{H}R + H\overline{R}\right)\right\}$$

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$$+\tfrac{1}{2}M\left\{\bar{H}H + \Omega^2\bar{H}H + i\Omega\left(\bar{H}H - \bar{H}H\right)\right\}^*$$

(7)

$$+\sum_{s=0}^{j-1} mr_s\,\theta q_s + \tfrac{1}{2}m \sum_{s=0}^{j-1}\left\{q^2 + \Omega^2\left(r^2 + q^2 - 2rd\right) + 2\Omega rq\right\}_s$$

$$+ J\left(\Omega\theta + \frac{\theta^2}{2}\right)$$

Where we have denoted,

H_0 by H

jm by M (Total blade mass)

$\sum_{s=0}^{j-1} mr_s^2$ by J (Moment of inertia of complete rotor about the axis of rotation)

* With ref to § 4 2 bracket (A), this is the total K E of a mass equal to the rotor mass situated at the hub This term can be omitted from this equation if we include it in the body impedance

Thus we define $T' = T - \left\{\quad\right\}^*$ (8)

4 4 *Total Potential Energy V*

With ref to Appendix 2, this becomes

$$V = \tfrac{1}{2}\sum_{s=0}^{j-1} bq_s^2 \qquad (9)$$

4 5 *Lagrange's Equations*

We define L (Lagrangian function) $= T' - V$

We define $\mathscr{L}q$ (Lagrangian operator) $= \dfrac{d}{dt}\left\{\dfrac{\partial}{\partial \dot{q}}\right\} - \dfrac{\partial}{\partial q}$

Thus Lagranges equations $\dfrac{d}{dt}\left\{\dfrac{\partial T'}{\partial \dot{q}}\right\} - \dfrac{\partial T'}{\partial q} + \dfrac{\partial V}{\partial q} = Q$

become $\mathscr{L}_q L = Q$ . . . (10)

4 6 *The Determination of the Generalised Forces*

The independent variables in L are —

$H_0 = x_{00} + iy_{00}, q_s, d_s, \theta$

The external forces are —F, Q_s ($s = o \quad , j-1$) see Fig 3

$$\text{Now } z_s = H_s + e^{i\theta}\left(i(r-d)-q\right)_s \qquad (1)$$

Also

$$\left.\begin{aligned} z_s &= x_s + iy_s \\ H_s &= x_{0s} + iy_{0s} \\ x_s &= x_{0s} + (d-r)_s \sin\theta - q_s\cos\theta \\ y_s &= y_{0s} - (d-r)_s\cos\theta - q_s\sin\theta \\ x_{0s} &= x_{00}\cos\alpha + y_{00}\sin\alpha \\ y_{0s} &= -x_{00}\sin\alpha + y_{00}\cos\alpha \end{aligned}\right] \qquad (11)$$

and $X = x_{0s} \cos \Omega t - y_{0s} \sin \Omega t$
$Y = x_{0s} \sin \Omega t + y_{0s} \cos \Omega t$ (12)

The virtual work during the elemental displacement becomes therefore,

$$\delta W = -F_x \delta X - F_y \delta Y + \sum_{s=0}^{j-1} -(Q_s \cos \theta\, \delta x_s + Q_s \sin \theta\, \delta y_s) \tag{13}$$

Using expressions (11) and (12) we have

$$\delta W = Q_{x00}\, \delta x_{00} + Q_{y00}\, \delta y_{00} + Q_{qs}\, \delta q_s + Q_\theta\, \delta\theta \tag{14}$$

Where

$$\left.\begin{aligned} Q_{x00} &= -F_x \cos(\Omega t - \alpha) - F_y \sin(\Omega t - \alpha) - \sum_{s=0}^{j-1} Q_s \cos(\alpha + \theta) \\ Q_{y00} &= +F_x \sin(\Omega t - \alpha) - F_y \cos(\Omega t - \alpha) - \sum_{s=0}^{j-1} Q_s \sin(\alpha + \theta) \\ Q_{qs} &= Q_s \\ Q_\theta &= rQ_s \\ Q_d &= o \end{aligned}\right] \tag{15}$$

Thus equation (15) define the generalised forces associated with the independent co-ordinates

