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Waves on a shear flow

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The propogation of disturbance when a shear flow with a free surface, in a channel of infinite horizontal extent and finite depth, is disturbed by the application of time-oscillatory pressure, is studied. The initial value problem is solved by using transform techniques and the steady state solution is obtained therefrom in the limit $t \to \infty$. The effect of the initial shear on the development of the wave system is investigated.

We consider an inviscid, incompressible fluid of constant density ρ moving in a uniform channel of constant depth h and infinite horizontal extent. The initial flow is horizontal and has a positive vertical gradient. The object of the present analysis is to study the generation of waves on the free surface when it is disturbed by the application of a time-periodic pressure of frequency ω' .

The principal interest in such studies emanates from some considerations pertaining to the formulation of steady state surface flow problems in the framework of linearized theory. The imposition of a free surface pressure in such a formulation leads to a non-homogeneous problem. Under certain circumstances, discussed in the sequel, the corresponding homogeneous problem has a non-trivial solution, thus leading, in general, to a non-unique solution of the complete problem. For a unique solution, one requires some extra conditions at $\pm\infty$, called the radiation conditions. In the absence of a systematic way of finding these, the nature of these conditions is, at most, the result of an intelligent guess [4]. Aside from this, there is also the basic question as to whether or

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not a steady state problem posed in the linearized theory has a solution. Indeed, it is shown below that under certain conditions such a solution is not possible.

A systematic way to resolve both these questions lies in first solving the initial value problem and then obtaining the steady state solution in the limit $t \to \infty$. The radiation conditions are then found by studying the behaviour of this solution for large $|x|$. It is rather a fortuitous circumstance that such a program can be carried out for problems in the theory of water waves. Attempts along these lines have been made before by Debnath and Rosenblat [1], Puri [5], and Stoker [6] under various assumptions. One common assumption is that the basic undisturbed flow has a constant velocity. In a physically realistic model, however, this is not true. We therefore study the problem when the undisturbed flow is characterized by a vertical gradient in the velocity. It turns out that the generation of waves is significantly affected by the presence of this shear in the initial flow.

1. Formulation of the problem

We consider a two dimensional flow of liquid in a uniform channel of infinite horizontal extent and finite depth h . The liquid is assumed to be inviscid and incompressible having a constant density ρ . We take the origin of the reference system (x', y') at the undisturbed free surface, the y' axis is directed vertically upward, and the x' axis is taken in the direction of the initial, horizontal flow $U' = b'y' + U_0$. Here b', U' are constants. We would like to study the linearized wave motion consequent upon the application of a pressure $p_0'(x')e^{i\omega' t'}$ at the free surface.

The equations of motion together with the attendant boundary and initial conditions are linearized about the velocity components $(U', 0)$ and the pressure $-\rho g y'$. Denoting by $u'(x', y', t')$, $v'(x', y', t')$ and $p'(x', y', t')$ the corresponding flow parameters of the disturbance, we solve the linearized Euler's equations:

$$(1') \qquad u'_{t'} + U'u'_{x'} + U'_{y'}v' = -\frac{1}{\rho}p'_{x'} ,$$

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(2')
$$v'_{t'} + U'v'_{x'} = -\frac{1}{\rho} p'_{x'}$$

and the equation of continuity

(3')
$$u'_{x'} + v'_{y'} = 0 ,$$

subject to the appropriate boundary and the initial conditions. Keeping in mind that in this theory the boundary conditions at the free surface may be applied at the equilibrium position [6], we have:

the kinematic condition:

(4')
$$v' = U'\xi'_{x'} + \xi'_{t'} \quad \text{at} \quad y' = 0 ;$$

the continuity of the pressure at the free surface:

(5')
$$p' = \rho g \xi' + p'_0(x')e^{i\omega' t'} \quad \text{at} \quad y' = 0 ;$$

and the condition at the rigid bottom:

(6')
$$v' = 0 \quad \text{at} \quad y' = -h .$$

Here $\xi'(x', t')$ is the elevation of the free surface and g is the gravitational constant. Also we have the initial conditions:

