

THE OPERATOR SHIFT SPACE

TIMUR OIKHBERG

*Department of Mathematics, University of California, Irvine,
Irvine, CA 92697, USA (toikhber@math.uci.edu)*

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Abstract We construct and examine an operator space X , isometric to ℓ_2 , such that every completely bounded map from its subspace Y into X is a compact perturbation of a linear combination of multiples of a shift of given multiplicity and their adjoints. Moreover, every completely bounded map on X is a Hilbert–Schmidt perturbation of such a linear combination.

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1. Introduction and the main result

In [8], Gowers and Maurey constructed a Banach space X_s such that any $T \in B(X_s)$ is a strictly singular perturbation of a Toeplitz operator. In this paper, we present a similar construction in the operator space case. In fact, the space X constructed below has certain additional properties. For instance, a linear operator on X is completely bounded if and only if it is a Hilbert–Schmidt perturbation of a Toeplitz operator with coefficients in a certain weighted ℓ_1 space.

Throughout the paper, we freely use standard operator space results and terminology. The reader is referred to [6, 23] for more information. We work mainly with operator spaces isometric to ℓ_2 . The *shift operator* $S \in B(\ell_2)$ is defined by setting $S\delta_i = \delta_{i+1}$ for $i \in \mathbb{N}$ (throughout the paper, $(\delta_i)_{i \in \mathbb{N}}$ denotes the canonical basis in ℓ_2). For the sake of brevity, we shall use the notation S^{-1} for S^* (that is, $S^{-1}\delta_1 = 0$, $S\delta_i = \delta_{i-1}$ for $i \geq 2$).

Some additional notation will be used in the paper. We use \mathcal{S}_p , $1 \leq p \leq \infty$, to denote a Schatten class of operators, while $\|\cdot\|_p$ is the corresponding norm. Throughout the paper, $C > 1$ and $m \in \mathbb{N}$ are fixed. $\ell_1(C^{|j|}, \mathbb{Z})$ denotes the weighted ℓ_1 space of all sequences $a = (a_j)_{j \in \mathbb{Z}}$, equipped with the norm $\|a\|_w = \sum_{j \in \mathbb{Z}} C^{|j|} |a_j|$. We define the Toeplitz operator T_a as

$$\tilde{T}_a = \sum_{j \geq 0} a_j S^{mj} + \sum_{j < 0} a_j S^{*mj} \in B(\ell_2).$$

Our main result is the following theorem.

Theorem 1.1. For every $m \in \mathbb{N}$ and $C > 1$ there exists an operator space $X = X(C, m)$, isometric to ℓ_2 , such that the following conditions hold.

- (i) (a) Any operator of the form $\tilde{T}_a + W \in B(X)$, with $a \in \ell_1(C^{|j|}, \mathbb{Z})$ and $W \in \mathcal{S}_2$, is completely bounded, with $\|\tilde{T}_a + W\|_{\text{cb}} \leq \|a\|_w + \|W\|_2$.
- (b) Conversely, every $T \in \text{CB}(X)$ can be written as $T = \tilde{T}_a + W$, with $a \in \ell_1(C^{|j|}, \mathbb{Z})$, $W \in \mathcal{S}_2$, $\|W\|_2 \leq \gamma \|T\|_{\text{cb}}$ and $\|a\|_w \leq 4\|T\|_{\text{cb}}$. Here, the constant γ depends only on C and m .
- (ii) If Y is an infinite-dimensional subspace of X , and $T \in \text{CB}(Y, X)$, then $T - \tilde{T}_a$ is compact for some $a \in \ell_1(C^{|j|}, \mathbb{Z})$ satisfying $\|a\|_w \leq 4\|T\|_{\text{cb}}$.
- (iii) Any infinite-dimensional subspace of X fails the operator approximation property (OAP).
- (iv) $K_0(\text{CB}(X)) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ and $K_1(\text{CB}(X)) = \{0\}$.

It is instructive to compare this statement with the results of Gowers and Maurey [8, 13, 14]. For any m , they construct a Banach space $X_s = X_s(m)$, equipped with a basis (e_i) , such that

- (1) the left and right shifts with respect to this basis, denoted by L and R , respectively, are contractive, and
- (2) for any $T \in B(Y, X_s)$ (Y is a subspace of X_s), there exists a sequence $(a_j)_{j \in \mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that

$$\left\| \left\| T - \left(\sum_{j \geq 0} a_j R^{mj} + \sum_{j < 0} a_j L^{-mj} \right) \right\| \right\| = 0.$$

Here, the semi-norm $\| \|W\| \|$ is described in terms the action of an operator W on block bases in X , in a rather involved way. It is known that W is compact $\Rightarrow \| \|W\| \| = 0 \Rightarrow W$ is strictly singular. However, we are not aware of any explicit descriptions of $B(X_s)$. In this way, Theorem 1.1 (ii) is similar to known results, and (i) is an improvement.

As noted above, the shift space of Gowers and Maurey has a basis; hence, it has the approximation property. We do not know if this property is inherited by all of its infinite-dimensional subspaces. Thus, (iii) underscores the difference between the classical and non-commutative cases.

Finally, Theorem 1.1 (iv) is similar to an exercise following [13, Theorem 12.1], stating that $K_0(\text{CB}(X_s)) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ and $K_1(\text{CB}(X_s)) = \{0\}$.

Motivation for this paper comes in part from the work on representations of Banach algebras as spaces of (completely) bounded maps. It was proved by Berkson and Porta [1] that $B(\ell_p)$ (for $p \in [1, 2) \cup (2, \infty]$) has no non-trivial representations on $B(\ell_2)$. Following the work of Størmer [26], Neufang, Popa, Ruan, Smith and Spronk [16, 24, 25] recently established that certain group algebras can be represented isometrically as subalgebras of $\text{CB}(B(H))$, where H is a suitable Hilbert space. In fact, in many cases the group

algebra can be identified with the algebra of module maps, satisfying certain continuity properties.

We are interested in a similar problem: suppose that \mathcal{A} is a unital Banach algebra and $\pi : \mathcal{A} \rightarrow B(H)$ is a unital representation. Can H be equipped with an operator space structure X such that $\text{CB}(X)$ consists of ‘small’ (usually, Hilbert–Schmidt) perturbations of elements of $\pi(\mathcal{A})$? The study of this question begun in [21], where we gave a positive answer when \mathcal{A} is a hyperfinite von Neumann subalgebra of $B(\ell_2)$, and π is the identity representation. In [19], we proved that the answer is positive when $H = \ell_2 \otimes_2 \ell_2$, \mathcal{A} is a dual Banach algebra, and $\pi = \rho \otimes I_{\ell_2}$, where $\rho : \mathcal{A} \rightarrow B(\ell_2)$ is a contractive, unital, weakly* continuous representation. A positive answer is given in [20] when \mathcal{A} is the bidual of the James quasi-reflexive space (viewed as a commutative Banach algebra).

In this paper, we consider the weighted convolution algebras $\ell(C^{|j|}, \Gamma)$, where the semigroup Γ is either \mathbb{Z} or $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $C > 1$. In other words, we work with the set of (bi-)infinite sequences

$$(a_j)_{j \in \Gamma} = \sum_{j \in \Gamma} a_j \delta_j,$$

equipped with the norm $\|(a_j)_{j \in \Gamma}\|_w = \sum_{j \in \Gamma} C^{|j|} |a_j|$, and with the multiplication $\delta_s \star \delta_t = \delta_{s+t}$ for $s, t \in \Gamma$ (see, for example, [4, § 4.6] for properties of such algebras). The map $\psi : \ell_1(C^{|j|}, \mathbb{Z}) \rightarrow B(\ell_2) : \delta_j \mapsto S^{mj}$ is not a representation, but ‘close to’, and $\text{CB}(X)$ consists of Hilbert–Schmidt perturbations of elements of $\psi(\ell_1(C^{|j|}, \mathbb{Z}))$. In § 8 we consider some faithful representations $\pi : \ell(C^{|j|}, \Gamma) \rightarrow B(H)$, where H is ℓ_2 or $\ell_2(\mathbb{Z})$. By modifying the reasoning of §§ 2–7, we equip H with an operator space structure $X(\pi)$ in such a way that $\pi : \ell(C^{|j|}, \Gamma) \rightarrow \text{CB}(X(\pi))$ becomes a complete isometry and, moreover, $\text{CB}(X(\pi)) = \pi(\ell(C^{|j|}, \Gamma)) + \mathcal{S}_2$. In Theorem 8.1, we list some properties of the spaces $X(\pi)$. These spaces do not seem to be as interesting as the space $X = X(C, m)$, and hence we mention them only briefly.

The rest of the paper is structured as follows. In § 2, we construct the space $X = X(C, m)$ (henceforth, this notation will be reserved for the operator space constructed there), establish the groundwork for the rest of the paper and prove Theorem 1.1 (i) (a). The key part is § 3, in which we show that for every $T \in \text{CB}(Y, X)$ (here Y is a subspace of X) there exists $a \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $T - \tilde{T}_a$ is compact. Based on this, we establish Theorem 1.1 (ii). In addition, we show that no coordinate subspace of X has non-trivial completely bounded (c.b.) projections (Theorem 3.1), and that a subspace $Y \hookrightarrow X$ is completely isomorphic to X if and only if $\dim X/Y$ is finite, and divisible by m (Theorem 3.2). Consequently, the space $X(C, 1)$ is completely prime (Corollary 3.3).

In § 4 we show that, for certain $Z \hookrightarrow X$, for any $T \in \text{CB}(Z, X)$ there exists $a \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $T - \tilde{T}_a$ is Hilbert–Schmidt. In particular, this happens when $Z = X$, thus completing the proof of Theorem 1.1 (i). Moreover, any infinite-dimensional $Y \hookrightarrow X$ contains an infinite-dimensional Z with the above property (Theorem 4.1).

Sections 5 and 6 are devoted to proving that infinite-dimensional subspaces of X fail the OAP, and to computing the K -groups of $\text{CB}(X)$ (parts (iii) and (iv) of Theorem 1.1, respectively). In § 7 we establish some additional properties of the space $X = X(C, m)$, and its subspaces. We show that, for general $Y \hookrightarrow X$, compact operators in $\text{CB}(Y, X)$

may fail to be Hilbert–Schmidt, or have certain non-commutative properties analogous to compactness (compare with Theorem 1.1 (i), stating that any compact $T \in \text{CB}(X)$ is Hilbert–Schmidt). Finally, in §8 we present a version of our construction, involving representations of weighted convolution algebras.

2. The construction of $X(C, m)$

In this section, we construct the space $X = X(C, m)$ as in Theorem 1.1. Recall that, by [21], there exists a family $(E_i)_{i=1}^\infty$ of finite-dimensional operator spaces such that

- (i) E_i is isometric to $\ell_2^{n_i}$ for some $n_i \in \mathbb{N}$, and $\{i \mid n_i = j\}$ is infinite for any $j \in \mathbb{N}$,
- (ii) for any operator $u : E_i^* \rightarrow E_j$, we have $\|u\|_1 / (4 + 2^{-i}) \leq \|u\|_{\text{cb}} \leq \|u\|_1$ if $i = j$,
 $\|u\|_{\text{cb}} = \|u\|_2$ if $i \neq j$.

Denote by \mathcal{K} the space of compact operators on ℓ_2 . Find a sequence of operators $u_i : \ell_2 \rightarrow \ell_2^{n_i}$ such that $\|u_i\|_2 = 1$ and, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and for any $u : \ell_2 \rightarrow \ell_2^n$ satisfying $\|u\|_2 = 1$, there exists $i \in \mathbb{N}$, for which $n_i = n$ and $\|u_i - u\|_1 < \varepsilon$. On the Banach space level, we identify the range of u_i with E_i described above.

We define the operator space $X = X(C, m)$ as follows: for $x \in \ell_2 \otimes \mathcal{K}$, let

$$\mathcal{U} = \left\{ C^{-(p+q)} S^{mp} S^{*mq} \mid p, q \in \mathbb{N} \cup \{0\} \right\}, \quad (2.1)$$

$$\|x\|_{X \otimes \mathcal{K}} = \sup \{ \|(u_i U \otimes I_{\mathcal{K}})x\|_{E_i \otimes \mathcal{K}} \mid i \in \mathbb{N}, U \in \mathcal{U} \}.$$

Clearly, X is an operator space (Ruan’s axioms are satisfied), X is isometric to ℓ_2 as a Banach space and $\|S^m\|_{\text{cb}}, \|S^{*m}\|_{\text{cb}} \leq C$ ($S^*S = I$, and hence a product of any number of copies of S and S^* in an arbitrary order can be written as $S^p S^{*q}$, where p and q are non-negative integers). Moreover, 2-summing operators with the range in X (and with arbitrary domain) are completely bounded (see the following lemma).