4 7 *The Equation of Motion*

$$\mathscr{L}_{x00} L = Q_{x00}$$

$$\mathscr{L}_{y00} L = Q_{y00}$$

Thus $Q_{x00} + iQ_{y00} = (\mathscr{L}_{x00} + i \mathscr{L}_{y00}) L$ (16)

Now it can be shown that

$$Q_{x00} + iQ_{y00} = -Fe^{-i(\Omega t - \alpha)} - \sum_{s=0}^{j-1} Q_s e^{i(\theta+\alpha)} \tag{17}$$

Also that

$$(\mathscr{L}_{x00} + i \mathscr{L}_{y00}) L = 2\mathscr{L}_{\bar{H}_0} L \tag{18}$$

Thus from (16), (17) and (18) we have

$$\left.\begin{aligned} \mathscr{L}_{\bar{H}_0} L &= \tfrac{1}{2}\left\{-Fe^{-i(\Omega t-\alpha)} - \sum_{s=0}^{j-1} Q_s e^{i(\theta+\alpha)}\right\} \\ \text{also} \quad \mathscr{L}_{qs} L &= Q_s \\ \mathscr{L}_\theta L &= Q_s r_s \end{aligned}\right] \tag{19}$$

Thus

$$-F - e^{i(\Omega t-a)}\sum_{s=0}^{j-1} Q_s\, e^{i(\theta+a)} = -m\,(j)^{\frac{1}{2}}\sigma \tag{20}$$

where $\sigma = D^2 \left\{Re^{+i(\Omega t-a)}\right\}$ (20a)

Also

$$Q_s=\left\{mr\theta+mq+(m\Omega^2+b)q-\tfrac{1}{2}m(j)^{\frac{1}{2}}\left[(D-i\Omega)^2\bar{H}_0+(D+i\Omega)^2H_0\right]\right\}_s \tag{21}$$

$$Q_s r = J\theta + \sum_{s=0}^{j-1} mr_s\, q_s \tag{22}$$

THE ABOVE EQUATIONS (20, 21 AND 22) ARE THE BASIC EQUATIONS OF MOTION FOR A ROTOR WITH " J " BLADES AND " ONE " MASS POINT PER BLADE

4 8 *Basic Rotor Equation of Motion for a Rotor with " j " Blades and " N " Mass Points Per Blade*

Writing equations in formal matrix notation we have

$$-F - e^{i(\theta+\Omega t-a)}\sum_{i=1}^{N}\sum_{s=0}^{j-1} Q_{is}\, e^{ia} = -(j)^{\frac{1}{2}}\,[\bar{m}]'[\sigma] \tag{23}$$

$$[Q_s] = [\lambda_s]\,\theta + [m]\,[q_s] + \left([s] + [c]\,\Omega^2\right)[q_s] \tag{24}$$

$$-\tfrac{1}{2}(j)^{\frac{1}{2}}[\bar{m}]\left\{(D - i\Omega)^2\,\bar{H}_0 + (D + i\Omega)^2\,H_0\right\}$$

$$T\ (\text{Total External Torque}) = [r]'\ [Q_s] = J\theta + \sum_{s=0}^{j-1}[\lambda_s]'\,[q_s] \tag{25}$$

Where

[m], [C], [S] are square matrices as defined in appendix 2

$$[Q_s]=\begin{bmatrix} Q_{1s} \\ \\ \\ Q_{Ns}\end{bmatrix},\ [r_s]=\begin{bmatrix} r_{1s} \\ \\ \\ r_{Ns}\end{bmatrix},\ [q_s]=\begin{bmatrix} q_{1s} \\ \\ \\ q_{Ns}\end{bmatrix},\ [\sigma]=\begin{bmatrix} \sigma_1 \\ \\ \\ \sigma_{\bar{N}}\end{bmatrix},\ [\bar{m}]=\begin{bmatrix} m_{1s} \\ \\ \\ m_{Ns}\end{bmatrix}$$

$$[\lambda_s] = [m]\,[r_s]$$

Equations (23), (24) and (25) have been given in terms of the physical co-ords (Rotating) associated with the system (e g H, q) and for certain special cases of the general solution this is the most convenient form

