ANOTHER PROOF OF THE CONTRACTION MAPPING PRINCIPLE

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In a recent note of Kolodner [2], the Cantor Intersection Theorem is used to give an alternative proof of the well known Contraction Mapping Principle. Kolodner applied Cantor's theorem first to a bounded metric space and then reduced the general case to this special case. Sometime ago, we found a somewhat different proof of the Contraction Mapping Principle using Cantor's theorem. Since our proof seems somewhat more direct we propose to present it here.

THEOREM. (Banach) Let T be a mapping of a complete metric space (X, ρ) into itself which satisfies $\rho(Tx, Ty) \leq k\rho(x, y)$ for some constant $k < 1$ and all $x, y \in X$. Then T has a unique fixed point ξ, and $\rho(T^n x, \xi) \to 0$ as $n \to \infty$ for each $x \in X$.

Proof. For $x \in X$, define $\varphi(x) = \rho(x, Tx)$.

Since T is a contraction, it easily follows that φ is a continuous function on X and $\varphi(T^n x) \to 0$ as $n \to \infty$, for each $x \in X$.

Define $C_m = \{x \in X \mid \varphi(x) \leq 1/m\}$.

From the above observation, we see that C_m is closed and non-empty for each $m = 1, 2, \ldots$.

Furthermore, we may estimate the diameter of the set C_m by the following device; if $x, y \in C_m$

$$\rho(x, y) \leq \rho(x, Tx) + \rho(Tx, Ty) + \rho(Ty, y) \leq 2/m + k\rho(x, y).$$

Hence, $\operatorname{diam} C_m \leq 2/m(1 - k)$. Thus, the family of sets $\{C_m\}_{m=1}^{\infty}$ is a nested family of closed sets for which $\operatorname{diam} C_m \to 0$ as $m \to \infty$. By Cantor's theorem the intersection of these sets contains a single point ξ.

Since $T(C_m) \subseteq C_m$ for all m, ξ is a fixed point of T and clearly unique.

For each $x \in X$, observe that

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$$\rho(T^n x, \xi) = \rho(T^n x, T^n \xi) \leq k^n \rho(x, \xi) \to 0, \text{ as } n \to \infty .$$

In fact, since $\rho(x, \xi) \leq \rho(x, Tx) / (1 - k)$, we have the following estimate for the rate of convergence: (cf [2]),

$$\rho(T^n x, \xi) \leq k^n \rho(x, Tx) / (1 - k) .$$

Remark. With little modification, the proof given above may be used to give an alternative proof of a theorem of Edelstein [1] concerning contraction mappings on ϵ-chainable spaces.

REFERENCES

1. M. Edelstein, An extension of Banach's Contraction Principle. Proc. Amer. Math. Soc., 12 (1961) 7-10.

2. I. I. Kolodner, On the proof of the Contraction Mapping Theorem. Amer. Math. Monthly, 74 (1967) 1212-1213.

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