Lemma 2.1. *If Y is an operator space, and $T : Y \rightarrow X$ is a 2-summing operator, then $\|T\|_{\text{cb}} \leq \pi_2(T)$. If Y is isometric to a Hilbert space, and $T : Y \rightarrow X$ is a Hilbert–Schmidt operator, then $\|T\|_{\text{cb}} \leq \|T\|_2$.*

Proof. By (2.1),

$$\|T\|_{\text{cb}} = \sup \{ \|u_i U T\|_{\text{cb}} \mid i \in \mathbb{N}, U \in \mathcal{U} \} \leq \sup \{ \|u_i U T\|_1 \mid i \in \mathbb{N}, U \in \mathcal{U} \}.$$

By [9, Statement 4.5], $\|u_i U T\|_1 \leq \|u_i\|_2 \|U\| \pi_2(T) \leq \pi_2(T)$ for such i and U . Thus, $\|T\|_{\text{cb}} \leq \pi_2(T)$. If Y is isometric to a Hilbert space, then $\pi_2(T) = \|T\|_2$ (see, for example, [9, Statement 3.9]) and we are done. \square

Thus, Theorem 1.1 (i) (a) has been established. To deal with the rest of the theorem, we need to obtain lower estimates for c.b. norms of operators on X . The following lemma is one of our main tools.

Lemma 2.2. *Suppose that Y is a subspace of X . Consider the operators $T : Y \rightarrow X$, $u : X \rightarrow \ell_2^n$ and $v : \ell_2^n \rightarrow Y$, such that $\|u\|_2 = 1 = \|v\|$. Let*

$$c = \sup\{C^{-(p+q)}\|uS^{mp}S^{*mq}v\|_1 \mid p, q \in \mathbb{N} \cup \{0\}\}.$$

Then $\|T\|_{cb} \geq \|uTv\|_1 / (4 \max\{c, 1\})$.

Proof. Fix $\varepsilon > 0$. By construction, there exist infinitely many $i \in \mathbb{N}$ such that $i > \varepsilon^{-1}$, $n_i = n$ and $\|u_i - u\|_1 < \varepsilon$ (we identify ℓ_2^n with E_i). We view u and v as maps from X to E_i and from E_i^* to X , respectively. By (2.1),

$$\|v\|_{cb} = \sup\{C^{-(p+q)}\|u_jS^{mp}S^{*mq}v\|_{cb} \mid j \in \mathbb{N}, p, q \in \mathbb{N} \cup \{0\}\}.$$

If $i = j$, then

$$C^{-(p+q)}\|u_jS^{mp}S^{*mq}v\|_{cb} \leq C^{-(p+q)}(\|uS^{mp}S^{*mq}v\|_1 + \|u - u_i\|_1) \leq c + \varepsilon$$

for any $p, q \in \mathbb{N} \cup \{0\}$. If $j \neq i$,

$$C^{-(p+q)}\|u_jS^{mp}S^{*mq}v\|_{cb} \leq \|u_jS^{mp}S^{*mq}v\|_2 \leq \|u_j\|_2\|S^{mp}S^{*mq}\| \|v\| = 1.$$

Therefore, $\|v\|_{cb} \leq \max\{c + \varepsilon, 1\}$.

By (2.1), $\|u_i\|_{cb} = 1$, and therefore $\|u\|_{cb} \leq \|u_i\|_{cb} + \|u - u_i\|_1 = 1 + \varepsilon$. By the definition of E_i , $\|w\|_{cb} \geq \|w\|_1 / (4 + \varepsilon)$ for any $w \in \text{CB}(E_i^*, E_i)$, and therefore

$$\|T\|_{cb} \geq \frac{\|uTv\|_{cb}}{\|u\|_{cb}\|v\|_{cb}} \geq \frac{\|uTv\|_1}{(4 + \varepsilon)(1 + \varepsilon) \max\{c + \varepsilon, 1\}}.$$

However, ε can be chosen to be arbitrarily small. □

3. Extracting the Toeplitz part

The main goal of this section is to prove part (ii) of Theorem 1.1, namely, that every element of $\text{CB}(Y, X)$ (Y is a subspace of X) is a compact perturbation of the restriction of a Toeplitz operator to Y . We also establish the following theorem.

Theorem 3.1. *Suppose that Y is a coordinate subspace of X (that is, $Y = \text{span}[\delta_i \mid i \in \mathcal{I}]$ for some $\mathcal{I} \subset \mathbb{N}$), and $P \in \text{CB}(Y)$ is a projection. Then P has finite-dimensional kernel or finite-dimensional range.*

Theorem 3.2. *Suppose that X' is a subspace of X of finite codimension and Y is a subspace of X . Then X' is completely isomorphic to Y if and only if $\dim X/Y$ is finite, and $\dim X/X' - \dim X/Y$ is divisible by m .*

Recall that an operator space Z is called *completely prime* if, for any projection $P \in \text{CB}(Z)$, either $\text{ran } P$ or $\text{ker } P$ is completely isomorphic to Z .

Corollary 3.3. *The space $X(C, 1)$ is completely prime for any $C > 1$.*

Proof. By Theorem 3.1, any projection $P \in \text{CB}(X(C, 1))$ with infinite rank has finite-dimensional kernel. By Theorem 3.2, $\text{ran } P$ is completely isomorphic to $X(C, 1)$. \square

For the proofs, we need certain special orthonormal bases (ξ_i) , modelled on lacunary sequences, in the sense that $\langle S^{mj_1} \xi_{i_1}, S^{mj_2} \xi_{i_2} \rangle = 0$ if $i_1 \neq i_2$, and $|j_1| + |j_2|$ is sufficiently small. More precisely, let $\alpha = \frac{1}{2} \min\{1, \ln C\}$. For $i \in \mathbb{N}$, let $K_i = \lfloor i/\alpha \rfloor$. A sequence $(\xi_i)_{i \in \mathbb{N}}$ in ℓ_2 is called *shift lacunary* if it is normalized, and

- (i) for each $i \in \mathbb{N}$, $\text{supp } \xi_i = \{r \in \mathbb{N} \mid \langle \xi_i, \delta_r \rangle \neq 0\} \subset [2m(K_i + 2), \infty)$,
- (ii) there exists a sequence of mutually orthogonal subspaces $(F_i)_{i \in \mathbb{N}}$ of ℓ_2 , such that $\dim F_i = 2K_i + 1$, and $S^{mj} \xi_i \in F_i$ whenever $|j| \leq K_i$.

Note that the above definition implies that $\langle \xi_i, \xi_k \rangle = 0$ if $i \neq k$. More properties of shift lacunary sequences are mentioned below, in (3.1).

Proposition 3.4. *Suppose that (Y_n) is a sequence of infinite-dimensional subspaces of ℓ^2 . There then exists a shift lacunary sequence $(\xi_n)_{n \in \mathbb{N}}$, so that $\xi_n \in Y_n$ for any n .*

Proof. Without loss of generality, we can assume that $\text{supp } \xi \subset [2m(K_i + 2), \infty)$ whenever $\xi \in Y_i$, $i \in \mathbb{N}$.

We shall select a shift lacunary sequence (ξ_i) inductively. Pick an arbitrary norm-1 $\xi_1 \in Y_1$, and an arbitrary subspace F_1 of ℓ^2 , containing $S^{mj} \xi_1$ for $|j| \leq K_1$. Now suppose that ξ_1, \dots, ξ_{n-1} and F_1, \dots, F_{n-1} , satisfying the definition above, have been chosen, and pick ξ_n and F_n . Pick an arbitrary norm-1 vector

$$\xi_n \in Y_n \cap \left(\bigcup_{\substack{|j| \leq K_n, \\ 1 \leq k \leq n-1}} S^{mj} F_k \right)^\perp.$$

Then $\text{supp } \xi_n \subset [2m(K_n + 2), \infty)$, and $\langle S^{mj} \xi_n, \eta \rangle = \langle \xi_n, S^{-mj} \eta \rangle = 0$ whenever $|j| \leq K_n$, and $\eta \in F_k$ for some $k \in \{1, \dots, n - 1\}$. To complete the construction, let F_n be an arbitrary subspace of ℓ_2 of dimension $2K_n + 1$, containing $S^{mj} \xi_n$ whenever $|j| \leq K_n$, and orthogonal to the spaces F_k for $1 \leq k \leq n - 1$. \square

We use shift lacunarity sequences to ‘localize’ the actions of Toeplitz operators \tilde{T}_a . Denote by P_i the orthogonal projection onto F_i . Observe that

$$\left. \begin{aligned} S^{mp} S^{*mq} \xi_i &= S^{m(p-q)} \xi_i && \text{if } p + q \leq 2(K_i + 1), \\ (I - P_i) S^{mp} S^{*mq} \xi_i &= 0 && \text{if } p + q \leq K_i, \\ \langle S^{mp} S^{*mq} \xi_i, \xi_k \rangle &= 0 && \text{if } p + q \leq \max\{K_i, K_k\} \end{aligned} \right\} \quad (3.1)$$

(here p and q are non-negative integers).

We begin by showing that ‘badly off-diagonal’ parts of operators can be neglected.

Lemma 3.5. *Suppose that (ξ_i) is a shift lacunary normalized sequence in $X(C, m) = X$, $Z = \text{span}\{\xi_i \mid i \in \mathbb{N}\}$ and $T \in \text{CB}(Z, X)$. Then, in the above notation, $W = T - \sum_{i=1}^\infty P_i T P_i|_Z$ is a Hilbert–Schmidt operator, satisfying $\|W\|_2 \leq 16\|T\|_{\text{cb}}$.*

Proof. For $n \in \mathbb{N}$ let $Q_n = \sum_{i=1}^n P_i$. By Lemma 2.1, it suffices to show that

$$\|Q_n W Q_n|_Z\|_2 \leq 16 \|T\|_{\text{cb}} \quad \text{for any } n.$$

A simple argument shows that

$$Q_n W Q_n = 4 \cdot 2^{-n} \sum_{\mathcal{F} \subset \{1,2,\dots,n\}} (Q_n - P_{\mathcal{F}}) W P_{\mathcal{F}} = 4 \cdot 2^{-n} \sum_{\mathcal{F} \subset \{1,2,\dots,n\}} (Q_n - P_{\mathcal{F}}) T P_{\mathcal{F}},$$

where $P_{\mathcal{F}} = \sum_{i \in \mathcal{F}} P_i$. Thus, it remains to prove that, for any \mathcal{F} , $c \leq 4 \|T\|_{\text{cb}}$, where $c = \|(Q_n - P_{\mathcal{F}}) T P_{\mathcal{F}}|_Z\|_2$. Define an isometry $v : \ell_2(\mathcal{F}) \rightarrow Z$ by setting $ve_i = \xi_i$ (the vectors $(e_i)_{i \in \mathcal{F}}$ form the canonical basis of $\ell_2(\mathcal{F})$). Find an operator $u : X \rightarrow \ell_2(\mathcal{F})$ such that $\|u\|_2 = 1$, $u(Q_n - P_{\mathcal{F}}) = u$ and $\|u(Q_n - P_{\mathcal{F}}) T P_{\mathcal{F}} v\|_1 = c$.

To estimate $C^{-(p+q)} \|u S^{pm} S^{*qm} v\|_1$, observe that

$$\|u S^{pm} S^{*qm} v\|_1 \leq \sum_{i \in \mathcal{F}} \|u S^{pm} S^{*qm} ve_i\| = \sum_{i \in \mathcal{F}} \|u(Q_n - P_{\mathcal{F}}) S^{pm} S^{*qm} \xi_i\|.$$

However, by (3.1), $(Q_n - P_{\mathcal{F}}) S^{mp} S^{*mq} \xi_i = 0$ if $p+q \leq i/\alpha$. For other values of j , we use the trivial estimate $\|u(Q_n - P_{\mathcal{F}}) S^{mp} S^{*mq} \xi_i\| \leq 1$. Therefore,

$$\sum_{i \in \mathcal{F}} \|u(Q_n - P_{\mathcal{F}}) S^{mp} S^{*mq} \xi_i\| \leq |\{i \in \mathcal{F} \mid i < \alpha(p+q)\}| \leq 1 + \alpha(p+q) \leq e^{\alpha(p+q)}.$$

By our choice of α , $C \leq e^{2\alpha}$; hence, $C^{-(p+q)} \|u S^{mp} S^{*mq} v\|_1 \leq 1$ for all values of p and q . By Lemma 2.2, $\|T\|_{\text{cb}} \geq \frac{1}{4} c$. □

Note that any c.b. map on the space Z is a Hilbert–Schmidt perturbation of a scalar (see [21]).