The general expansion however of the matrix equation (i e , expanding the square matrices) yields a set of simultaneous equations in which each

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equation contains all the N co-ords q_{is} ($i = 1 \quad N$) and their differential coefficients

To simplify the general solution therefore we apply the following transformation to q

Transformation of Co-ordinates

$$p_i = \frac{1}{\sqrt{j}} \Gamma q_i \tag{26}$$

Where

$$p_i = \begin{bmatrix} p_{i\,0} \\ \\ \\ \\ p_{i\,j\,1} \end{bmatrix} \quad q_i = \begin{bmatrix} q_{i\,0} \\ \\ \\ \\ q_{i\,j\,1} \end{bmatrix} \quad \Gamma = \begin{bmatrix} 1, & 1, & 1, & , & 1 \\ 1, & \gamma, & \gamma^2, & , & \gamma^{j\,1} \\ \\ 1, & \gamma^n, & \gamma^{2n}, & , & \gamma^{n^2} \\ \\ 1, & \gamma^{j\,1}, & \gamma^{2(j\,1)} & & ,\gamma^{(j\,1)^2} \end{bmatrix} \gamma = e^{\frac{2\pi i}{j}}$$

Modified equations of motion

If we now apply this transformation to the co-ords in L (equations (8) and (9)) we have

$$\mathscr{L}_{P_{i1}} L = \mathscr{L}_{R_i} L = Q_{R_i} \tag{26}$$

NOTE L is a function of R_i and $\bar{R}_i$ and if $j \neq 2$, R_i and $\bar{R}_i$ are independent, but when $j = 2$, $R_i = \bar{R}_i$ i e, not independent This results in two different equations for $\mathscr{L}_{P_{i1}}$ depending upon whether $j \geqslant 2$

The resultant matrix equation thus become —

Equation 23 remains unaltered ($j \geqslant 2$)

Equation 24 becomes

$$(j)^{\frac{1}{2}}[Q][G] = [m](D - i\Omega)^2[\sigma] + \left([s] + [c]\Omega^2\right)[\sigma] - \tfrac{1}{2}(j)^{\frac{1}{2}}[\bar{m}]'D^2W, (j > 2) \tag{27}$$

and

$$(j)^{\frac{1}{2}}[Q][G] = [m](D - i\Omega)^2[\sigma] + \left\{[s] + [c]\Omega^2\right\}[\sigma]$$

$$-\tfrac{1}{2}(j)^{\frac{1}{2}}[\bar{m}]'\left\{D^2W + (D - 2i\Omega)^2 \bar{H}_0 e^{i(\Omega t - \alpha)}\right\} (j = 2) \tag{28}$$

Where

$W = X + iY$,

$$[G] = \begin{bmatrix} 1 \\ g \\ \\ g^n \\ \\ g^{j\,1} \end{bmatrix}, g = e^{\frac{2\pi i}{j}}, \quad [Q] = \begin{bmatrix} Q_{10} & & , Q_{1\,j\,1} \\ \\ & Q_{is} & \\ \\ Q_{no} & & , Q_{n\,j\,1} \end{bmatrix}$$

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4 9 *Special Cases of the General Rotor Equations*

4 91 *Blade whose root is fixed against translation and rotation (with respect to rotating axis)*

$H_0 = \theta = 0$

Thus equation 24 becomes

$[Q_s] = [m] [\ddot{q}_s] + \left([s] + [c] \Omega^2\right) [q_s]$ (see also appendix 2)

From this equation values of blade deflection, shear and bending moment can be obtained by suitable application of matrix algebra

Frequency equation (Characteristic Equation)

The complimentary function solution for the above differential equation (i e , solution for $[Q_s] = 0$) yields the frequency equation

$$\Delta = \left| \left([s] + [c] \Omega^2\right) - [m] \lambda^2 \right| = 0$$

Where Δ denotes determinant

λ denotes the blade natural frequencies associated with a given value of Ω

NOTE

A standard S B A C digital computer programme has been prepared for obtaining the values of λ from the above equation by an iterative process

This result in a slightly modified form was also obtained by the Bristol Aircraft Co