(7')
$$u'(x', 0) = v'(x', 0) = p'(x', 0) = \xi'(x', 0) = 0 .$$

The pressure mechanism is switched on at $t' > 0$. In order to non-dimensionalize the above variables, we introduce

$$(x', y') = h(x, y) , \quad \xi' = \xi h , \quad U' = UU_0 , \quad t' = t\frac{h}{U_0} ,$$

$$p' = \rho p U_0^2 , \quad (u', v') = U_0(u, v) , \quad \omega' = \frac{U_0}{h}\omega , \quad b' = b\frac{U_0}{h} ,$$

$$p'_0(x') = \rho p_0(x)U_0^2 \quad \text{and} \quad \gamma = \frac{gh}{U_0^2} .$$

The equations (1') to (7') then may be written as

(1)
$$u_t + Uu_x + vU_y = -p_x ,$$

(2)
$$v_t + Uv_x = -p_y ,$$

(3)
$$u_x + v_y = 0 ,$$

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(4) $$v = U\xi_x + \xi_t \quad \text{at} \quad y = 0\ ,$$

(5) $$p = \gamma\xi + p_0 e^{i\omega t} \quad \text{at} \quad y = 0\ ,$$

(6) $$v = 0 \quad \text{at} \quad y = -1\ ,$$

(7) $$u(x,\ y,\ 0) = v(x,\ y,\ 0) = p(x,\ y,\ 0) = \xi(x,\ 0) = 0\ .$$

Also we have

(8) $$U = by + 1\ .$$

2. The unsteady problem

In order to solve the above equations, we shall use the Fourier transform with respect to x, defined by:

(9) $$\underline{\underline{F}}(\underline{\underline{f}}) = \tilde{\underline{\underline{f}}}(\lambda,\ \ldots) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underline{\underline{f}}(x,\ \ldots)e^{-i\lambda x}dx\ .$$

The equations (1) to (3), together with the appropriate conditions from (7), transformed according to (9), yield

(10) $$\tilde{v} = A(\lambda,\ t)\sinh\lambda(y+1)\ ,$$

(11) $$\tilde{u} = iA\cosh\lambda(y+1)\ ,$$

and

(12) $$\tilde{p} = -\frac{1}{\lambda}A_t\cosh\lambda(y+1) - iUA\cosh\lambda(y+1) + \frac{i\omega}{\lambda}A\cosh\lambda(y+1)\ ,$$

with $A(\lambda,\ t)$ as the 'constant' of integration. Also taking the transform of (4) and (5) and using the equations (10)-(12) to eliminate $\tilde{v}$ and $\tilde{p}$, we get, after simplifications:

(13) $$\tilde{\xi}_{tt} + (2i\lambda - ib\tanh\lambda)\tilde{\xi}_t + \left(-\lambda^2 + b\lambda\tanh\lambda + \gamma\lambda\tanh\lambda\right)\tilde{\xi} = -\tilde{p}_0\lambda\tanh\lambda e^{i\omega t}\ .$$

The solution of this equation subject to the initial conditions $\tilde{\xi}(\lambda,\ 0) = \tilde{\xi}_t(\lambda,\ 0) - 0$, is given by

(14) $$\tilde{\xi}(\lambda,\ t) = De^{ik_+t} + Be^{ik_-t} + \frac{\tilde{p}_0\lambda\tanh\lambda e^{i\omega t}}{(\omega-k_+)(\omega-k_-)}\ ,$$

where

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(15) $$D = -\frac{\tilde{p}_0\lambda\tanh\lambda}{(k_+-k_-)(\omega-k_+)}\ , \quad B = \frac{\tilde{p}_0\lambda\tanh\lambda}{(k_+-k_-)(\omega-k_-)}\ ,$$

and

(16) $$k_\pm = -\lambda + \tfrac{1}{2}b\tanh\lambda \pm \sqrt{\tfrac{1}{4}b^2\tanh^2\lambda+\gamma\lambda\tanh\lambda}\ .$$

The equations (14) and (16) then yield:

