ON SOME NEW GENERALIZATIONS OF THE FUNCTIONAL EQUATION OF CAUCHY

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1. Introduction. Examining certain problems in physics M. Hosszu [1] obtained the functional equation

$$f(x+y)^2 = [f(x) + f(y)]^2 , \tag{1}$$

where x, y, f are real.

In another paper M. Hosszu [2] proved that the equation (1) is equivalent to the functional equation of Cauchy; i.e., to the equation

$$f(x+y) = f(x) + f(y) \tag{2}$$

under the assumption that x is real and f is real and continuous.

H. Swiatak [3] examined a generalization of the equation (1) in the class of continuous functions.

A similar alternative functional equation is considered in a paper of J. Aczél, K. Fladt and M. Hosszu [4].

At the end of his paper M. Hosszu puts the question: what is the general real solution of the equation (1) ? E. Vincze was the first to give an answer to this question in his papers [5], [6], [7]. He proved that the functional equation

$$f(x+y)^n = [f(x) + f(y)]^n \tag{3}$$

is equivalent to the functional equation of Cauchy, where x, y are in an additive Abelian semi-group, f is an arbitrary complex-valued function and n is a natural number.

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In the paper [8] we also answered the problem of M. Hosszu independently of E. Vincze. The purpose of the present paper is to generalize the original problem to normed spaces. As our paper [8] was published in Hungarian we do not suppose here that our results, published there, are known.

2. Let Q be an arbitrary additively written semi-group, let H be a Hilbert space, and let f be a mapping of Q into H; this will be denoted by $f: Q \to H$.

THEOREM 1. The functional equation

$$\|f(x+y)\| = \|f(x) + f(y)\| \tag{4}$$

is equivalent to the equation of Cauchy

$$f(x+y) = f(x) + f(y) \tag{5}$$

if $f: Q \to H$.

Proof. Evidently it is sufficient to prove that every solution of the equation (4) satisfies the equation (5). In what follows, we shall suppose that $f: Q \to H$, and f satisfies the equation (4).

We are going to show that

$$f(2x) = 2f(x) . \tag{6}$$

The assertion is evident if $f(x) = 0$ (where 0 is the zero element of the Hilbert space).

From the equation (4) we obtain only

$$\|f(2x)\| = 2\|f(x)\| . \tag{7}$$

Further, it is true that

$$\|f(3x)\| = \|f(2x) + f(x)\| \leqq \|f(2x)\| + \|f(x)\| = 3\|f(x)\| . \tag{8}$$

$\|f(4x)\|$ can be obtained in two ways:

$$\|f(4x)\| = \|f(x) + f(3x)\| \leqq \|f(x)\| + \|f(3x)\| \leqq 4\|f(x)\| , \tag{9}$$

$$\|f(4x)\| = \|f(2x) + f(2x)\| = 4\|f(x)\| . \tag{10}$$

From the equations (9) and (10) we get

$$(11) \qquad \|f(x) + f(3x)\| = 4\|f(x)\| ;$$

hence

$$\|f(3x)\| = 3\|f(x)\| ,$$

that is, in the equation (8)

$$\|f(2x) + f(x)\| = \|f(2x)\| + \|f(x)\| .$$

As the Hilbert space has the property that the equality in the triangle inequality exists if and only if one summand is a non-negative scalar multiple of the other,

$$f(2x) = \gamma(x) f(x) \qquad (\gamma(x) \geq 0) .$$

From the equation (7) and $f(x) \neq 0$ we get $\gamma(x) = 2$.

We compute $\|f(2x+y)\|$ in two ways:

$$(12) \qquad \|f(2x+y)\| = \|2 f(x) + f(y)\| = \|f(x) + f(x) + f(y)\|$$

and

$$(13) \qquad \|f(2x+y)\| = \|f(x) + f(x+y)\| ,$$

from which we get

$$(14) \qquad \|f(x) + f(x+y)\| = \|f(x) + f(x) + f(y)\| .$$

Taking the second power of equation (14) and considering that f satisfies the equation (4) we have