4 92 *Frequency Equation for the Complete system (Blades and Body)*

Section 4 91 above considered the natural frequencies of the blade alone, and assumed an infinite rigid body If we now consider the system as a whole, we then obtain a frequency equation and the associated natural frequencies for the complete system

In this case however, some of the frequencies obtained are complex for a range of rotational speed and this implies divergent oscillations over this range

This problem is usually discussed under the heading of " Ground " or " Air " resonance

The equations for the complementary function solution then reduce to —

From equations (23), (27) and (28)

$$F = + (j)^{\frac{1}{2}} [\bar{m}]' [\sigma] \qquad (29)$$

$$\underline{[m] (D - i\Omega)^2 [\sigma] + \left([s] + [c] \Omega^2\right) [\sigma] - \tfrac{1}{2} (j)^{\frac{1}{2}} [\bar{m}]' D^2 W = 0, (j > 2)} \qquad (30)$$

$$\underline{[m] (D - i\Omega)^2 [\sigma] + \left([s] + [c] \Omega^2\right) [\sigma] - \tfrac{1}{2} (j)^{\frac{1}{2}} [\bar{m}]' \left\{D^2 W + (D - 2i\Omega) \bar{H}_0 e^{i\Omega t}\right\} = 0,}$$

$$(j = 2) \qquad (31)$$

Also $\underline{F = F_x + iF_y}$

$\underline{F_x = I_x X, \qquad F_y = I_y Y}$ (See Sec 3 1) (32)

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Thus

For a rotor with more than two blades ($j>2$) we have two equations (29) and (30) and two unknowns (W, σ), hence we can solve equations for the variables (fixed co-ords) W and σ

For a rotor with two blades ($j = 2$) we have two equations (29) and (31) and three unknowns (W, σ, H_0), hence we cannot solve the equations in their present form, and it is necessary to re-state them in rotating co-ords

Thus for a rotor with more than two blades, a solution can be obtained in fixed co-ordinates, and consequently a body impedance with directional properties (i e, $F_x \neq F_y$) may be introduced For a rotor with two blades however it is necessary to consider a body impedance with non-directional properties (i e, $F_x = F_y$) in order to obtain a transformation into rotating co-ordinates that does not include cyclic coefficients

If $F_x \neq F_y$ cyclic coefficients will occur for the two bladed rotor and the resulting equations then become insoluble (by normal methods at least)

4 921 *The Frequency Equation for a Rotor with more than Two Blades (Ground or Air Resonance Equations)*

For simplicity we will consider a rotor with "one" mass "m" per blade, "Coleman's" equation will be derived as a particular case (e g, Body with one degree of freedom)

We define

$$\left.\begin{aligned} (j)^{\frac{1}{2}}\sigma &= q_x + i q_y \\ F &= F_x + i F_y \end{aligned}\right]$$

Where

$$\left.\begin{aligned} q_x &= \cos(\Omega t - \alpha)\sum_{s=0}^{j-1} q_s \cos\alpha - \sin(\Omega t - \alpha)\sum_{s=0}^{j-1} q_s \sin\alpha \\ q_y &= \cos(\Omega t - \alpha)\sum_{s=0}^{j-1} q_s \sin\alpha + \sin(\Omega t - \alpha)\sum_{s=0}^{j-1} q_s \cos\alpha \\ F_x &= (K\xi/\eta)_x \quad X \\ F_y &= (K\xi/\eta)_y \quad Y \end{aligned}\right] \qquad (33)$$

NOTE $\left.\begin{aligned} [\bar{m}]' &= [m] = m \\ q_{is} &= q_s \end{aligned}\right]$ for $N = 1$ (one mass per blade)

The equations (29) and (30) thus become (dropping formal matrix notation, but matrix terms implied where relevant)

$$\left.\begin{aligned} -F_x + mD^2q_x &= 0 \\ -F_y + mD^2q_y &= 0 \\ m(D^2q_x + 2\Omega Dq_y - \Omega^2 q_x) + (S + C\Omega^2)\,q_x - \tfrac{1}{2} j m D^2 X &= 0 \\ m(D^2q_y - 2\Omega Dq_x - \Omega^2 q_y) + (S + C\Omega^2)\,q_y - \tfrac{1}{2} j m D^2 Y &= 0 \end{aligned}\right] \qquad (34)$$