(17) $$\tilde{\xi}(\lambda,\ t) = \frac{\tilde{p}_0\lambda\tanh\lambda}{k_+-k_-}\left[\frac{e^{i\omega t}-e^{ik_+t}}{\omega-k_+} - \frac{e^{i\omega t}-e^{ik_-t}}{\omega-k_-}\right]\ .$$

From now on we shall consider $p_0(x) = c\delta(x)$, where c is constant, so that $\tilde{p}_0(\lambda) = c/\sqrt{2\pi}$. Taking the inverse Fourier transform of (17), we get

(18) $$\xi(x,\ t) = \frac{c}{4\pi}\int_{-\infty}^{\infty} \frac{\lambda\tanh\lambda\cdot e^{i\lambda x}}{[\tfrac{1}{4}b^2\tanh^2\lambda+\gamma\lambda\tanh\lambda]^{1/2}}\left[\frac{e^{i\omega t}-e^{ik_+t}}{\omega-k_+} - \frac{e^{i\omega t}-e^{ik_-t}}{\omega-k_-}\right]d\lambda\ ,$$

which gives the amplitude of the disturbance for the unsteady problem. Also from (10) and the Fourier transform of (4) together with (14), we get $A(\lambda,\ t)$. Substituting its value in the equations (10)-(12), and taking their inverse Fourier transformations, we can get $u(x,\ y,\ t)$, $v(x,\ y,\ t)$, and $p(x,\ y,\ t)$, thus solving the nonsteady problem completely. In the sequel we shall confine ourselves to the amplitude function $\xi(x,\ t)$ only. Also it will be convenient to write

(19) $$\xi(x,\ t) = \frac{c}{4\pi}\left(e^{i\omega t}I+J\right)$$

where

$$I = \int_{-\infty}^{\infty} \psi(\lambda)e^{i\lambda x}\left\{\frac{1}{\omega-k_+} - \frac{1}{\omega-k_-}\right\}d\lambda\ ,$$

(21) $$J = \int_{-\infty}^{\infty} \psi(\lambda)e^{i\lambda x}\left\{\frac{e^{ik_-t}}{\omega-k_-} - \frac{e^{ik_+t}}{\omega-k_+}\right\}d\lambda\ ,$$

and

(22) $$\psi(\lambda) = \frac{\lambda\tanh\lambda}{[\tfrac{1}{4}b^2\tanh^2\lambda+\gamma\lambda\tanh\lambda]^{1/2}}\ .$$

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Our principal interest is to study $\xi(x, t)$ for large t to obtain the form of the steady state solution. The behaviour of the latter for large $|x|$ will determine the nature of the radiation conditions. This analysis has to be carried out subject to the condition $|x| << t$. Thus $|x| \to \infty$ implies $t \to \infty$.

3. The transients

The transients depend upon the integral J. The major contribution to its asymptotic evaluation for large t arises from the stationary points of the functions $k_{\pm}(\lambda)$ and from the poles of its integrand. Here we shall consider the former. The polar contribution subscribes to the steady state solution and will be discussed in the next section. It is shown in Appendix A that each of the two functions $k_{\pm}(\lambda)$ has precisely one stationary point; that is,

$$\left.\frac{dk_+}{d\lambda}\right|_{\lambda=\sigma} = 0 \quad \text{and} \quad \left.\frac{dk_-}{d\lambda}\right|_{\lambda=-\rho} = 0 \tag{23}$$

provided

$$\gamma \geq 1-b \,. \tag{A-4}$$

The presence of b in this condition reflects the contribution of the shear in the initial flow. Setting it equal to zero, we retrieve its analogue:

$$\gamma \geq 1 \tag{24}$$

obtained in the earlier works [5, 6]. We can now examine the effect of the initial shear.

If $b \geq 1$, that is, $b' \geq \frac{U_0}{h}$, in the dimensional units, (A-4) is satisfied for all γ. Taking, in particular, $\gamma = 0$ which implies $g = 0$, we conclude that steady flows in the presence of such a shear exist even when the gravity is absent. Simple calculations, on the other hand, show that this is not so when, besides, shear is absent. In this case the transients increase indefinitely as $O(t)$ for large t. Thus for $b' \geq \frac{U_0}{h}$, vorticity generated by the shear becomes more important than gravity and is responsible for the decay or otherwise of the transients.

