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SOME INTERTWINING RELATIONS MODULO OPERATOR IDEALS

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Abstract. Let $B(H)$ denote the algebra of all bounded linear operators on a separable, infinite-dimensional, complex Hilbert space H. Let I be a two-sided ideal in $B(H)$. For operators A, B and $X \in B(H)$, we say that *X intertwines A and B modulo I* if $AX - XB \in I$. It is easy to see that if X intertwines A and B modulo I, then it intertwines A^n and B^n modulo I for every integer $n > 1$. However, the converse is not true. In this paper, sufficient conditions on the operators A and B are given so that any operator X which intertwines certain powers of A and B modulo I also intertwines A and B modulo J for some two-sided ideal $J \supseteq I$.

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1. Introduction. Let H be a separable infinite-dimensional complex Hilbert space. Let $B(H) \supset K(H) \supset F(H)$ denote, respectively, the algebra of all bounded linear operators, the two-sided ideal of compact operators, and the two-sided ideal of finite rank operators on H. For any compact operator T, let $s_1(T), s_2(T) \ldots$ be the eigenvalues of $|T| = (T^*T)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. A compact operator T is said to be in the *Schatten p-class* $C_p (0 < p < \infty)$ if $\Sigma_i s_i(T)^p < \infty$. For $p \geq 1$, the Schatten p-norm of T is defined by $\|T\|_p = (\Sigma_i s_i(T)^p)^{1/p}$. This norm makes C_p into a Banach space. For all $p > 0$, C_p is a two-sided ideal in $B(H)$. Hence, C_1 is the trace class and C_2 is the Hilbert-Schmidt class. It is reasonable to let C_∞ denote the ideal of compact operators $K(H)$ and $\|\cdot\|_\infty$ stand for the usual operator norm. We refer to [**7**] for the general theory of the Schatten p-classes.

Throughout this paper, every ideal in $B(H)$ is assumed to be two-sided and proper. It is well known that if I is a non-trivial ideal in $B(H)$, then $F(H) \subseteq I \subseteq C_\infty$.

Let I be an ideal in $B(H)$; let A, B and $X \in B(H)$. We say that *X intertwines A and B modulo I* if $AX - XB \in I$. If X intertwines A and B modulo the trivial ideal, i.e., if $AX - XB = 0$, then we simply say that *X intertwines A and B*. It is easy to see that if X intertwines A and B modulo I, then it intertwines A^n and B^n modulo I for every integer $n > 1$. Of course, the converse is not true. Consider, for example, the case in which A and B are non-zero nilpotent operators.

It is the object of this paper to present some sufficient conditions on the operators A and B so that any operator X which intertwines certain powers of A and B modulo I, also intertwines A and B modulo J, where J is an ideal in $B(H)$ such that $I \subseteq J$. Our results generalize earlier results on this problem by Al-Moajil [**1**], Duggal [**5**], and the author [**10**].

2. Intertwining relations modulo arbitrary ideals. In [**8**], extending a result of Al-Moajil [**1**], the author has proved the following results.

THEOREM A. *Let* A, B *and* $X \in B(H)$, *where* A *and* B^* *are subnormal. If* $A^2X = XB^2$ *and* $A^3X = XB^3$, *then* $AX = XB$.

THEOREM B. *Let* A, B *and* $X \in B(H)$, *where* A *and* B^* *are subnormal. If* $A^2X - XB^2 \in F(H)$ *and* $A^3X - XB^3 \in F(H)$, *then* $AX - XB \in F(H)$.

THEOREM C. *Let* A, B *and* $X \in B(H)$, *where* A *and* B^* *are subnormal. If* $A^2X - XB^2 \in C_p$ *and* $A^3X - XB^3 \in C_p$, *for some* p *with* $1 \leq p \leq \infty$, *then* $AX - XB \in C_{8p}$.

By a slight modification of the proof of Theorem C given in [**10**], Duggal [**5**, Theorem 5] has extended this result to relatively prime powers other than 2 and 3.

The purpose of this section is to extend these results to larger classes of operators and to relatively prime powers other than 2 and 3. In fact, these theorems follow as immediate consequences of a general result (Theorem 1), which is valid for arbitrary ideals in $B(H)$.