Corollary 3.6. *Suppose that Z is as in Lemma 3.5 and $T \in \text{CB}(Z)$. There then exists a Hilbert–Schmidt operator $W \in B(Z)$, such that $\|W\|_2 \leq 16 \|T\|_{\text{cb}}$, and $\Lambda = T - W$ is a diagonal operator (that is, $\Lambda \xi_i = \lambda_i \xi_i$ for any i , with $\lambda_i \in \mathbb{C}$).*

Proof. Note that $T \xi_j = \sum_{i=1}^{\infty} \langle T \xi_j, \xi_i \rangle \xi_i$ for any j . By definition, $P_i \xi_j = \delta_{ij} \xi_j$ (δ_{ij} is Kronecker’s delta). Therefore, $\sum_{i=1}^{\infty} P_i T P_i \xi_j = \lambda_j \xi_j$, where $\lambda_j = \langle T \xi_j, \xi_j \rangle$. Let Λ be the diagonal operator on Z corresponding to $(\lambda_j)_{j \in \mathbb{N}}$, and note that $\Lambda = \sum_{i=1}^{\infty} P_i T P_i|_Z$. An application of Lemma 3.5 completes the proof. □

Next we show that, for any $n \geq 0$, every $T \in \text{CB}(Z, X)$ (with Z as above) can be approximated ‘reasonably well’ by linear combinations of S^{mj} , with $|j| \leq n$.

Lemma 3.7. *Suppose a normalized sequence $(\xi_i)_{i \in \mathbb{N}}$ is shift lacunary, and let $Z = \text{span}\{\xi_i \mid i \in \mathbb{N}\}$. For $N \in \mathbb{N}$ let $Z_N = \text{span}\{\xi_i \mid i > N\}$. Then for every $T \in \text{CB}(Z, X)$ and every $n \geq 0$ there exist complex numbers $(a_j)_{|j| \leq n}$, such that*

$$\sum_j |a_j| C^{|j|} \leq 4 \inf_N \|T|_{Z_N}\|_{\text{cb}}$$

and

$$\inf_N \left\| \left(T - \sum_j a_j S^{mj} \right) \Big|_{Z_N} \right\| \leq 4C^{-(n+1)} \inf_N \|T|_{Z_N}\|_{cb}.$$

Proof. By Lemma 3.5, $\sum_i P_i T P_i|_Z$ is a c.b. map, and $T - \sum_i P_i T P_i|_Z$ is Hilbert–Schmidt. By Lemma 2.1,

$$\inf_N \left\| \left(T - \sum_i P_i T P_i \right) \Big|_{Z_N} \right\|_{cb} \leq \inf_N \left\| \left(T - \sum_i P_i T P_i \right) \Big|_{Z_N} \right\|_2 = 0.$$

Therefore,

$$\inf_N \|T|_{Z_N}\|_{cb} = \inf_N \left\| \left(\sum_i P_i T P_i \right) \Big|_{Z_N} \right\|_{cb}.$$

Thus, it suffices to show that, for any $T = \sum_i P_i T P_i|_Z \in CB(Z, X)$ such that $\|T|_{Z_K}\|_{cb} < \frac{1}{4}$ for some K , we have $\inf_{N>n} c_{n,N} \leq C^{-(n+1)}$, where

$$c_{n,N} = \inf \left\{ \left\| \left(\sum_{j=-n}^n a_j S^{mj} - T \right) \Big|_{Z_N} \right\| \mid \sum_{j=-n}^n |a_j| C^{|j|} \leq 1 \right\}.$$

Denote by \mathcal{P}_N the set of operators $W : Z_N \rightarrow X$ such that $W\xi_i \in F_i$ for any $i > N$ (we equip \mathcal{P}_N with the norm inherited from $B(Z_N, X)$). By our assumption, $T|_{Z_N} \in \mathcal{P}_N$ for any N . Moreover, the definition of shift lacunarity implies that $S^{mj}|_{Z_N} \in \mathcal{P}_N$ if $|j| \leq N$. Therefore,

$$\left(\sum_{j=-n}^n a_j S^{mj} - T \right) \Big|_{Z_N} \in \mathcal{P}_N \quad \text{for } N \geq n.$$

It is easy to check that \mathcal{P}_N is a Banach space, which is isometric to the ℓ_∞ direct sum $(\bigoplus_{i>N} F_i)_\infty$. Indeed, the ranges of the projections P_i are mutually orthogonal; hence, the map $\text{Id} : \mathcal{P}_N \rightarrow (\bigoplus_i F_i)_\infty : W \mapsto (W\xi_i)_{i>N}$ is an isometry.

Suppose, for the sake of contradiction, that $\inf_N c_{N,n} > C^{-(n+1)}$. There then exists $c > C^{-(n+1)}$ such that $T|_{Z_N} \notin Y_{n,N}$ for any $N > \max\{n, K\}$. Here, $B_{\mathcal{P}_N}$ is the closed unit ball of \mathcal{P}_N , ‘conv’ denotes the closed convex hull of a set, and

$$Y_{n,N} = cC^{n+1} \text{conv} \left(C^{-(n+1)} B_{\mathcal{P}_N} \cup \left\{ \sum_{j=-n}^n a_j S^{mj} \Big|_{Z_N} \mid \sum_{j=-n}^n |a_j| C^{|j|} \leq 1 \right\} \right).$$

By the Hahn–Banach theorem, there exists $f_0 \in \mathcal{P}_N^*$ such that $|f_0(T|_{Z_N})| > c$, and $|f_0(W)| < c$ for any $W \in Y_{n,N}$. Consequently, $\|f_0\| < 1$, and $|f_0(S^{mj}|_{Z_N})| < C^{|j|-n-1}$ if $|j| \leq n$. By Helly’s theorem (or by the local reflexivity principle), there exists $f \in (\mathcal{P}_N)_*$ such that $\|f\| < 1$, and

$$f(S^{mj}|_{Z_N}) = f_0(S^{mj}|_{Z_N}) \quad \text{if } |j| \leq n$$

(consequently, $|f(S^{mj}|_{Z_N})| < C^{|j|-n-1}$ for such j). Here, we identify $(\mathcal{P}_N)_*$ (a predual of \mathcal{P}_N) with $(\bigoplus_{i>N} F_i)_1$. Thus, $f = (\eta_i)_{i>N}$, with $\eta_i \in F_i$, and $\sum_{i>N} \|\eta_i\| = \|f\| < 1$. The action of f on \mathcal{P}_N is defined by setting

$$f(W) = \sum_{i>N} \langle W\xi_i, \eta_i \rangle \quad \text{for } W \in \mathcal{P}_N.$$

Find a sequence $\max\{n, K\} < N_0 < N_1 < N_2 < \dots$ of positive integers and a sequence $(f_k)_{k=1}^\infty$ such that, for every k ,

$$f_k = (b_i^2 \eta_i)_{i=N_{k-1}}^{N_k-1}, \quad \text{where } \eta_i \in F_i, \|\eta_i\| = 1, b_i \geq 0, \sum_{i=N_{k-1}}^{N_k-1} b_i^2 < 1,$$

$$\left| \sum_{i=N_{k-1}}^{N_k-1} b_i^2 \langle T\xi_i, \eta_i \rangle \right| > c \quad \text{and} \quad \left| \sum_{i=N_{k-1}}^{N_k-1} b_i^2 \langle S^{mj} \xi_i, \eta_i \rangle \right| < C^{|j|-n-1} \quad \text{when } |j| \leq n.$$

Select $M > C^{2(n+1)}$, and define the operators $v : \ell_2^M \rightarrow Z_{N_0}$ and $u : X \rightarrow \ell_2^M$ by setting

$$ve_k = \sum_{i=N_{k-1}}^{N_k-1} b_i \xi_i \quad \text{and} \quad u^* e_k = \frac{1}{\sqrt{M}} \sum_{i=N_{k-1}}^{N_k-1} b_i \eta_i$$

(as usual, $(e_i)_{i=1}^M$ is the canonical basis from ℓ_2^M). Then $\|u\|_2 \leq 1$ and $\|v\| \leq 1$. Moreover,

$$|\langle uT|_{Z_{N_0}} ve_k, e_k \rangle| = \frac{1}{\sqrt{M}} \left| \sum_{i=N_{k-1}}^{N_k-1} b_i^2 \langle T\xi_i, \eta_i \rangle \right| > \frac{c}{\sqrt{M}}$$

for $1 \leq k \leq M$; hence, $\|uTv\|_1 \geq c\sqrt{M}$. It remains to estimate $C^{-(p+q)} \|uS^{mp}S^{*mq}v\|_1$ ($p, q \geq 0$) from above. For $p + q > n$, we have

$$C^{-(p+q)} \|uS^{mp}S^{*mq}v\|_1 \leq C^{-(p+q)} \sqrt{M} \leq C^{-(n+1)} \sqrt{M}.$$

If $p + q \leq n$, observe that, by (3.1),

$$\sqrt{M} |\langle uS^{mp}S^{*mq}ve_k, e_k \rangle| = \left| \sum_{i=N_{k-1}}^{N_k-1} b_i^2 \langle S^{m(p-q)} \xi_i, \eta_i \rangle \right| < C^{|p-q|-n-1} \leq C^{(p+q)-n-1}$$

for $1 \leq k \leq M$. Also by (3.1), $\langle uS^{mp}S^{*mq}ve_k, e_\ell \rangle = 0$ if $k \neq \ell$. Therefore,

$$\|uS^{mj}v\|_1 \leq \sum_{k,\ell=1}^M |\langle uS^{mj}ve_k, e_\ell \rangle| \leq C^{-(n+1)} \sqrt{M}.$$

By Lemma 2.2,

$$\|T|_{Z_{N_0}}\|_{cb} > \frac{c\sqrt{M}}{4 \max\{C^{-(n+1)}\sqrt{M}, 1\}} > \frac{1}{4},$$

which contradicts the assumption that $\|T|_{Z_{N_0}}\|_{cb} < \frac{1}{4}$. □

Lemma 3.8. *Suppose that Z and T are as in Lemma 3.7. There then exists a sequence $a \in \ell_1(C^{|j|}, \mathbb{Z})$ for which $\|a\|_w \leq 4\|T\|_{cb}$, and $T - \tilde{T}_a|_Z$ is compact.*

Proof. Assume that $\|T\|_{cb} < \frac{1}{4}$. By Lemma 3.7, for every $n \in \mathbb{N}$ there exist complex numbers $(a_j^{(n)})_{j=-n}^n$ and $N_n \in \mathbb{N}$ such that

$$\sum_{j=-n}^n |a_j^{(n)}|C^{|j|} < 1 \quad \text{and} \quad \left\| \left(\sum_{j=-n}^n a_j^{(n)} S^{mj} - T \right) \Big|_{Z_{N_n}} \right\| < C^{-(n+1)}$$

(we keep the notation of Lemma 3.7). Without loss of generality, we can assume that $N_1 < N_2 < \dots$.

For $|j| > n$, let $a_j^{(n)} = 0$, and regard $a^{(n)} = (a_j^{(n)})_{j \in \mathbb{Z}}$ as an element of $\ell_1(C^{|j|}, \mathbb{Z})$, with $\|a^{(n)}\|_w < 1$. By passing to a subsequence if necessary, we can assume that the sequence $(a^{(n)})$ converges weak* to $a = (a_j)_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z})$ (in other words, $\lim_n a_j^{(n)} = a_j$ for any $j \in \mathbb{Z}$). Clearly, $\|a\|_w \leq 1$. Moreover, $\lim_n \|\tilde{T}_a - \tilde{T}_{a^{(n)}}\| = 0$. Indeed, fix $\varepsilon > 0$, and find $K \in \mathbb{N}$ such that $C^{-(K+1)} < \frac{1}{3}\varepsilon$. Find $M \in \mathbb{N}$ such that $|a_j^{(n)} - a_j| < \varepsilon/9K$ for $|j| \leq K$ and $n \geq M$. For such n ,

$$\begin{aligned} \|\tilde{T}_a - \tilde{T}_{a^{(n)}}\| &= \left\| \sum_{j \in \mathbb{Z}} (a_j - a_j^{(n)}) S^{mj} \right\| \\ &\leq \sum_{|j| \leq K} |a_j - a_j^{(n)}| + \sum_{|j| > K} (|a_j| + |a_j^{(n)}|) \\ &< (2K + 1) \frac{\varepsilon}{9K} + \frac{2\varepsilon}{3} \\ &< \varepsilon. \end{aligned}$$

Therefore,

$$\|(\tilde{T}_a - T)|_{Z_{N_\ell}}\| \leq \|\tilde{T}_a - \tilde{T}_{a^{(\ell)}}\| + \|(\tilde{T}_{a^{(\ell)}} - T)|_{Z_{N_\ell}}\| < \varepsilon + C^{-\ell}$$

for $\ell > \max\{K, M\}$. Thus, $\lim_\ell \|(\tilde{T}_a - T)|_{Z_{N_\ell}}\| = 0$, and hence $T - \tilde{T}_a|_Z$ is compact. \square

Proof of Theorem 1.1 (ii). Suppose that Y is an infinite-dimensional subspace of X and that $T \in CB(Y, X)$ satisfies $\|T\|_{cb} < \frac{1}{4}$. First show that for every $c > 0$ there exists $a \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $\|a\|_w \leq 1$, and $\lim_N \|(\tilde{T}_a - T)|_{Y_N}\| \leq c$ (here $Y_N = Y \cap \text{span}[\delta_i \mid i \geq N]$). Indeed, suppose this is false for some $c \in (0, 1)$. Find a sequence $(a^{(n)})$ which is dense in the unit ball of $\ell_1(C^{|j|}, \mathbb{Z})$ and let $\tilde{T}^{(n)} = \tilde{T}_{a^{(n)}}$. For each n , Y contains an infinite-dimensional subspace Y_n such that $\|(\tilde{T}^{(n)} - T)\xi\| > \frac{1}{2}c$ for any $\xi \in Y_n$. By Proposition 3.4, there exists a shift lacunary sequence $(\xi_i)_{i \in \mathbb{N}}$ such that $\xi_i \in Y_n$ whenever $i \in \{2^{n-1}, 3 \cdot 2^{n-1}, 5 \cdot 2^{n-1}, \dots\}$. Therefore, $\limsup_i \|(\tilde{T}_a - T)\xi_i\| \geq \frac{1}{2}c$ whenever a belongs to the unit ball of $\ell_1(C^{|j|}, \mathbb{Z})$, which contradicts Lemma 3.8.