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For the asymptotic evaluation of J , we consider the following two cases:

(i) none of the stationary points, $x = \sigma, -\rho$ coincide with a pole of the integrand;

(ii) some may do so. This is called the critical case.

If case (i) holds, we may use the method of stationary phase [2], to write,

$$(25)\quad J \simeq \left\{\frac{2\pi}{t|k_+''(\sigma)|}\right\}^{1/2}\psi(\sigma)\exp\left\{itk_+(\sigma) - \frac{\pi i}{4}\,\mathrm{sgn}\left(k_t''(\sigma)\right)\right\} + \left\{\frac{2\pi}{t|k_-''(-\rho)|}\right\}^{1/2}\psi(-\rho)\exp\left\{itk_-(-\rho) - \frac{\pi i}{4}\,\mathrm{sgn}\left(k_-''(-\rho)\right)\right\} + O\left(\frac{1}{t}\right) .$$

It shows that the transients behave as $O\left(t^{-1/2}\right)$ for large t and the steady state solution results after the lapse of a long time.

Case (ii) entails the study of the roots of the equation

$$(26)\qquad (\omega-k_-)(\omega-k_+) = 0 .$$

Using (16) and simplifying, we can write it in the form

$$(27)\qquad \pm(\omega+\lambda) = [\tanh\lambda\{\omega b+\lambda(\gamma+b)\}]^{1/2} .$$

The required roots, which are the points of intersection of the curves

$$\Gamma_\pm : y = \pm(\omega+\lambda)$$

with the curve

$$\Gamma : y = [\tanh\lambda\{\omega b+\lambda(\gamma+b)\}]^{1/2} ,$$

are shown in Figure 1 on page 270. This shows that $\lambda = \sigma$ is the only possible stationary point that also satisfies equation (26). For this, we must have

$$(28)\qquad \omega - k_+(\sigma) \equiv \omega + \sigma - \tfrac{1}{2}b\tanh\sigma - \{\tfrac{1}{4}b^2\tanh^2\sigma+\gamma\sigma\tanh\sigma\}^{1/2} = 0 ,$$

and from (A-1),

$$(29)\qquad \left.\frac{dk_+}{d\lambda}\right|_{\lambda=\sigma} \equiv 1 - \frac{b}{2}\,\mathrm{sech}^2\sigma - \frac{1}{2}\,\frac{\gamma\tanh\sigma+\gamma\sigma\,\mathrm{sech}^2\sigma+\frac{1}{2}b^2\tanh\sigma\,\mathrm{sech}^2\sigma}{\left[\frac{1}{4}b^2\tanh^2\sigma+\gamma\sigma\tanh\sigma\right]^{1/2}} = 0 .$$

