CHARACTER SUMS AND THE SERIES $L(1, \chi)$

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Abstract

In this paper we derive a relation between character sums and partial sums of Dirichlet series.

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1. Introduction

Let k be a positive integer greater than 1, and let $\chi(n)$ be a real primitive character modulo k. The series

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of k consecutive terms. Let v be any nonnegative integer, j an integer such that $0 \leq j \leq k - 1$, and let

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk + n)}{vk + n} = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk + n}.$$

Then

$$L(1, \chi) = \sum_{v=0}^{\infty} T(v, 0, \chi) = \sum_{n=1}^{j} \chi(n)/n + \sum_{v=0}^{\infty} T(v, j, \chi).$$

The case $T(v, 0, \chi)$ was studied by Erdös and Davenport [4]. Preliminary results related to $T(v, j, \chi)$ can be found in papers by Davenport [4], the author [6, 7], and

Li and the author [8]. In [7], one can find that the behaviour of the sign of $T(v, j, \chi)$ has close relation to the class number problem for quadratic number fields.

In Section 2, Theorem 2.1 reveals a surprising relation between character sums and partial sums of Dirichlet series. As an application, Theorem 2.1 converts some classical problems on quadratic residues to the problems of deciding the sign of $T(v, j, \chi)$ (see Remark 2). In Section 3, we derive the imaginary version of Theorem 2.1. In Section 4, as a reference, we record a connection among Bernoulli numbers, Bernoulli polynomials and character sums.

We remind the reader that a real primitive character (mod k) exists only when either k or $-k$ is a fundamental discriminant, and that the character is then given by

$$\chi(n) = \left(\frac{d}{n}\right),$$

where d is k or $-k$, and the symbol is that of Kronecker (see, for example, Ayoub [1] for the definition of a Kronecker character).

Lastly, we state and prove a lemma which will be used in the proofs of Theorem 2.1 and Theorem 3.2.

LEMMA 1.1.

$$\sum_{l=1}^{k} \chi(j + l) e^{2\pi i m l / k} = \begin{cases} \chi(m)\sqrt{k} e^{-2\pi i m j / k}, & \text{if } \chi \text{ is even;} \\ i\chi(m)\sqrt{k} e^{-2\pi i m j / k}, & \text{if } \chi \text{ is odd.} \end{cases}$$

PROOF. Let us recall the definition of the Gauss sum

$$\sum_{n=1}^{k} \chi(n) e^{2\pi i m n / k} = \begin{cases} \chi(m)\sqrt{k}, & \text{if } \chi \text{ is even;} \\ i\chi(m)\sqrt{k}, & \text{if } \chi \text{ is odd.} \end{cases}$$

Multiplying by $e^{-2\pi i m j / k}$ on both sides of the Gauss sum, we obtain as the left-hand side

$$\sum_{n=1}^{k} \chi(n) e^{2\pi i m n / k} e^{-2\pi i m j / k} = \sum_{n=j+1}^{j+k} \chi(n) e^{2\pi i m n / k} e^{-2\pi i m j / k}$$

$$= \sum_{l=1}^{k} \chi(j + l) e^{2\pi i m (j+l)/k} e^{-2\pi i m j / k}$$

$$= \sum_{l=1}^{k} \chi(j + l) e^{2\pi i m l / k},$$

which completes the proof.   □

2. Even characters

In this section we show that the sign of $T(v, j, \chi)$ has close relation to the sign of character sum $\sum_{n=1}^{j} \chi(n)$, where χ is the real even primitive character modulo k.

For an integer j in the closed interval $[1, k - 1]$, write

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk + n} = \frac{1}{k} \sum_{l=1}^{k} \frac{\chi(j + l)}{w + l/k},$$

where $w = v + j/k$. For $w = v + j/k > 0$, consider the periodic function $f(x)$ of period 1, where $f(x)$ is defined by the equation

$$f(x) = 1/(w + x) \quad \text{for } 0 < x \le 1.$$

Over the interval $(0, 1)$, it has Fourier expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos 2\pi mx + b_m \sin 2\pi mx),$$

where

$$a_m = 2 \int_0^1 \frac{\cos 2\pi mx}{w + x} \, dx \quad \text{and} \quad b_m = 2 \int_0^1 \frac{\sin 2\pi mx}{w + x} \, dx$$