Thus, there exist a sequence of positive integers $N_1 < N_2 < \dots$ and a sequence $(a^{(n)})$ of elements of the unit ball of $\ell_1(C^{|j|}, \mathbb{Z})$, such that $\|(\tilde{T}^{(n)} - T)|_{Y_{N_n}}\| < 2^{-n}$ for any $n \in \mathbb{N}$ (here, as before, $\tilde{T}^{(n)} = \tilde{T}_{a^{(n)}}$). Passing to a subsequence if necessary (see the proof of

Lemma 3.8), we can assume the existence of $a \in \ell_1(C^{|j|}, \mathbb{Z})$, such that $\|a\|_w \leq 1$, and $\lim_n \|\tilde{T}_a - \tilde{T}^{(n)}\| < 2^{-n}$. Therefore,

$$\|(\tilde{T}_a - T)|_{Y_{N_n}}\| \leq \|(\tilde{T}^{(n)} - T)|_{Y_{N_n}}\| + \|\tilde{T}_a - \tilde{T}^{(n)}\| < 2^{1-n}$$

for any $n \in \mathbb{N}$. Thus, $\tilde{T}_a|_Y - T$ is compact. □

Remark 3.9. If Y is a coordinate subspace of X , then a as above is unique. Indeed, if $Y = \text{span}\{\delta_{i_k} \mid k \in \mathbb{N}\}$, then $a_j = \lim_{k \rightarrow \infty} \langle T\delta_{i_k}, \delta_{i_k+mj} \rangle$ for any $j \in \mathbb{Z}$. For a generic $Y \hookrightarrow X$ and $T \in \text{CB}(Y, X)$, there may be more than one $a \in \ell_1(C^{|j|}, \mathbb{Z})$ for which $T - \tilde{T}_a|_Y$ is compact (see § 7.2 for more details).

To prove Theorems 3.1 and 3.2, we follow in the footsteps of [8], and associate complex-valued functions with members of $\text{CB}(Y, X)$. To this end, identify $\ell_1(C^{|j|}, \mathbb{Z})$ with the set \mathcal{Y} of those analytic functions f on the annulus $\mathbb{A} = \{z \in \mathbb{C} \mid C^{-1} \leq |z| \leq C\}$, whose Laurent coefficients belong to $\ell_1(C^{|j|}, \mathbb{Z})$. More precisely, for $f \in \mathcal{Y}$ we have $f(z) = \sum_j a_j z^j$, with $\|(a_j)\|_w < \infty$. Let $\|f\|_{\mathcal{Y}} = \|(a_j)\|_w$. Clearly, \mathcal{Y} (equipped with pointwise multiplication) is isometric, and algebraically isomorphic, to $\ell_1(C^{|j|}, \mathbb{Z})$ with its convolution.

Suppose that Y is an infinite-dimensional coordinate subspace of X and that $T \in \text{CB}(Y, X)$. By Theorem 1.1 (ii) and Remark 3.9, there exists a unique $a = (a_j)_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z})$ for which $T - \tilde{T}_a$ is compact. We construct a function $\phi(T) \in \mathcal{Y}$ by setting

$$\phi(T)(z) = \sum_j a_j z^j \quad \text{for } z \in \mathbb{A}.$$

Note that, if Y_1 and Y_2 are infinite-dimensional coordinate subspaces of X , $T_1 \in \text{CB}(Y_1, Y_2)$ and $T_2 \in \text{CB}(Y_2, X)$, then $\phi(T_2 T_1) = \phi(T_1)\phi(T_2)$.

Thus, we can pass from $\text{CB}(X)$ to the well-understood, commutative Banach algebra \mathcal{Y} (see [4, Chapter 4.6] for its properties). This allows us to investigate c.b. projections on, and complete isomorphisms of, the space X .

Proof of Theorem 3.1. Suppose that Y is an infinite-dimensional coordinate subspace of X , $T \in \text{CB}(Y)$ and $T^2 = T$. Let $f = \phi(T)$. By the reasoning above, $f^2 = f$. That is, $f(z) = 0$ or $f(z) = 1$ for any $z \in \mathbb{A}$. However, f must be continuous, and hence either $f \equiv 0$, or $f \equiv 1$. In the former case, T is compact, and hence $\text{ran } T$ is finite dimensional. In the second case, $I - T$ is compact, and hence $\text{ran}(I - T)$ is finite dimensional. Therefore, $\ker T$ is finite dimensional. □

Remark 3.10. As we show in § 7.1, the statement of Theorem 3.1 fails for non-coordinate subspaces of X .

The next two lemmas allow us to prove a statement more general than Theorem 3.2. These results will be used again to compute the K -groups of $\text{CB}(X)$ in § 6.

Below, we define the direct sum $Y \oplus Z$ (Y and Z are operator spaces), for $y \in M_n(Y)$ and $z \in M_n(Z)$, by setting

$$\|y \oplus z\|_{M_n(Y \oplus Z)} = \max\{\|y\|_{M_n(Y)}, \|z\|_{M_n(Z)}, \|y \oplus z\|_{M_n(\text{MIN}(Y \oplus_2 Z))}\},$$

where $\text{MIN}(E)$ denotes the Banach space E , equipped with its minimal operator space structure. In the following, Y^n stands for $Y \oplus Y \oplus \dots \oplus Y$ (n times). If $n = 0$, we identify Y^n with $\{0\}$. $B(Y^n)$ is henceforth identified with $M_n(B(Y))$ (the space of $n \times n$ matrices with entries in $B(Y)$) as a vector space. Similarly, we identify $\text{CB}(Y^n)$ with $M_n(\text{CB}(Y))$.

Lemma 3.11. *Suppose that Y is an infinite codimensional subspace of X^n , $n \geq 1$. Then no completely bounded map from X^n into Y has a bounded inverse.*

Proof. An operator $T \in \text{CB}(X^n)$ can be represented as an $n \times n$ matrix $(T^{(pq)})_{p,q=1}^n$, whose entries are c.b. maps on X . By Theorem 1.1 (ii),

$$T^{(pq)} = \tilde{T}_{a^{(pq)}} + W^{(pq)},$$

with $W^{(pq)}$ compact, and

$$a^{(pq)} = (a_j^{(pq)})_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z}).$$

Denote the operator $(\tilde{T}_{a^{(pq)}})_{p,q=1}^n$ by \tilde{T}_a , where $a = (a_j)_{j \in \mathbb{Z}}$, and $a_j = (a_j^{(pq)})_{p,q=1}^n$ is an $n \times n$ matrix for any j ; clearly,

$$\sum_{j \in \mathbb{Z}} C^{|j|} \|a_j\| < \infty.$$

Similarly, denote the operator $(W^{(pq)})_{p,q=1}^n$ by W .

Consider a function $f = (I_{M_n} \otimes \phi)(T) : \mathbb{A} \mapsto M_n$. That is, $f = (f^{(pq)})_{p,q=1}^n$, with $f^{(pq)} = \phi(T^{(pq)})$ or, in other words,

$$f(z) = \sum_j a_j z^{mj},$$

where $a_j = (a_j^{(pq)})_{p,q=1}^n$ is an $n \times n$ matrix. Suppose first that $\det f(z) = 0$ for some $z \in \mathbb{T} = \{\omega \in \mathbb{C} \mid |\omega| = 1\}$. Fix $\varepsilon > 0$, and show that there exists a unit vector $\xi \in X^n$ with finite support, such that $\|T(I_{\ell_2^n} \otimes S)^\ell \xi\| < \varepsilon$ for any $\ell \geq 0$ (here $I_{\ell_2^n} \otimes S \in B(X^n)$ is defined via $I_{\ell_2^n} \otimes S(\xi_1, \dots, \xi_n) = (S\xi_1, \dots, S\xi_n)$ for $\xi_1, \dots, \xi_n \in X$). This would imply that T does not have a bounded inverse.

Denote by M_f the operator of multiplication by f , acting on $\ell_2^n(L_2(\mathbb{T}))$. By the density of trigonometric polynomials, there exists

$$p(z) = \sum_{|k| \leq N} b_k z^k, \quad b_k \in \ell_2^n,$$

such that $\|p\|_{\ell_2^n(L_2(\mathbb{T}))} = 1$, and $\|M_f p\|_{\ell_2^n(L_2(\mathbb{T}))} < \frac{1}{2}\varepsilon$. Let $\eta = \sum_{|k| \leq N} b_k \otimes \delta_{k+N}$, and note that $\|\tilde{T}_a(I_{\ell_2^n} \otimes S)^\ell \eta\| < \frac{1}{2}\varepsilon$ for any $\ell \geq 0$. By the compactness of W , there exists $K \geq 0$ such that $\|W(I_{\ell_2^n} \otimes S)^\ell \eta\| < \frac{1}{2}\varepsilon$ for any $\ell \geq K$. Let $\xi = (I_{\ell_2^n} \otimes S)^K \eta$. Then $\|T(I_{\ell_2^n} \otimes S)^\ell \xi\| < \varepsilon$ for any $\ell \geq 0$.

If $\det f$ does not vanish on \mathbb{T} , then the operator \tilde{T}_a is Fredholm (see [5, Chapter 1] or [15]); hence, so is $T = \tilde{T}_a + W$. In particular, $\dim(X^n/T(X^n)) < \infty$. \square

Corollary 3.12. *Suppose that X_1 is a finite codimensional subspace of X^n , $n \geq 1$, and Y is an infinite codimensional subspace of X^n . Then no completely bounded map from X_1 into Y has a bounded inverse.*

Proof. Consider $T \in \text{CB}(X_1, Y)$. Find a finite-dimensional $E \hookrightarrow X^n$ such that $X^n = \text{span}[E, X_1]$ and $E \cap X_1 = \{0\}$. Find $F \hookrightarrow X^n$ such that $\dim E = \dim F$, and $F \cap Y = \{0\}$. There then exists an invertible operator $U : E \rightarrow F$. Let $T' = TP + U(I - P)$, where P is a projection for which $\ker P = E$, $\text{ran } P = X_1$. Clearly, $T' \in \text{CB}(X^n, \text{span}[Y, F])$. If T has a bounded inverse, then so has T' , which contradicts Lemma 3.11. \square

Lemma 3.13. *Suppose that $n \geq 1$, X_1 and Y are subspaces of X^n and that $\dim X^n/X_1 < \infty$. Then X_1 is completely isomorphic to Y if and only if $\dim X^n/Y$ is finite, and m divides $\dim X^n/X_1 - \dim X^n/Y$.*

Proof. By the construction of X (2.1), the operators S^m and S^{*m} are completely bounded. Hence, any coordinate subspace of X is completely isomorphic to its subspaces of codimension $m, 2m, 3m, \dots$. Since all subspaces of X^n of given (finite) codimension are completely isomorphic, we have shown that X_1 and Y are completely isomorphic if $\dim X^n/X_1 - \dim X^n/Y$ is divisible by m .

Suppose, on the other hand, that X_1 and Y are subspaces of X^n , that X_1 is completely isomorphic to Y and that $\dim X/X_1 < \infty$. By Corollary 3.12, $\dim X^n/Y < \infty$. Suppose, without loss of generality, that $\dim X^n/X_1 \leq \dim X^n/Y$, and $T : X_1 \rightarrow Y$ is a complete isomorphism. By adding a finite-rank operator if necessary (as in the proof of Corollary 3.12), we can view T as a complete isomorphism from X^n onto its subspace of finite codimension.