The elimination of σ from these equations yields the condition for the

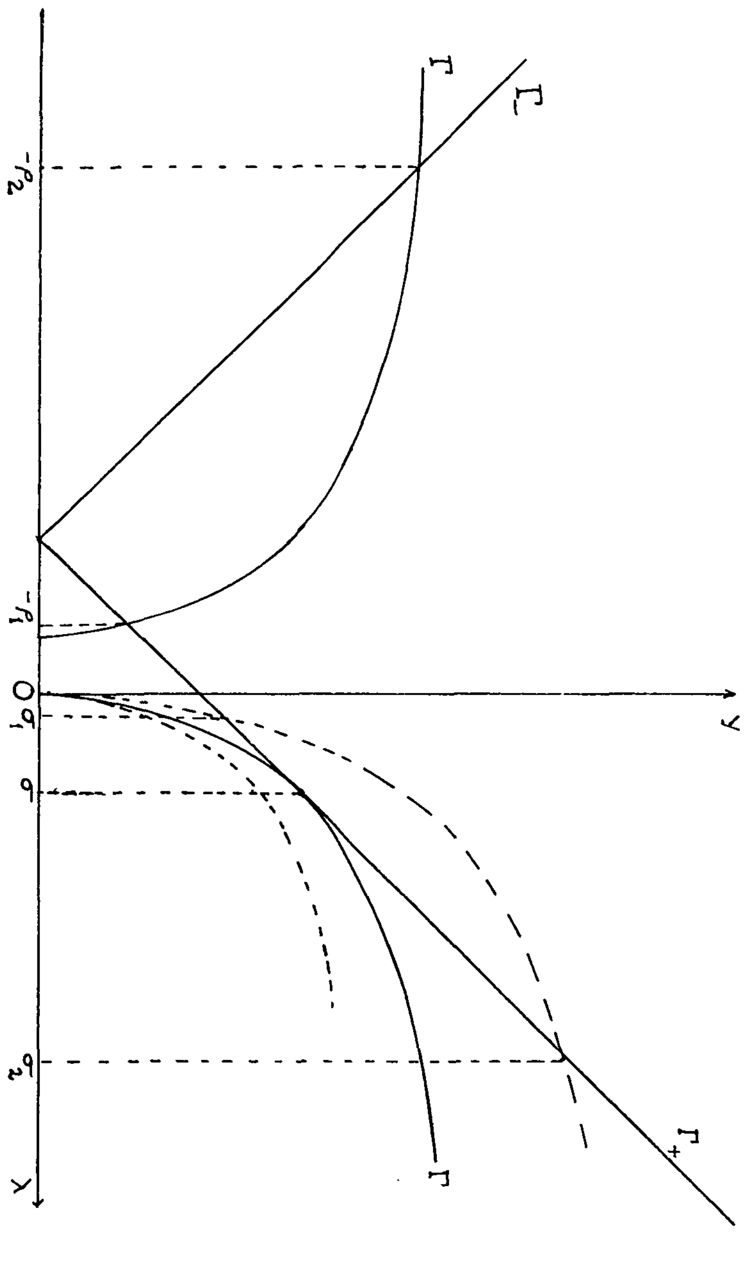

Figure 1.

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critical case in the form

(30) $$\gamma = \gamma_C(b,\ \omega)\ .$$

It is clear that the satisfaction of condition (30) implies the satisfaction of (A-4). This result subsumes the results of Stoker [6], Puri [5], and Debnath and Rosenblat [1]. We can obtain their results exactly by setting the appropriate parameter in (30) equal to zero.

The asymptotic approximation of J in this case also has the same contribution from $\lambda = -\rho$ as that in (25). To discuss the contribution from $\lambda = \sigma$, we write

(31) $$J = \int_{-\infty}^{\infty} \frac{\psi(\lambda)e^{i\lambda x}}{\omega-k_-} e^{itk_-}d\lambda$$
$$+ \left[\left(\int_{-\infty}^{\sigma-\varepsilon} + \int_{\sigma+\varepsilon}^{\infty} + \int_{\sigma-\varepsilon}^{\sigma} + \int_{\sigma}^{\sigma+\varepsilon}\right) \frac{\psi(\lambda)e^{i\lambda x}}{\omega-k_+} e^{itk_+}d\lambda\right]$$

where ε is a small positive number such that $0 < \sigma - \varepsilon$. It is clear that the maximum contribution to the value of J arises from the last two integrals. To evaluate these we use the transformation $K = \omega - k_+(\lambda)$. In order to see that it is not singular in $(\sigma-\varepsilon,\ \sigma)$ and $(\sigma,\ \sigma+\varepsilon)$, we expand K in the vicinity of σ :

$$K = \omega - k_+(\sigma) = -\frac{1}{2!}(\lambda-\sigma)^2 k_+''(\sigma) + \ldots\ .$$

From the appendix, we see that $k_+''(\sigma) = F'(\sigma) < 0$. Hence $K > 0$ and $\frac{dK}{d\lambda} \neq 0$ in $(\sigma-\varepsilon,\ \sigma)$ and $(\sigma,\ \sigma+\varepsilon)$. Indeed

$$\frac{dK}{d\lambda} \simeq |2Kk_+''(\sigma)|\,\mathrm{sgn}(\lambda-\sigma)\ .$$