$$(15) \qquad \mathrm{Re}[f(x), f(x+y)] = \mathrm{Re}[f(x), f(x) + f(y)]$$

(where Re[,] denotes the real part of the inner product), that is

$$(16) \qquad \mathrm{Re}[f(x), f(x+y) - (f(x) + f(y))] = 0 .$$

If we interchange the variables x and y in the equation (16) we get

$$(17) \qquad \mathrm{Re}[f(y), f(x+y) - (f(x) + f(y))] = 0 .$$

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From the equations (16) and (17) we find that

$$\operatorname{Re}[f(x) + f(y),\ f(x) + f(y) - f(x+y)] = 0 . \tag{18}$$

Let us write $f(x+y)$ in the form

$$f(x+y) = f(x) + f(y) + f(x+y) - (f(x) + f(y)) .$$

Taking to the second power the norm of both sides, and using the equation (18), we obtain

$$\|f(x+y)\|^2 = \|f(x) + f(y)\|^2 + \|f(x+y) - f(x) - f(y)\|^2 ,$$

that is

$$\|f(x+y) - f(x) - f(y)\|^2 = 0 ,$$

from which our assertion follows immediately.

CONSEQUENCE 1. If H is the field of the real numbers, then Theorem 1 gives an answer to the original question of M. Hosszu.

CONSEQUENCE 2. If H is the field of complex numbers, then Theorem 1 includes the result of E. Vincze mentioned in the introduction, as a special case, and, what is more, we did not use that the semi-group is commutative.

3. In what follows we treat the problem of how far the conditions of Theorem 1 can be weakened. We shall show that in general the Hilbert space can not be replaced by an arbitrary normed linear space.

In the following we suppose that H is a non-strictly normed space $(\dim H \geq 2)$, that is, there exist $a \neq 0$ and $b \neq 0$ $(a, b \in H)$ such that

$$\|a + b\| = \|a\| + \|b\| , \tag{19}$$

but $a \neq \lambda b$ for any positive λ. (It is easy to verify that in this case a and b are linearly independent.)

LEMMA 1. If there exist $a, b \in H$ which satisfy the equation (19), then for arbitrary non-negative λ and μ

$$\|\lambda a + \mu b\| = \lambda \|a\| + \mu \|b\| . \tag{20}$$

Proof. We can suppose that $\lambda \leqq \mu$; that is, on the one hand

$$(21) \qquad \|\lambda a + \mu b\| \leqq \lambda \|a\| + \mu \|b\| ,$$

and the other

$$(22) \qquad \begin{aligned} \|\lambda a + \mu b\| &= \|\mu(a+b) - (\mu-\lambda)a\| \geq |\, \mu \|a+b\| - (\mu-\lambda)\|a\| \,| \\ &= \lambda \|a\| + \mu \|b\| . \end{aligned}$$

Our assertion follows from the equations (21) and (22).

Now we shall prove the following theorem:

THEOREM 2. Let Q be an arbitrary semi-group. If the equation of Cauchy has a non-constant solution g in the class of real-valued functions, where g is defined on Q, then for this Q and for arbitrary non-strictly normed space H ($\dim H \geq 2$) there exists $f: Q \to H$ which satisfies the equation (4), but does not satisfy the equation of Cauchy.

Proof. Let a and b be chosen according to Lemma 1. We can suppose that $\|a\| = \|b\| = 1$.

Define f by means of g as follows:

$$f(x) = \begin{cases} g(x)a & , \text{ if } |g(x)| \leqq 1 \\ \operatorname{sign} g(x)a + (g(x) - \operatorname{sign} g(x))b & , \text{ if } |g(x)| > 1 . \end{cases}$$

If neither the coefficient of a nor that of b are zero, then according to the definition of $f(x)$, for every x it is true that the coefficients are of same signs. By the Lemma 1 we get

$$\|f(x)\| = |g(x)| .$$

This relation is valid also in the case when $|g(x)| \leq 1$, so

$$\|f(x+y)\| = |g(x+y)| = |g(x) + g(y)| .$$

Compute the $f(x) + f(y)$:

$$f(x) + f(y) = (g(x) + g(y))a \qquad , \text{ if } |g(x)| \leq 1 \text{ and } |g(y)| \leq 1 ,$$

$$f(x) + f(y) = (g(x) + \operatorname{sign} g(y))a + (g(y) - \operatorname{sign} g(y))b ,$$

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if $|g(x)| \leqq 1$ and $|g(y)| > 1$;

and symmetrically for $|g(x)| > 1$, $|g(y)| \leqq 1$,

$f(x) + f(y) = (\text{sign } g(x) + \text{sign } g(y))a + (g(x) - \text{sign } g(x) + g(y) - \text{sign } g(y))b$,

if $|g(x)| > 1$ and $|g(y)| > 1$.