Eliminating F_x, F_y using equ (33) and expressing in matrix form we have

$$\begin{bmatrix} -\tfrac{1}{2}jm^2\eta_x D^4 + mK_x\xi_x(D^2-\Omega^2) + K_x\xi_x(S+C\Omega^2), & 2\Omega m K_x\xi_x D \\ -2\Omega m K_y\xi_y D, & -\tfrac{1}{2}jm^2\eta_y D^4 + mK_y\xi_y(D^2-\Omega^2) + K_y\xi_y(S+C\Omega^2) \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = 0$$

Dividing throughout by m and collecting terms in Ω^2, and simplifying by introducing the parameters

$$\left.\begin{aligned} \lambda_1 = c/m, \qquad \lambda_2 = s/m, \qquad \lambda_3 = j\,m \\ A_x = (D^2+\lambda_2)K_x\xi_x - \tfrac{1}{2}\lambda_3\eta_x D^4 \\ A_y = (D^2+\lambda_2)K_y\xi_y - \tfrac{1}{2}\lambda_3\eta_y D^4 \end{aligned}\right] \quad (35)$$

We have,

$$\begin{bmatrix} A_x - (1-\lambda_1)\Omega^2 K_x\xi_x, & 2DK_x\xi_x\Omega \\ -2DK_y\xi_y\Omega, & A_y - (1-\lambda_1)\Omega^2 K_y\xi_y \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = 0$$

Thus since q_x, q_y are of the form $q_{x0}\,e^{i\nu t}$, $q_{y0}e^{i\nu t}$, then for D^2 we can write $-\nu^2$ and the characteristic equation then becomes

$$(\Omega^2)^2 - \left(B_x + B_y + \frac{4\nu^2}{(1-\lambda_1)^2}\right)(\Omega^2) + B_xB_y = 0 \quad (36)$$

which is a quadratic in (Ω^2)
where

$$\left.\begin{aligned} B_x = \frac{1}{(1-\lambda_1)}\,\frac{A_x}{K_x\xi_x} \\ B_y = \frac{1}{(1-\lambda_1)}\,\frac{A_y}{K_y\xi_y} \end{aligned}\right] \quad (37)$$

NOTE

The substitution of the following parameters in equations (29) and (30) yield " Coleman's " basic equations (Ref 5 equ 26)

Present Notation	*Coleman's Notation*
S	K_β/b^2
iW	z_f
jm	n m_b
$1/j\ \Sigma q_s e^{i\alpha}$	θ_j
$1/(j)^{\frac{1}{2}}\ \sigma$	ζ_i
F_x	$F_{y_f} = \{m_y + nm_b)D^2 + K_{y_f}\}\, y_f$
$-F_y$	$F_{x_f} = \{(m_x + nm_b)D^2 - K_{x_f}\}\, x_f$

4 922 *The Frequency Equation for a Rotor with Two Blades*

Introducing the following notation —

$D^* = D + i\Omega$
$D^{(*)} = D - i\Omega$

and if Φ (D) is a polynomial in D = Φ
then $\Phi(D + i\Omega), \Phi(D - i\Omega) = \Phi^*, \Phi^{(*)}$
Then from equation (29) we have

$$Fe^{-i(\Omega t-\alpha)} = f = \sqrt{2}\, m\, D^{*2} P \qquad (38)$$

and from equation (31) after simplification

$$m D^2 P + (s + c\Omega^2) P - \tfrac{1}{2}\sqrt{2}\, m (D^{*2} H_0 + D^{(*)2} \bar{H}_0) \qquad (39)$$

It can be shown that if
$I_x = I_y = I$ (say)
Then $f = \overset{(*)}{I} H_0 = K\xi^*/\eta^*$ (40)
Eliminating H_0, f between equations (38), (39) and (40) we have

$$\left[K \overset{*}{\xi} \overset{(*)}{\xi} \left\{mD^2 + (S + C\Omega^2)\right\} - m^2\left\{\overset{*}{D}{}^4 \eta \overset{(*)}{\xi} + \overset{(*)}{D}{}^4 \eta \overset{*}{\xi}\right\}\right] P = 0 \qquad (41)$$