From (31) then, we have

$$J \simeq e^{i\omega t}\int_0^{K_0} |2k_+''(\sigma)|^{-1/2}[\psi(x)e^{i\lambda x}]_{K=\omega-k_+(\sigma)}|K|^{-3/2}e^{-itK}dK$$
$$+ e^{i\omega t}\int_0^{K_1} |2k_+''(\sigma)|^{-1/2}[\psi(x)e^{i\lambda x}]_{K=\omega-k_+(\sigma)}|K|^{-3/2}e^{-itK}dK\ ,$$

where

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$$K_0 = \omega - k_+(\sigma+\varepsilon) > 0 \,,$$

$$K_1 = \omega - k_+(\sigma-\varepsilon) > 0 \,.$$

Following Lighthill [3] we now have

$$J \simeq 2(-3/2)!\,|2k_+''(\sigma)|^{-1/2}[\psi(\sigma)]t^{1/2}e^{i\omega t+i\sigma x+i\pi/4} \quad \text{as} \quad t \to \infty \,. \tag{32}$$

Thus the integral J and therefore the amplitude $\xi(x, t)$ increases indefinitely as $O\left(t^{1/2}\right)$ for large t. This shows that there is no steady state in the critical case. The linear theory thus fails and any further discussion of the problem in this case requires invoking the non-linear theory.

4. Radiation conditions

We shall now study the integrals I and J for large $|x|$ such that $|x| \ll t$. The main contribution to their evaluation arises from the poles of their integrands which are the solutions of the equation (26) shown graphically in Figure 1. Referring to it again, we find the following three cases.

CASE I. There are only two negative zeros, that is, at $\lambda = -\rho_1, -\rho_2$ obtained from the intersection of Γ_- with Γ. Also $\rho_2 > \rho_1$.

CASE II. There are four zeros. Two of these are as in Case I and two more, $\lambda = \sigma_1, \sigma_2$, result from the intersection of Γ_+ with Γ. Also it is clear that $0 < \sigma_1 < \sigma < \sigma_2$. Observing that Γ is a monotonically increasing function of γ, it is easy to see that

$$\gamma > \gamma_C(\omega, b) \tag{33}$$

is the necessary and sufficient condition for the additional points of intersection.

CASE III. This case follows from the second case in the event that the additional zeros coalesce, that is, when $\omega - k_+(\lambda) = 0$ and $\frac{dk_+}{d\lambda} = 0$ hold simultaneously. This is precisely the critical case characterized in the previous section by the condition

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(30) $$\gamma = \gamma_C(\delta,\ b)\ .$$

It was concluded there that this condition does not permit any steady state flow. *Hence this case will not be pursued any further.*

The integrals are now evaluated by using the well-known formula:

(34) $$\int_a^b f(\lambda)e^{i\lambda x}d\lambda \simeq \pi i(\operatorname{sgn}x)e^{i\lambda_0 x}\left[\text{residue of } f(\lambda) \text{ at } \lambda = \lambda_0\right] + O(1/|x|)$$

where $\lambda = \lambda_0$ is a simple pole of $f(\lambda)$ in $(a,\ b)$.

To evaluate the integral I , we write

(35) $$I = \int_{-\infty}^{\infty} \frac{\psi(\lambda)e^{i\lambda x}}{\omega-k_+}\,d\lambda - \int_{-\infty}^{\infty} \frac{\psi(\lambda)e^{i\lambda x}}{\omega-k_-}\,d\lambda\ .$$

In the first case, each of these integrals have one simple pole. Thus applying the formula (34), we obtain