For $f(x) + f(y)$ it is also valid that the coefficients are of the same sign if neither the coefficient of a nor that of b are zero. By means of Lemma 1 we find that

$$\|f(x) + f(y)\| = |g(x) + g(y)| = \|f(x+y)\| ,$$

that is, f satisfies the equation (4).

It will be proved that f does not satisfy the equation of Cauchy. In fact, there exist such x and y that $|g(x)| > 1$, $|g(y)| > 1$ and $|g(x+y)| > 1$. Then the coefficients of a in $f(x+y)$ and in $f(x) + f(y)$ are different, from which our assertion follows, as a and b are linearly independent.

We shall show that there exists a function defined on the additive semi-group of positive numbers which satisfies the equation (4), but does not satisfy the equation of Cauchy, and, what is more, our function will be also differentiable.

Let a and b be chosen according to Lemma 1. Let us choose the functions λ and μ so that

$$\lambda(x)\|a\| + \mu(x)\|b\| = x, \quad \text{if } x \geqq 0, \tag{23}$$

and moreover $\lambda(x) \geqq 0$ and $\mu(x) \geqq 0$ are differentiable functions; let λ be a non-additive function. By means of such λ and μ we can define f as follows:

$$f(x) = \lambda(x)a + \mu(x)b , \quad \text{if } x > 0 .$$

f does not satisfy the equation of Cauchy, because λ is a non-additive function and a and b are linearly independent. From the Lemma 1 and the equation (23) we get

$$\|f(x)\| = x ;$$

moreover

$$\|f(x) + f(y)\| = \|(\lambda(x) + \lambda(y))a + (\mu(x) + \mu(y))b\|$$
$$= (\lambda(x) + \lambda(y))\|a\| + (\mu(x) + \mu(y))\|b\| = x+y ,$$

that is, f satisfies the equation (4).

It is obvious that f is differentiable, and its derivative has the form

$$f'(x) = \lambda'(x)a + \mu'(x)b .$$

4. Now let H be a strictly normed space, Q an arbitrary semi-group, $f: Q \to H$, and let f satisfy the equation (4). We conjecture that f is also the solution of the equation of Cauchy. In the following we prove a special case of this conjecture.

THEOREM 3. Let H be a strictly normed space, Q the additive semi-group of positive numbers, $f: Q \to H$, and let

$$\|f(x)\| = \varphi(x) .$$

If there exists a measurable subset E of $(0, +\infty)$ with positive measure and a function g defined and measurable on E, and such that $\varphi(x) \leq g(x)$ $(x \in E)$ (this condition is fulfilled for example if φ is bounded from above in any open subinterval of $(0, +\infty)$ or if φ is measurable) and if f satisfies the equation (4), then f satisfies the equation of Cauchy.

Proof. In the proof of the Theorem 1, where we proved the equality

$$f(2x) = 2 f(x),$$

we used only that H is a strictly normed space, and did not suppose that it is a Hilbert space.

By means of induction it can be proved that

$$f(rx) = r f(x)$$

for every natural number r .

The function φ has the following properties:

a) $\varphi(x) \geq 0$,
b) $\varphi(x+y) \leq \varphi(x) + \varphi(y)$,

c) $\varphi(rx) = r \cdot \varphi(x)$

for every rational r. From b) and c) it follows that the function φ is Jensen-convex, that is, it satisfies also the inequality

$$\varphi\left(\frac{x+y}{2}\right) \leqq \frac{\varphi(x) + \varphi(y)}{2} .$$

From the condition of the theorem it follows that φ is continuous (see [9], [10], [11]). If x is continuous, then from c) we find that

$$\varphi(x) = \varphi(1)x ,$$

i.e.,

$$\|f(x)\| = \|f(1)\| x .$$

As H is a strictly normed space we obtain

$$f(x) = f(1)x ,$$

that is f satisfies the equation of Cauchy.

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