(see [5, Theorem 2.5]). Since $f(1-) = 1/(w + 1)$ and $f(1+) = 1/w$, by [5, Theorem 2.5], we have

$$(2.1) \qquad \frac{1}{2}\{f(1+) + f(1-)\} = \frac{1}{2w} + \frac{1}{2(w + 1)}$$

$$= \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos 2\pi m + b_m \sin 2\pi m).$$

Using integration by parts, we have, for $m \ge 1$,

$$a_m = \frac{2}{(2\pi m)^2}\left\{\frac{1}{w^2} - \frac{1}{(w + 1)^2}\right\} - \frac{4}{(2\pi m)^2} \int_0^1 \frac{\cos 2\pi mx}{(w + x)^3} \, dx.$$

Let

$$I_m = \frac{4}{(2\pi m)^2} \int_0^1 \frac{\cos 2\pi mx}{(w + x)^3} \, dx \quad \text{and} \quad J_m = \frac{2}{(2\pi m)^2}\left\{\frac{1}{w^2} - \frac{1}{(w + 1)^2}\right\}.$$

By looking at the graph of $y = (\sin 2\pi mx)/(w + x)^4$ on the interval $[0, 1]$, it is easy to see that

$$I_m = \frac{4}{(2\pi m)^2} \int_0^1 \frac{\cos 2\pi mx}{(w + x)^3} \, dx = \frac{12}{(2\pi m)^3} \int_0^1 \frac{\sin 2\pi mx}{(w + x)^4} \, dx > 0.$$

Since $0 < I_m = |I_m| < J_m$, we have $a_m = J_m - I_m = J_m \theta_m$, where $\theta_m = (J_m - I_m)/J_m$ and $0 < \theta_m < 1$. Similarly, we have

$$b_m = \frac{2}{2\pi m} \left(\frac{1}{w} - \frac{1}{w+1} \right) - \frac{4}{(2\pi m)^3} \left(\frac{1}{w^3} - \frac{1}{(w+1)^3} \right)$$
$$+ \frac{12}{(2\pi m)^3} \int_0^1 \frac{\cos 2\pi m x}{(w+x)^4} \, dx.$$

Let

$$X_m = \frac{12}{(2\pi m)^3} \int_0^1 \frac{\cos 2\pi m x}{(w+x)^4} \, dx \quad \text{and} \quad Y_m = \frac{4}{(2\pi m)^3} \left(\frac{1}{w^3} - \frac{1}{(w+1)^3} \right).$$

Then

$$X_m = \frac{12}{(2\pi m)^3} \int_0^1 \frac{\cos 2\pi m x}{(w+x)^4} \, dx = \frac{48}{(2\pi m)^4} \int_0^1 \frac{\sin 2\pi m x}{(w+x)^5} \, dx > 0$$

and $X_m < Y_m$. Hence, we have

$$b_m = \frac{2}{2\pi m} \left(\frac{1}{w} - \frac{1}{w+1} \right) - \frac{4}{(2\pi m)^3} \left(\frac{1}{w^3} - \frac{1}{(w+1)^3} \right) \eta_m,$$

where $\eta_m = (Y_m - X_m)/Y_m$ and $0 < \eta_m < 1$. Now

$$T(v, j, \chi) = \frac{1}{k} \sum_{l=1}^{k} \frac{\chi(j+l)}{w+l/k} \quad (w = v + j/k \text{ and } j \geq 1)$$

$$= \frac{1}{k} \sum_{l=1}^{k-1} \chi(j+l) f(l/k) + \frac{1}{k} \chi(j+k) \frac{1}{w+1}$$

$$= \frac{1}{k} \sum_{l=1}^{k-1} \chi(j+l) f(l/k)$$

$$+ \frac{2\chi(j+k)}{k} \left\{ \frac{1}{2} a_0 + \sum_{m=1}^{\infty} (a_m \cos 2\pi m + b_m \sin 2\pi m) \right\}$$

$$- \frac{2\chi(j+k)}{k} \frac{1}{2w} \qquad \text{(by (2.1))}$$

$$= \frac{1}{k} \sum_{l=1}^{k} \chi(j+l) \left\{ \frac{1}{2} a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{2\pi m l}{k} + b_m \sin \frac{2\pi m l}{k} \right) \right\}$$

$$+ \frac{\chi(j)}{k} \left(\frac{1}{2w} + \frac{1}{2(w+1)} \right) - \frac{2\chi(j)}{k} \frac{1}{2w}$$

$$= \frac{1}{k} \sum_{l=1}^{k} \chi(j+l) \sum_{m=1}^{\infty} \left(a_m \cos \frac{2\pi m l}{k} + b_m \sin \frac{2\pi m l}{k} \right) + \frac{\chi(j)}{k} \frac{-1}{2w(w+1)}$$