The following lemma, which is in the spirit of Lemma 1.1 in [**1**], indicates that in order to generalize Theorems A, B and C cited above, it is sufficient to consider two consecutive powers rather than two relatively prime powers.

LEMMA 1. *Let* I *be an ideal in* $B(H)$. *If* A, B *and* $X \in B(H)$ *are such that* $A^mX - XB^m \in I$ *and* $A^nX - XB^n \in I$, *for some relatively prime positive integers* m *and* n, *then* $A^kX - XB^k \in I$ *and* $A^{k+l}X - XB^{k+1} \in I$, *for some integer* $k > 1$.

Proof. Since m and n are relatively prime positive integers, there exist integers s and t such that $sm + tn = 1$ and st is negative; say s is negative and t is positive. The assumptions $A^mX - XB^m \in I$ and $A^nX - XB^n \in I$ imply that $A^{-ms}X - XB^{-ms} \in I$ and $A^{nt}X - XB^{nt} \in I$. Since $nt = -ms + 1$, the result now follows by letting $k = -ms$.

The following elegant factorization result, which is due to Douglas [**4**], will be essential for us to accomplish our goal.

LEMMA 2. *Let* T, $S \in B(H)$. *Then the following conditions are equivalent*:

(a) ran $T \subseteq$ ran S, *where* ran T *denotes the range of* T;

(b) $TT^* \leq cSS^*$, *for some constant* $c > 0$;

(c) $T = SR$, *for some* $R \in B(H)$.

It follows immediately from Lemma 2 that if $T \in B(H)$ is hyponormal, i.e., if $TT^* \leq T^*T$, then ran $T \subseteq$ ran T^*. Also, if ran $T \subseteq$ ran T^*, then $T = T^*R$ for some $R \in B(H)$, and hence $T^* = R^*T$. Thus, if ran $T \subseteq$ ran T^*, if $X \in B(H)$, and if I is an ideal in $B(H)$, then $TX \in I$ implies $T^*X \in I$. The particular case, that hyponormal operators have this property, has been observed by Weiss in [**12**].

If I and J are ideals in $B(H)$, let $I.\ J$ denote the ideal generated by products of the form TS with $T \in I$ and $S \in J$. Hence, by induction, I^n is defined as $I^n = I^{n-1}.\ I$ for every integer $n > 1$. It is well known (using the polar decomposition) that every ideal I in $B(H)$ is self-adjoint; i.e., $T \in I$ if and only if $T^* \in I$. Also, $|T|^2 \in I^2$ if and only if $|T| \in I$, and $|T| \in I$ if and only if $T \in I$. Consequently, $T^*T \in I$ if and only if $T \in I^{1/2}$, where $I^{1/2}$ is the unique ideal whose square is I. For any integer $n > 1$, $I^{1/n}$ is defined in the obvious way.

We are now in a position to prove the main result of this section.

THEOREM 1. *Let I be an ideal in $B(H)$. Let A, B and $X \in B(H)$ with* ran $A \subseteq$ ran A^* *and* ran $B^* \subseteq$ ran B. *If $A^nX - XB^n \in I$ and $A^{n+1}X - XB^{n+1} \in I$, for some integer $n > 1$, then $AX - XB \in I^{1/2^{n+1}}$.*

Proof. Let $C = AX - XB$. Then simple algebra shows that $A^nC \in I$ and $CB^n \in I$. Since ran $A \subseteq$ ran A^*, and since $A^nC \in I$, it follows that $A^*A^{n-1}C \in I$. Thus, $(A^{n-1}C)^*(A^{n-1}C) \in I$, and so $A^{n-1}C \in I^{1/2}$. Continuing down in this way, we obtain that $AC \in I^{1/2^{n-1}}$. Again, the assumption ran $A \subseteq$ ran A^* implies that $A^*C \in I^{1/2^{n-1}}$. Similarly, since ran $B^* \subseteq$ ran B, and since $B^{*n}C^* = (CB^n)^* \in I$, it follows that $B^*C^* \in I^{1/2^{n-1}}$. Hence, $BC^* \in I^{1/2^{n-1}}$, and so $CB^* \in I^{1/2^{n-1}}$. But then $CC^*C = C(X^*A^* - B^*X^*)C = CX^*A^*C - CB^*X^*C \in I^{1/2^{n-1}}$. Hence, $(C^*C)^2 \in I^{1/2^{n-1}}$, which implies that $C^*C \in I^{1/2^n}$, and so $C \in I^{1/2^{n+1}}$. This completes the proof.