(36) $$I \simeq \pi i(\operatorname{sgn}x)\left[-\psi_-(-\rho_2)e^{-i\rho_2 x}\right] + \psi_+(-\rho_1)e^{-i\rho_1 x} + O\left(\frac{1}{|x|}\right)$$

where

$$\psi_-(-\rho_2) = \lim_{\lambda\to-\rho_2} \left(\frac{\psi(\lambda)}{-dk_-/d\lambda}\right)$$

and

(38) $$\psi_+(-\rho_1) = \lim_{\lambda\to-\rho_2} \left(\frac{\psi(\lambda)}{-dk_+/d\lambda}\right)\ .$$

In the second case, we have to add to (36) the contribution from the other two positive poles at $\lambda = \sigma_1$, $\lambda = \sigma_2$. This again follows immediately with an application of (34),yielding

(39) $$\pi i(\operatorname{sgn}x)\left[\psi_+(\sigma_1)e^{i\sigma_1 x}\right] + \psi_+(\sigma_2)e^{i\sigma_2 x} + O\left(\frac{1}{|x|}\right)\ .$$

Similarly, the polar contribution to the value of J , in the first case, follows from the poles $\lambda = -\rho_1 < 0$ and $\lambda = -\rho_2 < -\rho$. To find it, we write

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$$(40)\quad J = \left[\left(\int_{-\infty}^{-\rho} + \int_{-\rho}^{\infty}\right) \frac{\psi(\lambda)e^{i\lambda x}\cdot e^{ik_- t}}{\omega - k_-}\, d\lambda\right] - \left[\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \frac{\psi(\lambda)e^{i\lambda x}\cdot e^{ik_+ t}}{\omega - k_+}\, d\lambda\right],$$

so that the main contribution comes from the first and the third integral. Setting $K = k_-(\lambda)$ in the first one and denoting it by J_1, we have, using equation (16),

$$J_1 = \int_{k_-(-\rho)}^{\infty} \left[\psi_-(\lambda)e^{i\lambda x}\right]_{\lambda=k_-^{-1}(K)} \frac{e^{iKt}}{\omega - K}\, dK .$$

Formula (34) then yields

$$(41)\qquad J_1 \simeq -i\pi e^{i\omega t}\left[\psi_-(-\rho_2)e^{-i\rho_2 x}\right] + O\left(\frac{1}{t}\right) .$$

Similarly, we treat the third integral. Transforming the variable of integration from λ to $k_+(\lambda)$ and using (34), it yields the contribution

$$(42)\qquad -\pi i e^{i\omega t}\left[\psi_+(-\rho_1)e^{-i\rho_1 x}\right] + O\left(\frac{1}{t}\right) .$$

From (40)-(42) we have

$$(43)\qquad J \simeq \pi i e^{i\omega t}\left[\psi_+(-\rho_1)e^{-i\rho_1 x} - \psi_-(-\rho_2)e^{-i\rho_2 x}\right] + O\left(\frac{1}{t}\right) .$$

In the second case, that is, when condition (33) holds, we have to add to (43) the contributions of the two additional positive poles at $\lambda = \sigma_1, \sigma_2$. This is given by the fourth integral in equation (40). Denoting it by J_2, we write it as

$$J_2 = \left(\int_{0}^{\sigma} + \int_{\sigma}^{\infty}\right) \frac{\psi(\lambda)e^{i\lambda x}\cdot e^{ik_+ t}}{\omega - k_+}\, d\lambda .$$

Again setting $\mu = k_+(\lambda)$, we get

$$J_2 = \left(-\int_{0}^{k_+(\sigma)} + \int_{-\infty}^{k_+(\sigma)} \left[\psi_+(\lambda)e^{i\lambda x}\right]_{\lambda=k_+^{-1}(\mu)}\right) \frac{e^{i\omega t}}{\omega - \mu}\, d\mu .$$

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We invoke formula (34) again to obtain

$$J_2 \simeq \pi i e^{i\omega t}\left[\psi_+(\sigma_1)e^{i\sigma_1 x} - \psi_+(\sigma_2)e^{i\sigma_2 x}\right] + O\left(\frac{1}{t}\right) . \tag{44}$$

We can now collect the various results from (36), (39), (40), (43), and (44) for the steady state problem.