(since $\sum_{l=1}^{k} \chi(j + l) = 0$)

$$= E + \frac{1}{k} \sum_{l=1}^{k} \sum_{m=1}^{\infty} \left(a_m \chi(j + l) \cos \frac{2\pi ml}{k} + b_m \chi(j + l) \sin \frac{2\pi ml}{k} \right)$$

(where $E = -\chi(j)/(2w(w + 1)k)$)

$$= E + \frac{1}{k} \sum_{m=1}^{\infty} \left(a_m \sum_{l=1}^{k} \chi(j + l) \cos \frac{2\pi ml}{k} + b_m \sum_{l=1}^{k} \chi(j + l) \sin \frac{2\pi ml}{k} \right)$$

$$= E + \frac{1}{k} \sum_{m=1}^{\infty} \left(a_m \chi(m) \sqrt{k} \cos \frac{2\pi mj}{k} - b_m \chi(m) \sqrt{k} \sin \frac{2\pi mj}{k} \right)$$

(by Lemma 1.1). Hence

$$\sqrt{k} T(v, j, \chi)$$

$$= \sqrt{k} E + \sum_{m=1}^{\infty} \left(a_m \chi(m) \cos \frac{2\pi mj}{k} - b_m \chi(m) \sin \frac{2\pi mj}{k} \right)$$

$$= \sqrt{k} E + \left(\frac{1}{w} - \frac{1}{w + 1} \right) \left\{ \frac{2}{(2\pi)^2} \left(\frac{1}{w} + \frac{1}{w + 1} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \theta_m}{m^2} \cos \frac{2\pi mj}{k} \right.$$

$$- \frac{2}{2\pi} \sum_{m=1}^{\infty} \frac{\chi(m)}{m} \sin \frac{2\pi mj}{k}$$

$$\left. + \frac{4}{(2\pi)^3} \left(\frac{1}{w^2} + \frac{1}{w(w + 1)} + \frac{1}{(w + 1)^2} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \eta_m}{m^3} \sin \frac{2\pi mj}{k} \right\}$$

(where $0 < \theta_m, \eta_m < 1$)

$$= \left(\frac{1}{w} - \frac{1}{w + 1} \right) \left\{ \frac{-\chi(j)}{2\sqrt{k}} + \frac{2}{(2\pi)^2} \left(\frac{1}{w} + \frac{1}{w + 1} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \theta_m}{m^2} \cos \frac{2\pi mj}{k} \right.$$

$$- \frac{1}{\sqrt{k}} \left(\sum_{m=1}^{j-1} \chi(m) + \frac{1}{2} \chi(j) \right)$$

$$\left. + \frac{4}{(2\pi)^3} \left(\frac{1}{w^2} + \frac{1}{w(w + 1)} + \frac{1}{(w + 1)^2} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \eta_m}{m^3} \sin \frac{2\pi mj}{k} \right\}$$

(by Proposition 2.3)

$$= \left(\frac{1}{w} - \frac{1}{w + 1} \right) \left\{ \frac{2}{(2\pi)^2} \left(\frac{1}{w} + \frac{1}{w + 1} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \theta_m}{m^2} \cos \frac{2\pi mj}{k} - \sum_{m=1}^{j} \frac{\chi(m)}{\sqrt{k}} \right.$$

$$\left. + \frac{4}{(2\pi)^3} \left(\frac{1}{w^2} + \frac{1}{w(w + 1)} + \frac{1}{(w + 1)^2} \right) \sum_{m=1}^{\infty} \frac{\chi(m) \eta_m}{m^3} \sin \frac{2\pi mj}{k} \right\}.$$