An important special case of the range inclusion requirement in Theorem 1 is that A and B^* are hyponormal operators (in particular subnormal operators). The most interesting ideals for which Theorem 1 is applied are $\{0\}$, $F(H)$ and $C_p(0 < p \leq \infty)$. Hence, Theorems A, B and C can be obtained as corollaries of Theorem 1 upon considering the following cases:

(a) $I = \{0\}$, and so $I^{1/2^{n+1}} = \{0\}$,

(b) $I = F(H)$, and so $I^{1/2^{n+1}} = F(H)$,

(c) $I = C_p$, and so $I^{1/2^{n+1}} = C_{2^{n+1}p}$ $(0 < p \leq \infty)$.

Here we have, respectively, used the facts that $T = 0$ if and only $T^*T = 0$, $T \in F(H)$ if and only if $T^*T \in F(H)$, and $T \in C_{2p}$ if and only if $T^*T \in C_p(0 < p \leq \infty)$.

We should like to close this section by remarking that if $I = \{0\}$ then, in Theorem 1, the conditions that ran $A \subseteq$ ran A^* and ran $B^* \subseteq$ ran B can be weakened so that $\overline{\text{ran}}\, A \subseteq \overline{\text{ran}}\, A^*$ and $\overline{\text{ran}}\, B^* \subseteq \overline{\text{ran}}\, B$, or equivalently, (by taking orthogonal complements) ker $A \subseteq$ ker A^* and ker $B^* \subseteq$ ker B, where ker A and $\overline{\text{ran}}\, A$ denote the kernel of A and the closure (in the usual Hilbert space topology) of ran A, respectively. However, these conditions cannot be replaced by the symmetric conditions that ran $A \subseteq$ ran A^* and ran $B \subseteq$ ran B^*, or even more strongly, that A and B are hyponormal operators. This may be concluded from Remark 3.2 (a) in [**1**] or by considering the following example.

EXAMPLE 1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Let U be the unilateral shift operator defined by $Ue_n = e_{n+1}$, for all n, and let P be the orthogonal projection on the subspace spanned by e_1 and e_2; i.e., $Pe_1 = e_1$, $Pe_2 = e_2$ and $Pe_n = 0$ for $n > 2$. On $H \oplus H$, let $T = [\begin{smallmatrix} 0 & 0 \\ 0 & U \end{smallmatrix}]$ and $X = [\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}]$. Then T is hyponormal, $T^2X = XT^2$ and $T^3X = XT^3$, but $TX \neq XT$. In fact, every product involved here is zero except $XT = [\begin{smallmatrix} 0 & PU \\ 0 & 0 \end{smallmatrix}]$, which is non-zero because $PUe_1 = e_2$.

3. Intertwining relations modulo non-trivial ideals. In [**5**], Duggal has proved the following two results.

THEOREM D. *Let A, B and $X \in B(H)$ such that A is semi-Fredholm with* ind $A \leq 0$ *or B is semi-Fredholm with* ind $B \geq 0$. *If $A^mX - XB^m \in C_p$ and $A^nX - XB^n \in C_p$, for some relatively prime positive integers m and n, and some p with $1 \leq p \leq \infty$, then $AX - XB \in C_p$.*

THEOREM E. *Let A, B and $X \in B(H)$ such that $1 - A^*A \in C_p$ or $1 - B^*B \in C_p$, for some p with $1 \leq p \leq \infty$. If $A^mX - XB^m \in C_p$ and $A^nX - XB^n \in C_p$, for some relatively prime positive integers m and n, then $AX - XB \in C_p$.*