Writing $\eta(x) = \lim_{t\to\infty} \xi(x, t)$, we have the following.

(i) Let $\gamma < \gamma_C$,

$$\eta(x) \simeq \begin{cases} \frac{ic}{2} e^{i\omega t}\left[\psi_+(-\rho_1)e^{-i\rho_1 x} - \psi_-(-\rho_2)e^{-i\rho_2 x}\right] , & x > 0 , \\ 0 , & x < 0 . \end{cases} \tag{45}$$

(ii) Let $\gamma > \gamma_C$,

$$\eta(x) \simeq \begin{cases} \frac{ic}{2} e^{i\omega t}\left[\left\{\psi_+(-\rho_1)e^{-i\rho_1 x} - \psi_-(-\rho_2)e^{-i\rho_2 x}\right\} + \psi_+(\sigma_2)e^{i\sigma_2 x}\right] , & x > 0 , \\ -\frac{ic}{2}\psi_+(\sigma_1)e^{i(\sigma_1 x + \omega t)} , & x < 0 . \end{cases} \tag{46}$$

Thus, in the first case there are two progressive waves moving downstream with velocities ω/ρ_1 and ω/ρ_2 and no disturbance upstream.

In the second case, aside from the above two waves, there are two more, moving with velocities ω/σ_1 and ω/σ_2 towards the negative x-direction. The latter is on $x > 0$ and so, seemingly have started at ∞ . Physically it does not make sense because there is no source of energy present at $+\infty$. The real explanation is that the wave is actually moving towards $+\infty$ when viewed in a reference frame moving with the undisturbed velocity at the free surface. These results are looked upon as representing the radiation conditions at $\pm\infty$ in both the cases; that is, when $\gamma < \gamma_C$ and $\gamma > \gamma_C$.

Appendix A

The stationary points of $k_+(\lambda)$ are given by the solutions of the equation $\frac{dk_+}{d\lambda} = 0$, that is,

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(A-1) $$1 = \frac{b}{2}\operatorname{sech}^2\lambda + \frac{1}{2}\frac{\gamma\tanh\lambda+\gamma\lambda\operatorname{sech}^2\lambda+\frac{1}{2}b^2\tanh\lambda\operatorname{sech}^2\lambda}{\left[\frac{1}{4}b^2\tanh\lambda+\gamma\lambda\tanh\lambda\right]^{1/2}}$$
$$= F(\lambda)\ , \text{ say.}$$

We observe that

$$F(\lambda) \to 0 \quad \text{as} \quad \lambda \to \pm\infty\ ,$$

and to

(A-2) $$\tfrac{1}{2}b \pm \sqrt{\gamma+\tfrac{1}{4}b^2} \quad \text{as} \quad \lambda \to 0_{\pm}\ .$$

Also

(A-3) $$F'(\lambda) = -\frac{\gamma^2\left(\tanh\lambda-\lambda\operatorname{sech}^2\lambda\right)^2}{\left[\frac{1}{4}b^2\tanh^2\lambda+\gamma\lambda\tanh\lambda\right]^{1/2}} - 2\operatorname{sech}^2\lambda\tanh\lambda\left\{b + \frac{\gamma\lambda+\frac{1}{2}b^2\tanh\lambda}{\left[\frac{1}{4}b^2\tanh\lambda+\gamma\lambda\tanh\lambda\right]^{1/2}}\right\} .$$

It can easily be seen that $f'(\lambda) < 0$ for $\lambda > 0$ and hence $F(\lambda)$ is monotonically decreasing in $(0, \infty)$ and $(-\infty, 0)$. It follows from (A-1)-(A-2) that $k_+(\lambda)$ has precisely one stationary point, say $\lambda = \sigma > 0$ provided

(A-4) $$\tfrac{1}{2}b + \sqrt{\gamma+\tfrac{1}{4}b^2} \geq 1\ , \text{ that is, } \gamma \geq 1-b\ .$$

Similarly, we can show that $\frac{dk_-}{d\lambda} = 0$ for only one value, $\lambda = -\rho < 0$, provided the condition (A-4) holds.

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