Let

$$S(v, j) = \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\left|\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m}{m^2}\cos\frac{2\pi mj}{k}\right|$$

$$+ \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\left|\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m}{m^3}\sin\frac{2\pi mj}{k}\right|$$

and

$$P_j = \frac{1}{\sqrt{k}}\left|\sum_{m=1}^{j}\chi(m)\right|.$$

Then we have

$$S(v, j) \leq \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\zeta(2) + \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\zeta(3)$$

$$\leq \frac{2}{(2\pi)^2}\frac{2}{w}\frac{\pi^2}{6} + \frac{4}{(2\pi)^3}\frac{3}{w^2}\frac{\pi^2}{6} \qquad (\zeta(2) = \pi^2/6 > \zeta(3))$$

$$= (1/6 + 1/(4\pi w))/w < 1/(4w) \qquad \text{(if } v \geq 1\text{)}.$$

If $|M| = \left|\sum_{m=1}^{j}\chi(m)\right| \geq 1$, then, for an integer $v \geq \sqrt{k}/(4|M|)$, we have $P_j = |M|/\sqrt{k} > 1/(4w)$.

We have proved the following theorem.

THEOREM 2.1. *Let $k \geq 5$, $k \equiv 0$ or 1 (mod 4) be a fundamental discriminant and χ the real even primitive character modulo k. If $M = \sum_{m=1}^{j}\chi(m) \neq 0$, then*

$$T(v, j, \chi)\sum_{m=1}^{j}\chi(m) < 0$$

for an integer $v \geq \max\{1, \sqrt{k}/(4|M|)\}$.

REMARK 1. Theorem 2.1 can replace Theorem 2.1 and Theorem 2.2 of [7], a weaker result of author's earlier work.

REMARK 2. Let $S_{ji} = S_{ji}(\chi) = \sum_{(i-1)k/j < n < ik/j}\chi(n)$, where i and j are natural numbers, and k is the modulus of χ. In [3], Berndt expressed S_{ji} as a linear combination of L-functions evaluated at $s = 1$. For some classes of real primitive characters, he proved that S_{ji} is of a constant sign for particular j and i. Some cases when S_{ji} is of a constant sign are still open problems. For example, let $\chi_p(n)$ denote the Legendre symbol modulo p, where prime $p \equiv 1, 7$ (mod 8). Although numerical computations suggest that $S_{51} > 0$, Berndt [3] was unable to say anything about the

sign of S_{51}. As an application of Theorem 2.1, the following Theorem 2.2 provides a new approach to attack the problem whether $S_{51}(\chi_p)$ is of a constant sign for prime $p \equiv 1 \pmod 8$.

THEOREM 2.2. *Let prime $p \equiv 1 \pmod 8$ and $j = [p/5]$ be odd. Then*

$$S_{51}(\chi_p) > 0 \quad \text{if and only if} \quad T(v, j, \chi) < 0$$

for some integer $v \geq \max\{1, \sqrt{k}/(4|M|)\}$.

PROOF. By assumption $j = [p/5]$ is odd, where $[x]$ denotes the greatest integer less than or equal to x. Hence $S_{51}(\chi_p) \neq 0$. Now, by applying Theorem 2.1, the theorem follows.   □

Finally, to close this section, we need to state Proposition 2.3. The proofs of Proposition 2.3 and Proposition 3.1 are very similar. Since the proof of Proposition 2.3 can be found in [7, Proposition 2.4], we only provide here the proof of Proposition 3.1 (see Section 3).

PROPOSITION 2.3. *Let j be any integer in the closed interval $[1, k - 1]$ and χ the real even primitive character modulo k. Then*

$$\sum_{n=1}^{j-1} \chi(n) + \frac{1}{2}\chi(j) = \frac{\sqrt{k}}{\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \sin \frac{2\pi nj}{k}.$$

3. Odd characters

In this section we derive the imaginary version of Theorem 2.1. First, we prove the following crucial proposition.

PROPOSITION 3.1. *Let j be any integer in the closed interval $[1, k - 1]$ and χ the real odd primitive character modulo k. Then*

$$\sum_{n=1}^{j-1} \chi(n) + \frac{1}{2}\chi(j) = h(-k) - \frac{\sqrt{k}}{\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \cos \frac{2\pi nj}{k},$$

where $h(-k)$ denotes the class number of $\mathbb{Q}(\sqrt{-k})$.