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Theorem D is a generalization of Theorem 7 in [**10**]. It should be mentioned here that the condition $1 - B^*B \in C_p$ in Theorem E should be replaced by $1 - BB^* \in C_p$. To see this, let $H^{(\infty)} = \oplus_{n=1}^{\infty} H_n$, where $H_n = H$ for all n, and let B be the operator valued weighted shift

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

$A = B^*$ and

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then $1 - AA^* = 1 - B^*B = 0 \in C_p$ for all p with $0 < p \leq \infty$, and

$$1 - A^*A = 1 - BB^* = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \notin C_p$$

for all p with $0 < p \leq \infty$. Moreover, $A^2X = XB^2 = A^3X = XB^3 = 0$ and

$$AX - XB = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \notin C_p$$

for all p with $0 < p \leq \infty$.

In this section we refine these results by extending them in two directions: to larger classes of operators and to all non-trivial ideals in $B(H)$.

Note that $A \in B(H)$ is a *semi-Fredholm operator* if ran A is closed and either ker A or ker A^* is finite-dimensional. It is well known that ran A is closed if and only if ran A^* is closed. Thus, A is a semi-Fredholm operator if and only if A^* is semi-Fredholm. The index of a semi-Fredholm operator A is given by ind $A =$ dim ker $A -$ dim ker A^*. Hence, ind $A^* = -$ind A. A semi-Fredholm operator A is a *Fredholm operator* if $-\infty <$ ind $A < \infty$; i.e., A is a Fredholm operator if ran A is closed and both ker A and ker A^* are finite-dimensional. An operator $A \in B(H)$ is said to be *left invertible modulo an ideal* I in $B(H)$, if there exists an operator $B \in B(H)$ such that $1 - BA \in I$; i.e., the coset $\nu(A)$ is left invertible in the quotient algebra $B(H)/I$, where ν is the canonical homomorphism of $B(H)$ onto $B(H)/I$. The (two-sided) invertibility modulo I is defined in the obvious way. It has been shown in [**6**, Theorem 1.1] that for $A \in B(H)$,

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ran A is closed and ker A is finite-dimensional if and only if A is left invertible modulo C_∞. The following lemma asserts that in this characterization C_∞ can be replaced by any non-trivial ideal in $B(H)$.

LEMMA 3. *Let I be a non-trivial ideal in $B(H)$. Then for $A \in B(H)$, ran A is closed and ker A is finite-dimensional if and only if A is left invertible modulo I.*

Proof. Since $I \subseteq C_\infty$, the "if" part follows by Theorem 1.1 in [**6**]. Now we prove the "only if" part. Assume that ran A is closed and ker A is finite-dimensional. Then ind $A \neq \infty$. If ind A is finite, then A is a Fredholm operator. Hence, by Atkinson's theorem [**8**, p. 96], A is invertible modulo $F(H)$. Since I is non-trivial, and hence $F(H) \subseteq I$, it follows that A is invertible modulo I. If, on the other hand, ind $A = -\infty$, then it follows from Proposition XI.3.21 in [**3**] that there exists a finite rank operator $F \in F(H)$ such that $T = A + F$ is left invertible. Let $S \in B(H)$ be a left inverse of T. Then $SA + SF = 1$, and so $1 - SA \in F(H)$. Consequently, $1 - SA \in I$. Thus, in either case A is left invertible modulo I and the proof is complete.

One version of our main result of this section can be stated as follows.

THEOREM 2. *Let I be a non-trivial ideal in $B(H)$. Let A, B and $X \in B(H)$, where either* ran A *is closed and* ker A *is finite-dimensional or* ran B *is closed and* ker B^* *is finite-dimensional. If $A^nX - XB^n \in I$ and $A^{n+l}\,X - XB^{n+l} \in I$, for some integer $n > 1$, then $AX - XB \in I$.*

Proof. As in the proof of Theorem 1, let $C = AX - XB$. Then $A^nC \in I$ and $CB^n \in I$. If ran A is closed and ker A is finite-dimensional then, by Lemma 3, A is left invertible modulo I, and so there exists an operator $S \in B(H)$ such that $1 - SA \in I$. But then $1 - S^nA^n \in I$. This, together with $A^nC \in I$, implies that $C \in I$. On the other hand, if ran B is closed (and hence ran B^* is closed) and ker B^* is finite-dimensional then, by Lemma 3, B^* is left invertible modulo I. In view of this, $B^{*n}C^* = (CB^n)^* \in I$ now implies that $C^* \in I$. Hence, $C \in I$, and the proof is complete.