As in the proof of Lemma 3.11, consider a function $f = (I_{M_n} \otimes \phi)(T) : \mathbb{A} \mapsto M_n$, corresponding to $a \in M_n \otimes \ell_1(C^{|\mathbb{J}|}, \mathbb{Z})$. Then $W = T - \tilde{T}_a$ is compact. T is a Fredholm operator of index $K = \dim X/X_1 - \dim X/Y$, and hence so is \tilde{T}_a . By [5] (or [15]), $-K$ is equal to the winding number of the function $z \mapsto \det f(z^m)$ around 0, which must be divisible by m . \square

Proof of Theorem 3.2. Apply the previous lemma with $n = 1$. \square

4. When is $T - \tilde{T}_a$ Hilbert–Schmidt?

In the previous section, we showed that every c.b. map from Y to X ($Y \hookrightarrow X$) is a compact perturbation of a Toeplitz operator \tilde{T}_a , with $a \in \ell_1(C^{|\mathbb{J}|}, \mathbb{Z})$. Below we show that, for certain subspaces, this perturbation is actually Hilbert–Schmidt. In particular, we establish this for $Y = X$ (Theorem 1.1 (i) (b)). In addition, we prove the following theorem.

Theorem 4.1. *Any infinite-dimensional subspace Y of X has an infinite-dimensional subspace Z such that any $T \in \text{CB}(Z, X)$ can be written as $T = \tilde{T}_a|_Z + W$, with $a \in \ell_1(C^{|\mathbb{J}|}, \mathbb{Z})$, $W \in \mathcal{S}_2$, $\|W\|_2 \leq 20\sqrt{2}\|T\|_{\text{cb}}$, and $\|a\|_w \leq 4\|T\|_{\text{cb}}$. Moreover, such a Z can be selected so that any $T \in \text{CB}(Z)$ can be written as $T = \lambda I_Z + W$, with $\lambda \in \mathbb{C}$, $W \in \mathcal{S}_2$, $\|W\|_2 \leq 700\|T\|_{\text{cb}}$ and $|\lambda| \leq \|T\|$. In fact, any infinite-dimensional Y contains*

an uncountable family of subspaces Z , which have the property described above, and are not completely isomorphic to each other.

To do this, we need a definition.

Definition 4.2. A sequence $(\xi_i)_{i \in \mathbb{N}} \subset X$ is called *c-nice* (here $c \geq 1$) if it is orthonormal and, for any $M \in \mathbb{N}$, any $\varepsilon > 0$ and any finite set $(\eta_k)_{k=1}^L \in X$, there exists $N_0((\eta_k)_{k=1}^L, \varepsilon, M) \in \mathbb{N}$ with the property that, for every $N \geq N_0((\eta_k)_{k=1}^L, \varepsilon, M)$, there exists an orthogonal projection Q for which the following conditions apply.

- (i) $\|QS^{mp}S^{*mq}v\|_2 \leq cC^{p+q}$ for any $p, q \geq 0$. Here, $v : \ell_2^M \rightarrow \text{span}[\xi_i \mid i \in \mathbb{N}]$ is defined by setting $ve_i = (\xi_i + \xi_{i+N})/\sqrt{2}$ (as usual, $(e_i)_{i=1}^M$ is the canonical basis for ℓ_2^M).
- (ii) $\|Q\eta_k\| \geq (\|\eta_k\| - \varepsilon)/\sqrt{2}$ for $1 \leq k \leq L$.

A subspace Y of X is called *c-nice* if it has a *c-nice* orthonormal basis.

Ubiquity of *c-nice* sequences will be proved below (Lemmas 4.4 and 4.8). First we present a result underscoring their importance for this section.

Lemma 4.3. *Suppose that Y is a c-nice subspace of X and that $T \in \text{CB}(Y, X)$ is compact. Then T is Hilbert–Schmidt, with $\|T\|_2 \leq 4\sqrt{2}c\|T\|_{\text{cb}}$.*

Proof. It suffices to show that, for every $\varepsilon > 0$ and $M \in \mathbb{N}$,

$$\|T\|_{\text{cb}} \geq \frac{1}{4c} \left(\sum_{i=1}^M \frac{(\|T\xi_i\| - \varepsilon)^2}{2} \right)^{1/2}, \tag{4.1}$$

where (ξ_i) is a *c-nice* orthonormal basis in Y .

Taking $N > N_0((T\xi_i)_{i=1}^M, \varepsilon, M)$ to be sufficiently large, and perturbing T slightly, we can assume that $T\xi_{i+N} = 0$ for $1 \leq i \leq M$. Let v be as in the definition of niceness, and find $u : X \rightarrow \ell_2^M$ such that $u = uQ$, $\|u\|_2 = 1$ and

$$\|uQTv\|_1 = \|QTv\|_2 = \left(\sum_{i=1}^M \|QTve_i\|^2 \right)^{1/2} \geq \left(\sum_{i=1}^M \frac{(\|T\xi_i\| - \varepsilon)^2}{2} \right)^{1/2}.$$

By the definition of niceness again,

$$\|uS^{mp}S^{*mq}v\|_1 = \|uQS^{mp}S^{*mq}v\|_1 \leq \|QS^{mp}S^{*mq}v\|_2 \leq cC^{p+q}.$$

An application of Lemma 2.2 establishes (4.1). □

Next we give examples of *c-nice* sequences. We say that a sequence $(\xi_i)_{i \in \mathbb{N}}$ is *regular* if it is normalized, shift lacunary, and there exists a sequence $(\alpha_j)_{j \geq 0}$ of complex numbers, such that

$$|\langle S^{mj}\xi_i, \xi_i \rangle - \alpha_j| < \frac{\alpha^{2i}}{2^{2i+4}(1 + i/\alpha)^4}$$

if $1 \leq j \leq 2(1 + i/\alpha)$ (here, as before, $\alpha = \min\{1, \ln C\}/2$). Note that, for $-1 \geq j \geq -2(1 + i/\alpha)$,

$$\langle S^{mj}\xi_i, \xi_i \rangle = \overline{\langle \xi_i, S^{mj}\xi_i \rangle} = \overline{\langle S^{-mj}\xi_i, \xi_i \rangle}.$$

Therefore, defining $\alpha_0 = 1$ and $\alpha_j = \bar{\alpha}_{-j}$ for $j < 0$, we conclude that, for a regular sequence, we have

$$|\langle S^{mj}\xi_i, \xi_i \rangle - \alpha_j| < \frac{\alpha^{2i}}{2^{2i+4}(1 + i/\alpha)^4} \quad \text{whenever } |j| \leq 2(1 + i/\alpha). \tag{4.2}$$

Lemma 4.4. *Every infinite-dimensional subspace of ℓ_2 contains a regular sequence.*

Proof. Suppose that Y is an infinite-dimensional subspace of X . By Proposition 3.4, it contains a shift lacunary sequence $(\xi_i)_{i \in \mathbb{N}}$. By a diagonalization argument (and passing to a subsequence if necessary), we can assume that $\alpha_j = \lim_k \langle S^{mj}\xi_i, \xi_i \rangle$ exists for each $j > 0$. Passing to a further subsequence, we can assure ourselves that the appropriate bound for $|\langle S^{mj}\xi_i, \xi_i \rangle - \alpha_j|$ holds if $1 \leq j \leq 2(1 + i/\alpha)$. \square

Lemma 4.5. *Any regular sequence is 1-nice.*

Proof. Suppose, for $1 \leq k \leq L$, that $\eta_k \in X$, $\|\eta_k\| = 1$ and $\text{supp } \eta_k \subset [1, N_0]$. We shall show that, for any $N > \max\{N_0, M\}$, there exists an orthogonal projection Q such that $\|Q\eta_k\| \geq \|\eta_k\|/\sqrt{2}$, and $\|QS^{mp}S^{mq}v\|_2 \leq C^{p+q}$ for any $p, q \geq 0$ (as before, $(e_i)_{i=1}^M$ is the canonical basis for ℓ_2^M , and $ve_i = (\xi_i + \xi_{i+N})/\sqrt{2}$).

For $1 \leq i \leq M$ define the operators $w_i : \ell_2^{2K_i+1} \rightarrow F_i$ and $w'_i : \ell_2^{2K_i+1} \rightarrow F_{i+N}$ by setting $w_i f_j = S^{mj}\xi_i$ and $w'_i f_j = S^{mj}\xi_{i+N}$ (here $K_i = \lfloor i/\alpha \rfloor$, $(f_j)_{j=-K_i}^{K_i}$ is an orthonormal basis in $\ell_2^{2K_i+1}$, and F_1, F_2, \dots are as in §3). Consider the polar decompositions $w_i = U_i A_i$ and $w'_i = U'_i A'_i$, where U_i and U'_i are isometries from $\ell_2^{2K_i+1}$ into F_i and F_{i+N} , respectively, and $A_i = (w_i^* w_i)^{1/2}$, $A'_i = (w_i'^* w_i')^{1/2}$ act on $\ell_2^{2K_i+1}$.

Now observe that, for $-K_i \leq j, k \leq K_i$,

$$\langle w_i^* w_i f_k, f_j \rangle = \langle w_i f_k, w_i f_j \rangle = \langle S^{mk}\xi_i, S^{mj}\xi_i \rangle,$$

and therefore, $w_i^* w_i = \sum_{j,k=1}^n \langle S^{m(k-j)}\xi_i, \xi_i \rangle E_{jk}$ (here E_{jk} is the matrix with 1 on the intersection of the j th row and k th column, and zeros elsewhere). A similar expansion can be written for $w_i'^* w_i'$. By (4.2),

$$\begin{aligned} & |\langle S^{m(k-j)}\xi_i, \xi_i \rangle - \langle S^{m(k-j)}\xi_{i+N}, \xi_{i+N} \rangle| \\ & \leq |\langle S^{m(k-j)}\xi_i, \xi_i \rangle - \alpha_{k-j}| + |\langle S^{m(k-j)}\xi_{i+N}, \xi_{i+N} \rangle - \alpha_{k-j}| \\ & < \frac{\alpha^{2i}}{2^{2i+3}(1 + i/\alpha)^4} \end{aligned}$$

for such k and j , and hence

$$\|w_i^* w_i - w_i'^* w_i'\|_2^2 \leq \sum_{j,k=0}^{K_i} |\langle S^{m(k-j)}\xi_i, \xi_i \rangle - \langle S^{m(k-j)}\xi_{i+N}, \xi_{i+N} \rangle|^2 \leq \frac{\alpha^{4i}}{2^{4i+6} K_i^2}.$$

Moreover, $\|w_i^* w_i + w_i'^* w_i'\| \leq 2(2K_i + 1)$, and therefore, by [2, Theorem X.2.3],

$$\|A_i - A_i'\|^2 \leq \|w_i^* w_i - w_i'^* w_i'\| \|w_i^* w_i + w_i'^* w_i'\| \leq \left(\frac{\alpha}{2}\right)^{2i}.$$

Now let $G_i = \text{span}[(U_i + U_i')A_i f_j \mid -K_i \leq j \leq K_i]$ and $G = \text{span}[G_1, \dots, G_M]$. Note that $G_i \hookrightarrow F_i + F_{i+N}$, and hence the spaces G_i are mutually orthogonal. We claim that the orthogonal projection Q , whose kernel is precisely G , satisfies the definition of 1-niceness. First note that any $\xi \in G$ can be written as $\xi = \xi_1 + \xi_2$, with $\xi_1 \in F_1 + \dots + F_M$, $\xi_2 \in F_{N+1} + \dots + F_{N+M}$, and $\|\xi_1\| = \|\xi_2\| = \|\xi\|/\sqrt{2}$. By the choice of N , $\langle \eta_k, \xi_2 \rangle = 0$ for $1 \leq k \leq L$, and hence $\|\eta_k - \xi\| \geq \|\eta_k\|/\sqrt{2}$ for any $\xi \in G$. Thus, $\|Q\eta_k\| \geq \|\eta_k\|/\sqrt{2}$.