PROOF. For fixed integer j, we define the periodic function ϕ with period 2π as follows:

$$\phi(x) = \begin{cases} 1 & \text{if } 0 < x < 2\pi j/k; \\ 1/2 & \text{if } x = 0 \text{ or } x = 2\pi j/k \text{ or } x = 2\pi; \\ 0 & \text{if } 2\pi j/k < x < 2\pi. \end{cases}$$

Then $\sum_{n=1}^{j-1} \chi(n) + \chi(j)/2 = \sum_{n=1}^{k} \phi(2\pi n/k)\chi(n)$. Since $(\phi(0_+) + \phi(0_-))/2 = (1+0)/2 = \phi(0)$ and $(\phi((2\pi j/k)_+) + \phi((2\pi j/k)_-))/2 = (0+1)/2 = \phi(2\pi j/k)$, by [5, Theorem 2.5], over the interval $[0, 2\pi]$, ϕ has a Fourier expansion

$$\phi(x) = \frac{j}{k} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_0/2 = \frac{j}{k}, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} \phi(x)\cos nx\,dx = \frac{1}{n\pi}\sin\frac{2\pi nj}{k}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\phi(x)\sin nx\,dx = \frac{1}{n\pi} - \frac{1}{n\pi}\cos\frac{2\pi nj}{k}.$$

Now

$$\sum_{n=1}^{j-1}\chi(n) + \frac{1}{2}\chi(j) = \sum_{m=1}^{k}\phi\left(\frac{2\pi m}{k}\right)\chi(m)$$

$$= \sum_{m=1}^{k}\left\{\frac{j}{k} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi mn}{k} + b_n\sin\frac{2\pi mn}{k}\right)\right\}\chi(m)$$

$$= \sum_{m=1}^{k}\left\{\sum_{n=1}^{\infty}\left(\chi(m)a_n\cos\frac{2\pi mn}{k} + \chi(m)b_n\sin\frac{2\pi mn}{k}\right)\right\}$$

$\left(\text{since } \sum_{m=1}^{k}\chi(m) = 0\right)$

$$= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{k}\chi(m)a_n\cos\frac{2\pi mn}{k} + \sum_{m=1}^{k}\chi(m)b_n\sin\frac{2\pi mn}{k}\right)$$

$$= \sum_{n=1}^{\infty}\left(a_n\sum_{m=1}^{k}\chi(m)\cos\frac{2\pi mn}{k} + b_n\sum_{m=1}^{k}\chi(m)\sin\frac{2\pi mn}{k}\right)$$

$$= \sum_{n=1}^{\infty}b_n\chi(n)\sqrt{k} = \sum_{n=1}^{\infty}\left(\frac{1}{n\pi} - \frac{1}{n\pi}\cos\frac{2\pi nj}{k}\right)\chi(n)\sqrt{k}$$

$$= \frac{\sqrt{k}}{\pi}L(1,\chi) - \frac{\sqrt{k}}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\cos\frac{2\pi nj}{k}$$

$$= h(-k) - \frac{\sqrt{k}}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\cos\frac{2\pi nj}{k}.$$

Here we use the class number formula for imaginary quadratic fields

$$h(-k) = (\sqrt{k}/\pi)L(1,\chi)$$

and the Gauss sum $\sum_{m=1}^{k}\chi(m)\exp(2\pi imn/k) = i\chi(n)\sqrt{k}$ for $\chi(-1) = -1$. $\square$

THEOREM 3.2. *Let $k \geq 5$, $-k \equiv 0$ or 1 (mod 4) be a fundamental discriminant and χ the real odd primitive character modulo k. If $M = h(-k) - \sum_{m=1}^{j} \chi(m) \neq 0$, then*

$$T(v, j, \chi) \left(h(-k) - \sum_{m=1}^{j} \chi(m) \right) > 0$$

for some integer $v \geq \max \{ 1, \sqrt{k}/(4|M|) \}$.

PROOF. As in Section 2, for an integer j in the closed interval $[1, k-1]$, write

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk + n} = \frac{1}{k} \sum_{l=1}^{k} \frac{\chi(j + l)}{w + l/k},$$

where $w = v + j/k$. For $w = v + j/k > 0$, consider the periodic function $f(x)$ of period 1 such that $f(x) = 1/(w + x)$ on $(0, 1]$, $f(1-) = 1/(w + 1)$ and $f(1+) = 1/w$. By [5, Theorem 2.5], over the interval $(0, 1)$, $f(x)$ has the Fourier expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos 2\pi mx + b_m \sin 2\pi mx),$$

and

$$\frac{f(1+) + f(1-)}{2} = \frac{1}{2w} + \frac{1}{2(w + 1)} = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos 2\pi m + b_m \sin 2\pi m),$$

where

$$a_m = 2 \int_0^1 \frac{\cos 2\pi mx}{w + x} dx \quad \text{and} \quad b_m = 2 \int_0^1 \frac{\sin 2\pi mx}{w + x} dx.$$