If we now write $P = P_0 e^{i\nu t}$, the above equation yields the frequency equation for the two bladed rotor
e g

$$K \overset{*}{\xi} \overset{(*)}{\xi} (\lambda_2 + \lambda_1 \Omega^2 - \nu^2) - \frac{\lambda_3}{2}\left[(\nu + \Omega)^4 \eta \overset{(*)}{\xi} + (\nu - \Omega)^4 \eta \overset{*}{\xi}\right] = 0 \qquad (42)$$

NOTE

In a body/blade system as defined by " Feingold " (Ref 4) equation (42) becomes

$$\overset{*}{I} \overset{(*)}{I} (\lambda_2 + \lambda_1 \Omega^2 - \nu^2) - \frac{\lambda_3}{2}\left[(\nu + \Omega)^4 \overset{(*)}{I} + (\nu - \Omega)^4 \overset{*}{I}\right] = 0 \qquad (43)$$

Substitute the " Coleman notation " as given in Section 4 921 and noting that

$$\overset{*}{I} = \left\{(m + 2m_b) D^{*2} + K\right\}$$

$$\overset{(*)}{I} = \left\{(m + 2m_b) D^{(*)2} + K\right\}$$

equation (43) gives " Feingold's " frequency equation for a two bladed rotor (Ref 4 equ 7)

Appendix 4

The life estimation of a component subject to a combined sheet and rivet load as shown in Fig 9

Two examples of the method outlined in Section 4 221 are given,

Example 1 *Typical Skin/Spar Joint*

(*High sheet stress—Low rivet load*)

Data

Sheet	20G
Sheet Stress	± 17,400 lb /sq in
Rivet Load	± 16 lb
t/D	0 288

Thus K_F (Unloaded Hole) = 1 55 (Standard Data)

$\bar{K}_F$ (Loaded Hole) = 1 05 (From Fig 11)

f_u = 27,000 lb /sq in

f_l = 3,720 lb /sq in

$f_u + f_l$ = 30,720 lb /sq in

Thus from Fig (10)

N = 1 5 × 10^6 cycles mean curve

= 0 35 × 10^6 cycles —5% scatter curve

Actual failure of specimen 2 58 × 10^6 *cycles*

Example 2 *Typical Skin/Stringer Joint*

(Low sheet stress—High rivet load)

Data

Sheet	20G
Sheet Stress	± 11,200 lb /sq in
Rivet Load	± 80 lb
t/D	0 288

Thus (as example 1)

f_u = 17,300 lb /sq in

f_l = 18,700 lb sq in

$f_u + f_l$ = 36,000 lb /sq in

Thus from Fig 10 N = 0 17 × 10^6 cycles mean line

= 0 12 × 10^6 cycles —5% scatter line

Actual failure of specimen 0 21 × 10^6 *cycles*

NOTE

In example 2, the value of f_l is of the same order as f_u, consequently any increase in the rivet load would substantially reduce the general stress level allowable for a given life

Discussion

Mr P E Q Shunker (*Westland Aircraft Ltd*), who opened the discussion, congratulated the Author on a most interesting paper on the very difficult subject of rotor blade design and stressing Like any other structure, the rotor blade must be analysed in two parts

(i) the critical external loading which must be derived, and

(ii) the strength assessment of the blade under that loading

Since the blades were part of a dynamical system, the external load analysis was concerned not only with the air loads themselves, but also with their associated inertia loads

The paper demonstrated a method for the solution of the dynamic problem, using the elegant devices of the matrix algebra The Author had made a valuable contribution on this particular aspect For example, he had placed the so-called "ground resonance" phenomenon in its correct perspective as being part, albeit extremely important, of the general dynamic picture In discussing the effects of dynamic phenomena he had pointed out the pitfalls to which the unwary were prone and in his analysis he had demonstrated the importance of the "body freedoms" One would imagine that this was more important in the case of the ultra-light than in heavier aircraft

The first difficulty with which one was faced, however, in blade analysis was the question of the aerodynamic loading While the Author acknowledged that much remained to be done on this point, the paper could have been considerably enhanced