Next we estimate

$$\|QS^{mp}S^{*mq}v\|_2^2 = \sum_{i=1}^M \|QS^{mp}S^{*mq}v e_i\|^2 = \frac{1}{2} \sum_{k=1}^M \|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\|^2.$$

For $i < \alpha(p + q)$, we trivially have $\|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\| \leq \sqrt{2}$. For $i \geq \alpha(p + q)$, let $j = p - q$, and observe that

$$S^{mp}S^{*mq}\xi_i = S^{mj}\xi_i = U_i A_i f_j \quad \text{and} \quad S^{mp}S^{*mq}\xi_{i+N} = S^{mj}\xi_{i+N} = U_i' A_i' f_j.$$

Moreover,

$$\begin{aligned} (U_i + U_i')A_i f_j &= (U_i A_i f_j + U_i' A_i' f_j) + U_i'(A_i - A_i')f_j \\ &= S^{mp}S^{*mq}(\xi_i + \xi_{i+N}) + U_i'(A_i - A_i')f_j \in G = \ker Q. \end{aligned}$$

Hence, $QS^{mp}S^{*mq}(\xi_i + \xi_{i+N}) = -QU_i'(A_i - A_i')f_j$ and

$$\|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\| \leq \|A_i - A_i'\| < \left(\frac{\alpha}{2}\right)^i.$$

Thus, for any $p, q \geq 0$,

$$\begin{aligned} 2\|QS^{mp}S^{*mq}v\|_2^2 &= \sum_{i \geq \alpha(p+q)} \|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\|^2 \\ &\quad + \sum_{i < \alpha(p+q)} \|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\|^2. \end{aligned}$$

By the above,

$$\begin{aligned} \sum_{i < \alpha(p+q)} \|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\|^2 &\leq 2|\{i \in \mathbb{N} \cup \{0\} \mid i < \alpha(p + q)\}| \\ &\leq \begin{cases} 2(1 + \alpha(p + q)), & p + q \geq 1, \\ 0, & p = q = 0, \end{cases} \end{aligned}$$

and

$$\sum_{i \geq \alpha(p+q)} \|QS^{mp}S^{*mq}(\xi_i + \xi_{i+N})\|^2 < \sum_{i = \lceil \alpha(p+q) \rceil}^{\infty} \left(\frac{\alpha}{2}\right)^i < \alpha.$$

Therefore,

$$2\|Q S^{mp} S^{*mq} v\|_2^2 \leq 2(1 + 2\alpha(p + q)) \leq 2e^{2\alpha(p+q)} \leq 2C^{p+q}$$

for any $p, q \geq 0$. Thus, the sequence (ξ_i) is 1-nice. □

Remark 4.6. The proof of Lemma 4.5 actually yields a stronger result: suppose that (ξ_i) is a regular sequence. Then for any $(\eta_k)_{k=1}^L \subset X$, $M \in \mathbb{N}$ and $\varepsilon > 0$, there exist $N'_0((\eta_k)_{k=1}^L, \varepsilon, M)$ such that for any $N \geq N'_0((\eta_k)_{k=1}^L, \varepsilon, M)$ and for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there exists an orthogonal projection Q_λ for which $\|Q_\lambda \eta_k\| > (\|\eta_k\| - \varepsilon)/\sqrt{2}$ whenever $1 \leq k \leq L$ and $\|Q_\lambda S^{mp} S^{*mq} v_\lambda\|_2 \leq C^{p+q}$. Here, the operator $v_\lambda : \ell_2^M \rightarrow \text{span}\{\xi_i \mid i \in \mathbb{N}\}$ is defined by setting $v_\lambda e_i = (\xi_i + \lambda \xi_{i+N})/\sqrt{2}$. Indeed, in the notation of the lemma, we let Q_λ be the projection with the kernel $\text{span}\{(U_i + \lambda U'_i)A_i f_j \mid 1 \leq i \leq M, |j| \leq K_i\}$. The proof of Lemma 4.5 shows that Q_1 satisfies the above conditions. The case of a general λ is dealt with in the same manner.

Proof of Theorem 4.1. By Lemma 4.4, every infinite subspace Y of X contains a regular sequence $(\xi_i)_{i \in \mathbb{N}}$. Let $Z = \text{span}\{\xi_i \mid i \in \mathbb{N}\}$, and consider $T \in \text{CB}(Z, Y)$ satisfying $\|T\|_{\text{cb}} < \frac{1}{5}$. By Theorem 1.1 (v), there exists $a \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $W = T - \tilde{T}_a$ is compact, and $\|a\|_w < \frac{4}{5}$. Consequently, $\|W\|_{\text{cb}} < 1$. By Lemmas 4.3 and 4.5, $\|W\|_2 < 4\sqrt{2}$.

Now consider $T_1 \in \text{CB}(Z)$, with $\|T_1\|_{\text{cb}} < \frac{1}{85}$. By Lemma 3.6, there exists a diagonal operator $T \in \text{CB}(Z)$ (that is, $T\xi_i = \lambda_i \xi_i$ for any $i \in \mathbb{N}$) such that $\|T\|_{\text{cb}} < \frac{1}{5}$ and $\|T - T_1\|_2 < \frac{16}{85}$. As above, there exists $a = (a_j)_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $\|a\|_w < 1$ and $\|W\|_2 < 4\sqrt{2}$, where $W = T - \tilde{T}_a|_Z$. Let $c = \sum_{j \in \mathbb{Z}} a_j \alpha_j$, where (as in the definition of regularity) $\alpha_j = \lim_i \langle S^{mj} \xi_i, \xi_i \rangle$. Clearly, the sum above converges, and $|c| \leq \|a\|_w < 1$. We shall show that $cI_Z - \tilde{T}_a|_Z$ is a Hilbert–Schmidt operator. Indeed, for any i ,

$$\lambda_i = \langle T\xi_i, \xi_i \rangle = \sum_{j \in \mathbb{Z}} a_j \langle S^{mj} \xi_i, \xi_i \rangle + c_i,$$

where $c_i = \langle W\xi_i, \xi_i \rangle$. However,

$$|\langle S^{mj} \xi_i, \xi_i \rangle - \alpha_j| < \frac{\alpha^{2i}}{2^{2i+4}(1 + i/\alpha)^4}$$

for $|j| \leq K_i$, while for other values of j we have $|\langle S^{mj} \xi_i, \xi_i \rangle - \alpha_j| \leq 2$. Therefore,

$$\begin{aligned} |\lambda_i - c| &\leq \sum_{j \in \mathbb{Z}} |a_j| |\langle S^{mj} \xi_i, \xi_i \rangle - \alpha_j| + |c_i| \\ &\leq \sum_{j=-K_i}^{K_i} |a_j| \frac{\alpha^{2i}}{2^{2i+4}(1 + i/\alpha)^4} + 2 \sum_{|j| > K_i} |a_j| + |c_i|. \end{aligned}$$

However, $\sum_{j \in \mathbb{Z}} C^{|j|} |a_j| \leq 1$, and hence $\sum_{|j| > K_i} |a_j| \leq C^{-(K_i+1)} \leq C^{-i/\alpha}$. On the other hand, $\sum_{j=-K_i}^{K_i} |a_j| \leq 1$. Hence, by definition of α ,

$$|\lambda_i - c| \leq \frac{\alpha^{2i}}{2^{2i+4}(1 + i/\alpha)^4} + 2C^{-i/\alpha} + |c_i| \leq \frac{1}{2^{4i+4}} + 2e^{-2i} + |c_i| < 2.1e^{-2i} + |c_i|.$$

Therefore,

$$\|T - cI_Z\|_2^2 = \sum_{i=1}^n |\lambda_i - c|^2 \leq \sum_{i=1}^n (2.1e^{-2i} + |c_i|)^2 < 2 \sum_{i=1}^n (4.5e^{-4i} + |c_i|^2).$$

We have $\sum_i |c_i|^2 \leq \|W\|_2^2 < 32$, and hence $\|T - cI_Z\|_2 < \sqrt{64 + 9/(e^4 - 1)}$, and

$$\|T_1 - cI_Z\|_2 \leq \|T - cI_Z\|_2 + \|T_1 - T\|_2 \leq \sqrt{64 + \frac{9}{e^4 - 1}} + \frac{16}{85} < \frac{700}{85}.$$

Finally, $|c| = \limsup |\langle T_1 \xi_i, \xi_i \rangle| \leq \|T_1\|$.

We next modify our proof to show that Y contains uncountably many subspaces Z with the above properties. For each infinite $\mathcal{F} \subset \mathbb{N}$, let $Z_{\mathcal{F}} = \text{span}[\xi_i \mid i \in \mathcal{F}]$. Any subsequence of a regular sequence is regular, and hence $\text{CB}(Z_{\mathcal{F}}, X)$ and $\text{CB}(Z_{\mathcal{F}})$ are as in the statement of the theorem.

It is well known (see, for example, [21]) that there exists an uncountable family \mathcal{I} of subsets of \mathbb{N} such that $\mathcal{F} \Delta \mathcal{G}$ is infinite if \mathcal{F} and \mathcal{G} are distinct members of \mathcal{I} . We shall show that $Z_{\mathcal{F}}$ and $Z_{\mathcal{G}}$ are not completely isomorphic to each other. Indeed, suppose positive integers $p_1 < p_2 < \dots$ are such that $\{p_1, p_2, \dots\} = \mathcal{F} \setminus \mathcal{G}$. We shall show that no $T \in \text{CB}(Z_{\mathcal{F}}, Z_{\mathcal{G}})$ has a bounded inverse.

Indeed, write $T = \sum_{j \in \mathbb{Z}} a_j S^{mj} + W$, where $\sum_{j \in \mathbb{Z}} C^{|j|} |a_j| < \infty$ and $W \in \mathcal{S}_2$. Let P be the orthogonal projection onto $Z_{\mathcal{G}}$. For each i , we have

$$T\xi_{p_i} = PT\xi_{p_i} = \sum_{j \in \mathbb{Z}} a_j P S^{mj} \xi_{p_i} + PW\xi_{p_i}.$$

The sequence (ξ_i) is shift lacunary. Hence, $S^{mj} \xi_{p_i}$ is orthogonal to ξ_k whenever $p_i \neq k$, and $|j| \leq K_{p_i}$. In this case, $S^{mj} \xi_{p_i}$ is orthogonal to $Z_{\mathcal{G}}$ or, in other words, $P S^{mj} \xi_{p_i} = 0$. As $K_{p_i} \geq i$ for $i > i_0$, we have

$$\lim_i \|T\xi_{p_i}\| \leq \lim_i \left(\sum_{|j| > i} |a_j| + \|W\xi_{p_i}\| \right) = 0$$

(here we use the compactness of W). Thus, T is not invertible. □

Remark 4.7. In fact, we have shown that the spaces $Z_{\mathcal{F}}$ and $Z_{\mathcal{G}}$ are not semi-isomorphic (in the sense of [22]).

Coordinate subspaces of X give rise to a class of c -nice sequences. We say that an increasing sequence of positive integers (n_i) is *self-repeating* if, for any $N_0 \in \mathbb{N}$, there exist $N, K > N_0$ such that $n_{i+N} = n_i + K$ whenever $1 \leq i \leq N_0$.

Lemma 4.8. *Suppose an increasing sequence (n_i) is self-repeating. Then the sequence of coordinate vectors $(\delta_{n_i})_{i \in \mathbb{N}}$ is γ_1 -nice, where*

$$\gamma_1 = \gamma_1(C, m) = \max \left\{ 1, \sup_{j \in \mathbb{N}} \frac{\sqrt{mj}}{\sqrt{2C^j}} \right\} \leq \max \left\{ 1, \frac{\sqrt{m}}{2\sqrt{\epsilon \ln C}} \right\}.$$

Proof. Suppose that $\eta \in X$ and $\text{supp } \eta_k \in [1, N_1]$ for $1 \leq k \leq L$. Find N_2 such that $C^{N_2} > \sqrt{M}$. Find $N, K > \max\{n_M + mN_2, N_1\}$ so that $n_{i+K} = n_i + N$ for $1 \leq i \leq M$. Let Q be the orthogonal projection with the kernel $G = \text{span}[\delta_k + \delta_{k+K} \mid 1 \leq k \leq N]$. As in the proof of Lemma 4.5, we conclude that $\|Q\eta_k\| \geq \|\eta_k\|/\sqrt{2}$ for $1 \leq k \leq L$.