Applying exactly the same argument as in Section 2, we have the same expressions, as in Section 2, for symbols a_m, θ_m, b_m, and η_m. Now

$$T(v, j, \chi)$$

$$= \frac{1}{k} \sum_{l=1}^{k} \frac{\chi(j + l)}{w + l/k} = \frac{1}{k} \sum_{l=1}^{k-1} \chi(j + l) f(l/k) + \frac{1}{k} \frac{\chi(j + k)}{w + 1}$$

$$= \frac{-\chi(j)}{2w(w+1)k} + \frac{1}{k} \sum_{m=1}^{\infty} \left(a_m \sum_{l=1}^{k} \chi(j + l) \cos \frac{2\pi ml}{k} + b_m \sum_{l=1}^{k} \chi(j + l) \sin \frac{2\pi ml}{k} \right)$$

(up to now, details are the same as in Section 2)

$$= \frac{-\chi(j)}{2w(w + 1)k} + \frac{1}{k} \sum_{m=1}^{\infty} \left(a_m \chi(m) \sqrt{k} \sin \frac{2\pi mj}{k} + b_m \chi(m) \sqrt{k} \cos \frac{2\pi mj}{k} \right)$$

(by applying Lemma 1.1 to odd character). Hence

$$\sqrt{k}\,T(v, j, \chi)$$

$$= \frac{-\chi(j)}{2w(w+1)\sqrt{k}} + \sum_{m=1}^{\infty}\left(a_m\chi(m)\sin\frac{2\pi mj}{k} + b_m\chi(m)\cos\frac{2\pi mj}{k}\right)$$

$$= \frac{-\chi(j)}{2w(w+1)\sqrt{k}} + \sum_{m=1}^{\infty}\left(\frac{2}{(2\pi m)^2}\left(\frac{1}{w^2} - \frac{1}{(w+1)^2}\right)\chi(m)\theta_m\sin\frac{2\pi mj}{k}\right.$$

$$+ \frac{2}{2\pi m}\left(\frac{1}{w} - \frac{1}{w+1}\right)\chi(m)\cos\frac{2\pi mj}{k}$$

$$\left. - \frac{4}{(2\pi m)^3}\left(\frac{1}{w^3} - \frac{1}{(w+1)^3}\right)\chi(m)\eta_m\cos\frac{2\pi mj}{k}\right)$$

(where $0 < \theta_m, \eta_m < 1$)

$$= \left(\frac{1}{w} - \frac{1}{w+1}\right)\left(\frac{-\chi(j)}{2\sqrt{k}} + \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m}{m^2}\sin\frac{2\pi mj}{k}\right.$$

$$+ \frac{2}{2\pi}\sum_{m=1}^{\infty}\frac{\chi(m)}{m}\cos\frac{2\pi mj}{k}$$

$$\left. - \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m}{m^3}\cos\frac{2\pi mj}{k}\right)$$

$$= \left(\frac{1}{w} - \frac{1}{w+1}\right)\left(\frac{-\chi(j)}{2\sqrt{k}} + \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m}{m^2}\sin\frac{2\pi mj}{k}\right.$$

$$+ \frac{1}{\sqrt{k}}\left(h(-k) - \sum_{n=1}^{j-1}\chi(n) - \frac{1}{2}\chi(j)\right)$$

$$\left. - \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m}{m^3}\cos\frac{2\pi mj}{k}\right)$$

(by Proposition 3.1)

$$= \left(\frac{1}{w} - \frac{1}{w+1}\right)\left(\frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m}{m^2}\sin\frac{2\pi mj}{k}\right.$$

$$+ \frac{1}{\sqrt{k}}\left(h(-k) - \sum_{n=1}^{j}\chi(n)\right)$$

$$\left. - \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m}{m^3}\cos\frac{2\pi mj}{k}\right).$$

Let $M = h(-k) - \sum_{n=1}^{j} \chi(n)$, $P_j = |M|/\sqrt{k}$ and

$$S(v, j) = \frac{2}{(2\pi)^2} \left(\frac{1}{w} + \frac{1}{w+1} \right) \left| \sum_{m=1}^{\infty} \frac{\chi(m)\theta_m}{m^2} \sin \frac{2\pi mj}{k} \right|$$

$$+ \frac{4}{(2\pi)^3} \left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2} \right) \left| \sum_{m=1}^{\infty} \frac{\chi(m)\eta_m}{m^3} \cos \frac{2\pi mj}{k} \right|.$$