In terms of the index function, the hypotheses on A and B in Theorem 2 can be restated so that either A is semi-Fredholm with ind $A \neq \infty$ or B is semi-Fredholm with ind $B \neq -\infty$. Now, in view of Lemma 3, Theorem D and the corrected version of Theorem E follow as special cases of Theorem 2.

At the end of this section, we should like to give the following example, which shows that Theorem 2 is not valid for the trivial ideal $I = \{0\}$.

EXAMPLE 2. Let U and P be the operators defined in Example 1. On $H \oplus H$, let $T = \left[\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right]$ and $X = \left[\begin{smallmatrix} 0 & 0 \\ P & 0 \end{smallmatrix}\right]$. Then ran T is closed and both ker T and ker T^* are one-dimensional subspaces of $H \oplus H$; i.e., T is a Fredholm operator with ind $T = 0$. Since $U^{*2}P = PU^2 = 0$ and $U^*P \neq PU$, simple matrix computations show that $T^2\,X = XT^2 = 0$, and so $T^3X = XT^3 = 0$, but $TX \neq XT$.

4. Intertwining relations modulo the trivial ideal. This section is mainly devoted to the case $I = \{0\}$. Let $A \in B(H)$ and let $\sigma(A)$ denote its spectrum. Then, by the Riesz functional calculus, any $X \in B(H)$ that commutes with A also commutes with $f(A)$ for every function f that is analytic on some neighbourhood of $\sigma(A)$ Recall that the operator $f(A)$ is defined by $f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)(z - A)^{-1}dz$, where Γ is any Jordan system in the domain of f that contains $\sigma(A)$ in its 'inside'.

For the reader's convenience, a proof of the following "folk" result is included.

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THEOREM 3. *Let $A, X \in B(H)$. If f is a function that is one-to-one and analytic on some neighbourhood of $\sigma(A)$, then $f(A)X = Xf(A)$ if and only if $AX = XA$.*

Proof. We have only to prove the "only if" part. Assume that $f(A)X = Xf(A)$ and let Ω be the domain of analyticity of f such that $\sigma(A) \subset \Omega$. Then f^{-1} is one-to-one and analytic on $f(\Omega)$. By the spectral mapping theorem, $\sigma(f(A)) = f(\sigma(A)) \subset f(\Omega)$. Now $f(A)X = Xf(A)$ implies that $f^{-1}(f(A))X = Xf^{-1}(f(A))$. But the basic properties of the Riesz functional calculus show that $A = f^{-1}(f(A))$. Hence, $AX = XA$, as required.

An intertwining version of Theorem 3 is now presented.

THEOREM 4. *Let A, B and $X \in B(H)$. If f is a function that is one-to-one and analytic on some neighbourhood of $\sigma(A) \cup \sigma(B)$, then $f(A)X = Xf(B)$ if and only if $AX = XB$.*

Proof. Define operators T and Y on the Hilbert space $H \oplus H$ by $T = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$, $Y = \left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$. Then $\sigma(T) = \sigma(A) \cup \sigma(B)$ and $f(T) = \left[\begin{smallmatrix} f(A) & 0 \\ 0 & f(B) \end{smallmatrix}\right]$. Since, by simple algebra, $AX = XB$ if and only if $TY = YT$, and $f(A)X = Xf(B)$ if and only if $f(T)Y = Yf(T)$, the result now follows by applying Theorem 3 to the operators T and Y.

For our purpose, the most interesting cases of Theorem 4 are demonstrated in the following corollaries.

COROLLARY 1. *Let A, B and $X \in B(H)$, where A and B are self-adjoint. Then, for every odd positive integer n, $A^n X = XB^n$ if and only if $AX = XB$.*

The requirement that n is an odd positive integer in Corollary 1 can be dropped, if the condition on A and B is strengthened as follows.