It remains to estimate $\|QS^{mp}S^{*mq}v\|_2$. If $p + q > N_2$, then

$$\|QS^{mp}S^{*mq}v\|_2 \leq \|v\|_2 = \sqrt{M} \leq C^{N_2} \leq \gamma_1 C^{p+q}.$$

If $p + q < N_2$, use the equality

$$2\|QS^{mp}S^{*mq}v\|_2^2 = 2\sum_{i=1}^M \|QS^{mp}S^{*mq}ve_i\|^2 = \sum_{i=1}^M \|QS^{mp}S^{*mq}(\delta_{n_i} + \delta_{n_{i+N}})\|^2.$$

If $mq \geq n_i$, then $S^{*mq}\delta_{n_i} = 0$, and hence $\|QS^{mp}S^{*mq}(\delta_{n_i} + \delta_{n_{i+N}})\| \leq 1$. Otherwise,

$$S^{mp}S^{*mq}(\delta_{n_i} + \delta_{n_{i+N}}) = \delta_{n_i+m(p-q)} + \delta_{n_{i+N}+m(p-q)} \in G,$$

and $QS^{mp}S^{*mq}(\delta_{n_i} + \delta_{n_{i+N}}) = 0$. Therefore,

$$2\|QS^{mp}S^{*mq}v\|_2^2 \leq |\{i \in \mathbb{N} \mid mq \geq n_i\}| \leq mq \leq m(p + q).$$

By our choice of γ_1 , $\|QS^{mp}S^{*mq}v\|_2 \leq \sqrt{\frac{1}{2}m(p + q)} \leq \gamma_1 C^{p+q}$. □

Proof of Theorem 1.1 (i), (ii). Consider $T \in \text{CB}(X)$. By the proof of Theorem 1.1 (ii), and by Remark 3.9, there exists a unique $a \in \ell_1(C^{|j|}, \mathbb{Z})$ such that $\|a\|_w \leq 4\|T\|_{\text{cb}}$, and $T - \tilde{T}_a$ is compact. By Lemma 4.8 (applied to the sequence $n_i = i$), we conclude that $\|T - \tilde{T}_a\|_2 \leq 4\sqrt{2}\gamma_1\|T - \tilde{T}_a\|_{\text{cb}} \leq 20\sqrt{2}\gamma_1\|T\|_{\text{cb}}$. □

Remark 4.9. Observe that \mathbb{N} contains uncountably many self-repeating sequences. To see this, call an infinite subset of \mathbb{N} self-repeating if the corresponding sequence is self-repeating. Suppose that \mathcal{F} is a subset of $\mathcal{B} = \{10^n \mid n \in \mathbb{N}\}$. Let $\mathcal{F}_1 = \mathcal{F} \cup (\mathcal{F} \cap [1, 10] + 30)$ and, for $n \geq 2$, $\mathcal{F}_{n+1} = \mathcal{F}_n \cup (\mathcal{F}_n \cap [1, 10^n] + 3 \cdot 10^n)$. Then the set $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is self-repeating (indeed, $\mathcal{F}' \cap [1, 10^n] + 3 \cdot 10^n = \mathcal{F}' \cap [1 + 3 \cdot 10^n, 4 \cdot 10^n]$ for every $n \in \mathbb{N}$), and $\mathcal{F}' \cap \mathcal{B} = \mathcal{F}$. Thus, $\mathcal{F} \mapsto \mathcal{F}'$ is an injection from $2^{\mathcal{B}}$ to the family of self-repeating sets.

For a self-repeating set \mathcal{F} , let $Y_{\mathcal{F}} = \text{span}[\delta_i \mid i \in \mathbb{N}]$. If \mathcal{F} and \mathcal{G} are subsets of \mathcal{B} , and $|\mathcal{F} \Delta \mathcal{G}| = \infty$, then, arguing as in the last part of Theorem 4.1, we conclude that $Y_{\mathcal{F}'}$ and $Y_{\mathcal{G}'}$ are not completely isomorphic. Thus, X contains an uncountable family of coordinate subspaces that are not completely isomorphic to each other.

5. Failure of the OAP

Proof of Theorem 1.1 (iii). Suppose that Y is an infinite-dimensional subspace Y of X . By Lemma 4.4, Y contains a regular sequence (ξ_i) . Select $N \geq 21$. For every

$n \geq N$ and $s = \pm 1$ there exists $r = r(n, s)$ such that $n_r = \dim E_r = 2^{n+1}$, and $\|u_r - w_{ns}\|_1 < 2^{-(n+1)}$, where $w_{ns} : X \rightarrow E_r$ is defined by

$$w_{ns}\xi = 2^{-(n+1)/2} \sum_{i=1}^{2^n} \langle \xi, \xi_i + s\xi_{i+2^n} \rangle e_i \quad \text{for any } \xi \in X$$

(here $(e_i)_{i=1}^{2^n}$ is the canonical basis for E_r). Define $v_{ns} : E_r^* \rightarrow Z$ by letting $v_{ns}e_i = (\xi_i - s\xi_{i+2^n})/\sqrt{2}$. We show first that $\|v_{ns}\|_{cb} \leq 1$. Indeed, by (2.1) (see the proof of Lemma 2.2),

$$\begin{aligned} \|v_{ns}\|_{cb} &\leq \max \left\{ 1, \sup_{p,q \geq 0} C^{-(p+q)} \|u_r S^{mp} S^{*mq} v_{ns}\|_1 \right\} \\ &\leq \max \left\{ 1, \sup_{p,q \geq 0} C^{-(p+q)} (\|w_{ns} S^{mp} S^{*mq} v_{ns}\|_1 + \|u_r - w_{ns}\|_1) \right\}. \end{aligned} \tag{5.1}$$

However,

$$\begin{aligned} 2\|w_{ns} S^{mp} S^{*mq} v_{ns}\|_1 &\leq 2 \sum_{k,\ell=1}^{2^n} |\langle w_{ns} S^{mp} S^{*mq} v_{ns} e_k, e_\ell \rangle| \\ &= \sum_{k,\ell=1}^{2^n} |\langle (\xi_k + s\xi_{k+2^n}), S^{mp} S^{*mq} (\xi_\ell - s\xi_{\ell+2^n}) \rangle| \\ &\leq \sum_{i=1}^{2^n} |\langle \xi_i, S^{mp} S^{*mq} \xi_i \rangle - \langle \xi_{i+2^n}, S^{mp} S^{*mq} \xi_{i+2^n} \rangle| \\ &\quad + \sum_{k \neq \ell, 1 \leq k, \ell \leq 2^{n+1}} |\langle \xi_k, S^{mp} S^{*mq} \xi_\ell \rangle|. \end{aligned} \tag{5.2}$$

Using the definition of regularity as in the proof of Lemma 4.5, and recalling that $\alpha \leq \frac{1}{2}$, we conclude that

$$|\langle \xi_i, S^{mp} S^{*mq} \xi_i \rangle - \langle \xi_{i+2^n}, S^{mp} S^{*mq} \xi_{i+2^n} \rangle| \leq \frac{\alpha^{2i}}{2^{2i+3}(1+i/\alpha)^4} < \frac{1}{2^{4i+3}}$$

if $\max\{p, q\} \leq 2(1+i/\alpha)$. Thus,

$$\begin{aligned} &\sum_{i=1}^{2^n} |\langle \xi_i, S^{mp} S^{*mq} \xi_i \rangle - \langle \xi_{i+2^n}, S^{mp} S^{*mq} \xi_{i+2^n} \rangle| \\ &\leq \sum_{i=1}^{\lfloor \alpha(p+q)/2 \rfloor} (|\langle \xi_i, S^{mp} S^{*mq} \xi_i \rangle| + |\langle \xi_{i+2^n}, S^{mp} S^{*mq} \xi_{i+2^n} \rangle|) \\ &\quad + \sum_{i=\lfloor \alpha(p+q)/2 \rfloor + 1}^{2^n} |\langle \xi_i, S^{mp} S^{*mq} \xi_i \rangle - \langle \xi_{i+2^n}, S^{mp} S^{*mq} \xi_{i+2^n} \rangle| \\ &\leq \alpha(p+q) + \sum_{i=1}^{2^n} \frac{1}{2^{4i+3}} < \alpha(p+q) + \frac{1}{100}. \end{aligned}$$

On the other hand, by (3.1), $\langle \xi_k, S^{mp} S^{*mq} \xi_\ell \rangle = 0$ if $p + q \leq \max\{k, \ell\}/\alpha$, and hence

$$\sum_{k \neq \ell, 1 \leq k, \ell \leq 2^{n+1}} |\langle \xi_k, S^{mp} S^{*mq} \xi_\ell \rangle| \leq |\{(k, \ell) \mid k \neq \ell, \max\{k, \ell\} < \alpha(p + q)\}| < (\alpha(p + q))^2.$$

Thus, by (5.2),

$$\|w_{ns} S^{mp} S^{*mq} v_{ns}\|_1 \leq \frac{1}{2} \left(\frac{1}{100} + \alpha(p + q) + (\alpha(p + q))^2 \right),$$

and therefore, by (5.1),

$$\|v_{ns}\|_{cb} \leq \max \left\{ 1, \sup_{p, q \geq 0} \frac{1}{2} C^{-(p+q)} (1 + \alpha(p + q) + (\alpha(p + q))^2) \right\} = 1,$$

since $1 + \alpha(p + q) + (\alpha(p + q))^2 \leq (1 + \alpha(p + q))^2 \leq e^{2\alpha(p+q)} \leq C^{(p+q)}$.

If Y has the OAP, there exists, by [21, Lemma 6.1], a finite-rank map $T : Z \rightarrow Y$ such that $T\xi_i = \xi_i$ for $1 \leq i \leq 2^N$ and $\|Tv_{ns}\|_{cb} \leq 2^{n/4-4}$ for any $n \geq N$ and $s = \pm 1$. By the above, $\|v_{ns}\|_{cb} \leq 1$ and, moreover, $\|w_{ns}\|_{cb} \leq \|u_r - w_{ns}\|_1 + \|u_r\|_{cb} < \frac{8}{7}$. Thus,

$$\|w_{ns} Tv_{ns}\|_1 \leq 5 \|w_{ns} Tv_{ns}\|_{cb} < 7 \|Tv_{ns}\|_{cb} < 2^{n/4-1},$$

and therefore

$$\begin{aligned} 2^{n/2+1} \sum_{i=1}^{2^n} |\langle w_{ns} Tv_{ns} e_i, e_i \rangle| &= \sum_{i=1}^{2^n} |\langle T\xi_i, \xi_i \rangle - \langle T\xi_{i+2^n}, \xi_{i+2^n} \rangle + s(\langle T\xi_i, \xi_{i+2^n} \rangle - \langle T\xi_{i+2^n}, \xi_i \rangle)| \\ &\leq 2^{n/2+1} \|w_{ns} Tv_{ns}\|_1 \leq 2^{3n/4}. \end{aligned}$$

Averaging over s , we obtain

$$\begin{aligned} 2^{n/2} \sum_{i=1}^{2^n} |\langle w_{n,+1} Tv_{n,+1} e_i, e_i \rangle + \langle w_{n,-1} Tv_{n,-1} e_i, e_i \rangle| &= \sum_{i=1}^{2^n} |\langle T\xi_i, \xi_i \rangle - \langle T\xi_{i+2^n}, \xi_{i+2^n} \rangle| \leq 2^{3n/4}. \end{aligned}$$

Let $A_n = \text{Re} \sum_{i=1}^{2^n} \langle T\xi_i, \xi_i \rangle$. Then $A_N = 2^N$ and, by the triangle inequality, $A_{n+1} \geq 2A_n - 2^{3n/4}$. It is easy to show by induction that $A_n > 3 \cdot 2^{n-2} + 7 \cdot 2^{3n/4}$ for any $n \geq N$. On the other hand, $\sup_n A_n \leq \|T\|_1 < \infty$: a contradiction. \square

6. K -theory of $\text{CB}(X(C, m))$

In this section, we prove Theorem 1.1 (iv). First, recall the basics. Suppose that \mathcal{A} is a Banach algebra. We denote by $M_n(\mathcal{A})$ the algebra of $n \times n$ matrices with entries from \mathcal{A} , with obvious multiplication. For $a_i \in M_{n_i}(\mathcal{A})$, $i = 1, 2$, we denote by $a_1 \oplus a_2$ the element

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in M_{n_1+n_2}(\mathcal{A}).$$

To define the K -groups, first assume that \mathcal{A} has identity I . For $n \in \mathbb{N}$, $\text{IP}_n(\mathcal{A})$ denotes the set of idempotents (or projections) in $M_n(\mathcal{A})$. For $k < n$, we view $\text{IP}_k(\mathcal{A})$ as a subset of $\text{IP}_n(\mathcal{A})$, by identifying $p \in \text{IP}_k(\mathcal{A})$ with $p \oplus 0^{(n-k)} \in \text{IP}_n(\mathcal{A})$ (here and below, $0^{(s)}$ stands for the zero element of $M_s(\mathcal{A})$). We let $\text{IP}_\infty(\mathcal{A}) = \bigcup_{n \in \mathbb{N}} \text{IP}_n(\mathcal{A})$.

We say that $p, q \in \text{IP}_n(\mathcal{A})$ are *algebraically equivalent* ($p \sim_a q$) if there exist $u, v \in M_n(\mathcal{A})$ satisfying $vu = p$, $uv = q$. p and q are *0-equivalent* ($p \sim_0 q$) if $p \oplus I^{(k)} \sim_a q \oplus I^{(k)}$ for some $k \geq 0$ (here $I^{(k)}$ is the element of $M_k(\mathcal{A})$ consisting of k copies of I on the diagonal and zeros elsewhere). Equivalently, $p \sim_0 q$ if $p \oplus r \sim_a q \oplus r$ for some $r \in \text{IP}_\infty(\mathcal{A})$.