If $M \neq 0$, then, by applying the same argument as in Section 2, we have $P_j > 1/(4w) > S(v, j)$ for an integer $v \geq \max\{1, \sqrt{k}/(4|M|)\}$. Hence, the theorem is proved.   □

REMARK 3. In [8, Theorem 6 (1)], we proved that $T(v, [k/2], \chi) < 0$ for a real odd primitive character χ of modulus $k \not\equiv 7 \pmod 8$ and an integer $v > k^{1/4}$. Combining Theorem 6 (1) of [8] and Theorem 3.2, we have $h(-k) - \sum_{n=1}^{[k/2]} \chi(n) \leq 0$. Since class number $h(-k)$ is a positive integer, we derive $\sum_{n=1}^{[k/2]} \chi(n) > 0$ for modulus $k \not\equiv 7$ (mod 8), a well-known result of Dirichlet. For Dirichlet's classical result, the reader can find a concise proof in the paper of Moser [9].

4. Bernoulli numbers and polynomials

As a reference, we state a connection among Bernoulli numbers, Bernoulli polynomials and character sum. Let $B_n(x)$ denote the nth Bernoulli polynomial and $B_n = B_n(0)$ the nth Bernoulli number. Then we have the following equation.

LEMMA 4.1. $\sum_{n=1}^{N-1} n^q = (B_{q+1}(N) - B_{q+1})/(1+q)$.

The proof can be found in [10].

For a prime integer $p > 3$ and any positive integer $n < p$, we have $(n/p) \equiv n^{(p-1)/2}$ (mod p), where (n/p) denotes the Legendre symbol. By Lemma 4.1, for an integer m in the interval $[1, p-1]$, we obtain

$$\sum_{n=1}^{m} \left(\frac{n}{p} \right) \equiv \sum_{n=1}^{m} n^{(p-1)/2} = \frac{2}{p+1} \{B_{(p+1)/2}(m+1) - B_{(p+1)/2}\} \equiv \alpha \quad (\text{mod } p)$$

for some integer α, $0 \leq \alpha < p$. Now, we have the following criteria.

THEOREM 4.2.

(1) $\sum_{n=1}^{m} \left(\frac{n}{p} \right) = \alpha$ if and only if $0 \leq \alpha < (p-1)/2$.

$$(2) \qquad \sum_{n=1}^{m} \left(\frac{n}{p} \right) = \alpha - p \quad \text{if and only if} \quad (p-1)/2 < \alpha < p.$$

Where $0 \le \alpha < p$ *is the integer such that* $2(B_{(p+1)/2}(m+1) - B_{(p+1)/2})/(p+1) \equiv \alpha$ (mod p).

References

[1] R. Ayoub, *An introduction to the analytic theory of numbers*, Math. Surveys Monographs 10 (Amer. Math. Soc., Providence, 1963).
[2] B. C. Berndt, 'Character analogues of the Poisson and Euler-MacLaurin summation formulas with applications', *J. Number Theory* **7** (1975), 413–445.
[3] ———, 'Classical theorems on quadratic residues', *Enseig. Math. (2)* **22** (1976), 261–304.
[4] H. Davenport, 'On the series for $L(1)$', *J. London Math. Soc.* **24** (1949), 229–233.
[5] E. A. González-Velasco, *Fourier analysis and boundary value problems* (Academic Press, San Diego, 1995).
[6] M.-G. Leu, 'On $L(1, \chi)$ and class number formula for the real quadratic fields', *Proc. Japan Acad.* **72A** (1996), 69–74.
[7] ———, 'Character sums and the series $L(1, \chi)$ with applications to real quadratic fields', *J. Math. Soc. Japan* **51** (1999), 151–166.
[8] M.-G. Leu and W.-C. Winnie Li, 'On the series for $L(1, \chi)$', *Nagoya Math. J.* **141** (1996), 125–141.
[9] L. Moser, 'A theorem on quadratic residues', *Proc. Amer. Math. Soc.* **2** (1951), 503–504.
[10] H. Rademacher, *Topics in analytic number theory* (Springer, New York, 1973).

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