COROLLARY 2. *Let A, B and $X \in B(H)$, where A and B are positive. Then, for every positive real number r, $A^r X = XB^r$ if and only if $AX = XB$.*

Using the simple (but very useful) observation that, for any $A \in B(H)$, the matrix $\left[\begin{smallmatrix} 0 & A \\ A^* & 0 \end{smallmatrix}\right]$ defines a self-adjoint operator on $H \oplus H$, enables us to prove the main result of this section.

THEOREM 5. *Let A, B and $X \in B(H)$. Then, for every positive integer n, $(AA^*)^n AX = X(BB^*)^n B$ and $(A^*A)^n A^*X = X(B^*B)^n B^*$ if and only if $AX = XB$ and $A^*X = XB^*$.*

Proof. On $H \oplus H$, let $T = \left[\begin{smallmatrix} 0 & A \\ A^* & 0 \end{smallmatrix}\right]$, $S = \left[\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right]$ and $Y = \left[\begin{smallmatrix} X & 0 \\ 0 & X \end{smallmatrix}\right]$. Then T and S are self-adjoint. Simple algebra shows that

$$T^{2n+1} = \begin{bmatrix} 0 & (AA^*)^n A \\ (A^*A)^n A^* & 0 \end{bmatrix} \quad \text{and} \quad S^{2n+1} = \begin{bmatrix} 0 & (BB^*)^n B \\ (B^*B)^n B^* & 0 \end{bmatrix}.$$

Now the conditions $(AA^*)^n AX = X(BB^*)^n B$ and $(A^*A)^n A^*X = X(B^*B)^n B^*$ are equivalent to saying that $T^{2n+1}Y = YS^{2n+1}$. But this last condition is equivalent, by Corollary 1, to saying that $TY = YS$, which is also equivalent to saying that $AX = XB$ and $A^*X = XB^*$. The proof is now complete.

COROLLARY 3. *Let A, B and $X \in B(H)$, where A and B are normal. Then, for every positive integer n, $(AA^*)^n AX = X(BB^*)^n B^*$ if and only if $AX = XB$.*

Proof. We first observe that the adjoint of $(AA^*)^n A$ is $(A^*A)^n A^*$. Now the result follows from Theorem 4 and the Fuglede-Putnam theorem.

The normality assumption in Corollary 3 is essential, even in the finite-dimensional setting. For example, consider $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $X = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ acting on a two-dimensional Hilbert space. Then $(AA^*A)X = X(AA^*A)$, but $AX \neq XA$.

If, in Corollary 3, we take X to be the identity operator, then we have the following generalization of a finite-dimensional result of Khatri [**9**, Theorem 3(iii)].

Corollary 4. *Let $A, B \in B(H)$. Then, for every positive integer n, $(AA^*)^n A = (BB^*)^n B$ if and only if $A = B$. In particular, $(AA^*)^n A = (A^*A)^n A^*$ if and only if $A = A^*$; i.e., $(AA^*)^n A$ is self-adjoint if and only if A is self-adjoint.*

Proof. If $(AA^*)^n A = (BB^*)^n B$ then, by taking the adjoints of both sides of this equation, we also have $(A^*A)^n A^* = (B^*B)^n B^*$. The result now follows from Theorem 5.

We conclude with the following two remarks concerning Section 4.

Remarks. (1) The intertwining relations in this section can also be taken modulo C_∞. Just consider the Calkin algebra $B(H)/C_\infty$, which is a C^*-algebra, and hence it can be represented as an operator algebra.

(2) It follows from Theorem 3 in [**2**] that if $A, B \in B(H)$ are self-adjoint, and if $X \in B(H)$ is such that $A^n X - XB^n \in C_p$, for some odd positive integer n and some p with $1 \leq p \leq \infty$, then $AX - XB \in C_{np}$. It also follows from Theorem 7 in [**2**] that if $A, B \in B(H)$ are positive, and if $X \in B(H)$ is such that $A^r X - XB^r \in C_p$, for some real number $r \geq 1$ and some p with $1 \leq p \leq \infty$, then $AX - XB \in C_{rp}$. Moreover, it follows from Theorem 3.1 in [**11**] that if either A or B is invertible, then $AX - XB \in C_p$.

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