The 0-equivalence class of $p \in \text{IP}_\infty(\mathcal{A})$ is denoted by $[p]_0$. Following [3, §5], we introduce the (abelian) group $K_0(\mathcal{A})$ as the set of all differences $[p]_0 - [q]_0$, with $p, q \in \text{IP}_\infty(\mathcal{A})$. The equality $[p]_0 - [q]_0 = [p']_0 - [q']_0$ holds if $p \oplus q' \sim_0 p' \oplus q$. Addition is defined by $[p]_0 + [q]_0 = [p \oplus q]_0$, and $[0]_0$ serves as the neutral element of this group.

To define the group $K_1(\mathcal{A})$, consider the set $\text{GL}_n(\mathcal{A})$ of all invertible elements of $M_n(\mathcal{A})$. For $\ell < n$, identify $U \in \text{GL}_\ell(\mathcal{A})$ with $U \oplus I^{(n-\ell)} \in \text{GL}_n(\mathcal{A})$. Let $\text{GL}_\infty(\mathcal{A}) = \bigcup_n \text{GL}_n(\mathcal{A})$. We say that $U \in \text{GL}_\ell(\mathcal{A})$ and $V \in \text{GL}_n(\mathcal{A})$ are *homotopically equivalent* ($U \sim_1 V$, or $U \sim_h V$) if, for some $k \geq \max\{\ell, n\}$, there exists a continuous path $h : [0, 1] \rightarrow \text{GL}_k(\mathcal{A})$ such that $h(0) = U \oplus I^{(k-\ell)}$ and $h(1) = V \oplus I^{(k-n)}$. $[U]_1$ denotes the equivalence class of U . The abelian group $K_1(\mathcal{A})$ is generated by the set of equivalence classes $\text{GL}_\infty(\mathcal{A}) / \sim_1$, equipped with the addition $[U]_1 + [V]_1 = [U \oplus V]_1$ and with the neutral element $[I]_1$.

If \mathcal{A} is a non-unital Banach algebra, denote by \mathcal{A}^\sharp its unitization, with the identity I . $\mathbf{s} : \mathcal{A}^\sharp \rightarrow \mathbb{C}$ denotes the *scalar map*: $\mathbf{s}(a + \lambda I) = \lambda$, for any $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. \mathbf{s}_n stands for $I_{M_n} \otimes \mathbf{s}$ (viewed as a map from $M_n(\mathcal{A}^\sharp)$ to M_n). Clearly, if $a \in M_n(\mathcal{A}^\sharp)$ is an idempotent (or invertible), then so is $\mathbf{s}_n(a) \in M_n$. The K -groups of \mathcal{A} can be described as follows: $K_1(\mathcal{A}) = K_1(\mathcal{A}^\sharp)$, and $K_0(\mathcal{A})$ is the subgroup of $K_0(\mathcal{A}^\sharp)$ generated by $[p]_0 - [\mathbf{s}_n(p)]_0$, for $p \in \text{IP}_n(\mathcal{A}^\sharp)$.

Now suppose that π is a bounded algebraic homomorphism between Banach algebras \mathcal{A} and \mathcal{B} . Clearly, $I_{M_n} \otimes \pi$ maps idempotents (and, in the case of unital algebras, invertible elements) of $M_n(\mathcal{A})$ into idempotents (respectively, invertible elements) of $M_n(\mathcal{B})$. Attaching the unit if necessary, we define the group homomorphisms $K_\ell(\pi) : K_\ell(\mathcal{A}) \rightarrow K_\ell(\mathcal{B}) : [a]_\ell \mapsto [I_{M_n} \otimes \pi(a)]_\ell$ for $\ell = 0, 1$ and appropriate $a \in M_n(\mathcal{A})$.

The reader is referred to [3, 13] for more information on the K -theory for Banach algebras. Applications of K -theory to algebras of bounded operators can be found in [11, 13, 28]. We quote some specific results later on.

By Theorem 1.1 (i), we have a short exact sequence

$$0 \rightarrow \mathcal{S}_2 \xrightarrow{i} \text{CB}(X) \xrightarrow{\phi} \mathcal{Y} \rightarrow 0, \quad (6.1)$$

where \mathcal{Y} is the algebra of analytic functions on the annulus \mathbb{A} with coefficients in $\ell_1(C^{|j|}, \mathbb{Z})$, i is the natural injection and ϕ is the quotient map. In fact, ϕ was described explicitly in the paragraphs following Remark 3.9. There, we observed that for any $T \in \text{CB}(X)$ there exist unique $a = (a_j)_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z})$ and $W \in \mathcal{S}_2$ for which $T = \tilde{T}_a + W$. We defined the function $\phi(T)$ by setting, for $z \in \mathbb{A}$, $\phi(T)(z) = \sum_{j \in \mathbb{Z}} a_j z^j$. Moreover, it was observed that ϕ is an algebraic isomorphism, that is, $\phi(T_1 T_2) = \phi(T_1) \phi(T_2)$.

We first calculate the K -groups of the algebras \mathcal{S}_2 and \mathcal{Y} , and then use Bott periodicity [3, Chapter 9] to prove Theorem 1.1 (iv).

To deal with \mathcal{Y} , observe that idempotents of $M_n(\mathcal{Y})$ are precisely those functions $f : \mathbb{A} \rightarrow M_n$ whose entries belong to \mathcal{Y} and such that $f(z)$ is a projection (not necessarily self-adjoint) for any $z \in \mathbb{A}$. For $f \in \text{IP}_n(\mathcal{Y})$ define $\text{rank } f = \text{rank } f(z)$ (clearly, the latter does not depend on the choice of $z \in \mathbb{A}$).

To classify the equivalence classes of $f \in \text{GL}_n(\mathcal{Y})$, observe that, since the Banach algebra \mathcal{Y} is commutative, $f \in M_n(\mathcal{Y})$ is invertible if and only if $\det f$ is an invertible element of \mathcal{Y} . By [4, Theorem 4.6.12], the spectrum of \mathcal{Y} is identified with \mathbb{A} via the evaluation map. Thus, $f \in M_n(\mathcal{Y})$ is invertible if and only if $\det f$ does not vanish anywhere on \mathbb{A} .

If $g \in \mathcal{Y}$ does not vanish on \mathbb{A} , denote by $\text{wind } g$ the winding number of the function $\mathbb{T} \rightarrow \mathbb{C} : z \mapsto g(rz)$, $C^{-1} \leq r \leq C$, around 0. By continuity, this winding number does not depend on r .

Lemma 6.1.

- (a) The group $K_1(\mathcal{Y})$ is homomorphic to \mathbb{Z} , via $[f]_1 \mapsto -\text{wind } \det f$, $f \in \text{GL}_n(\mathcal{Y})$.
- (b) The group $K_0(\mathcal{Y})$ is homomorphic to \mathbb{Z} , via $[f]_0 \mapsto \text{rank } f$, $f \in \text{IP}_n(\mathcal{Y})$.

Proof. First we tackle (a). It is well known in K -theory that $K_1(C(\mathbb{T})) = \mathbb{Z}$, with the group homomorphism given by $[f]_1 \mapsto -\text{wind } \det f$ for $f \in \text{GL}_n(C(\mathbb{T}))$ (this follows, for instance, from [10, Proposition 8.3.1]). Moreover, the restriction map $j : C(\mathbb{A}) \rightarrow C(\mathbb{T}) : f \mapsto f|_{\mathbb{T}}$ determines a group homomorphism between $K_1(C(\mathbb{A}))$ and $K_1(C(\mathbb{T}))$. Indeed, suppose $f \in \text{GL}_n(C(\mathbb{A}))$ is such that there exists a continuous map $\Phi : [0, 1] \rightarrow \text{GL}_n(C(\mathbb{T}))$ such that $\Phi(0) = jf$ and $\Phi(1) = \mathbf{1}_{M_n}$ (that is, $\Phi(1)(z) = I_{M_n}$ for any $z \in \mathbb{T}$). Define the maps $\Phi_1, \Phi_2 : [0, 1] \rightarrow \text{GL}_n(C(\mathbb{A}))$, for $t \in [0, 1]$ and $z \in \mathbb{A}$, by setting $\Phi_1(t)(z) = \Phi(t)(z/|z|)$ and

$$\Phi_2(t)(z) = \begin{cases} f(z), & tC^{-1} + (1-t) \leq |z| \leq tC + (1-t), \\ f(z(tC^{-1} + (1-t))/|z|), & C^{-1} \leq |z| < tC^{-1} + (1-t), \\ f(z(tC + (1-t))/|z|), & tC + (1-t) < |z| \leq C. \end{cases}$$

Clearly, both Φ_1 and Φ_2 are continuous, $\Phi_1(1) = \mathbf{1}_{M_n}$ (as a function on \mathbb{A}), $\Phi_2(1) = f$ and $\Phi_1(0) = \Phi_2(0)$.

This shows that f and g lie in the same connected component of $\text{GL}_n(C(\mathbb{A}))$ if and only if jf and g lie in the connected component of $\text{GL}_n(C(\mathbb{T}))$. Therefore, the map $K_1(C(\mathbb{A})) \rightarrow \mathbb{Z} : [f]_1 \mapsto -\text{wind } \det f$ is a group homomorphism.

Moreover, by [27, Theorem 3.8], the Gelfand transform induces a homomorphism between $K_1(\mathcal{Y})$ and $K_1(C(\mathbb{A}))$. That is, $f, g \in \text{GL}_\infty(\mathcal{Y})$ are connected by a path lying in $\text{GL}_\infty(\mathcal{Y})$ if and only if they are connected by a path lying in $\text{GL}_\infty(C(\mathbb{A}))$. Conversely, every equivalence class of $\text{GL}_\infty(C(\mathbb{A}))$ has a representative in $\text{GL}_\infty(\mathcal{Y})$ (in fact, $f : z \mapsto z^{-k}$ can work as such a representative, for an appropriate $k \in \mathbb{Z}$). Thus, $K_1(\mathcal{Y}) \rightarrow \mathbb{Z} : [f]_1 \mapsto -\text{wind } \det f$ is a group homomorphism.

Part (b) is proved in a similar fashion. The only additional ingredient is the connection between homotopy equivalence and algebraic equivalence between classes of idempotents. More precisely, suppose \mathcal{A} is a Banach algebra. By [3, Propositions 4.3.1 and 4.4.1], if $p \in \text{IP}_n(\mathcal{A})$ and $q \in \text{IP}_\ell(\mathcal{A})$ are algebraically equivalent, then, for sufficiently large k , there exists a continuous map $h : [0, 1] \rightarrow \text{IP}_k(\mathcal{A})$ such that $h(0) = p \oplus 0^{(k-n)}$ and $h(1) = q \oplus 0^{(k-m)}$. Conversely, if such a map h exists, then $p \sim_a q$ [3, Proposition 4.3.3]. \square

To tackle \mathcal{S}_2 , we need a technical lemma (which may be known to specialists).

Lemma 6.2. *If W is a Hilbert–Schmidt map on a Hilbert space H , and $I + W$ is invertible, then $(I + W)^{-1} - I$ is Hilbert–Schmidt.*

Proof. Let $T = I + W$. Find a subspace $E \hookrightarrow H$ of finite codimension, for which $\|W|_E\| < 1/2$. Let $F = H \ominus E$, $E' = T(E)$, and $F' = T(F)$. Clearly, E' and F' are closed, and $\dim H/E' = \dim H/E$ (T has Fredholm index 0). By the invertibility of T , $E' \cap F' = \emptyset$, and hence $H = \text{span}[E', F']$. Let $W_0 = WP_E$, where P_E is the orthogonal projection onto E . Then $W_1 = \sum_{k=1}^{\infty} (-1)^k W_0^k$ is Hilbert–Schmidt, and $I + W_1 = (I + W_0)^{-1}$. Therefore, $T(I + W_1)|_{E'} = I_{E'}$ (the identity map on E').

Denote by Q the projection onto E' along F' (that is, $\text{ran } Q = E'$ and $\ker Q = F'$). Let $W_2 = W_1Q + ((T|_F)^{-1} - I_{F'})(I - Q)$. Clearly, W_2 is Hilbert–Schmidt, and

$$T(I + W_2) = T(I + W_2)Q + T(I + W_2)(I - Q) = Q + (I - Q) = I.$$

Since both T and $I + W_2$ are Fredholm operators of index 0, $I + W_2 = T^{-1}$. \square

Our treatment of the K -groups of \mathcal{S}_2 owes much to [11]. We view the unitization of \mathcal{S}_2 (denoted by \mathcal{S}_2^\sharp) as the subalgebra of $\text{CB}(X)$, generated by \mathcal{S}_2 and the identity I . By definition, X^n is isometric to $\ell_2^n(\ell_2)$, and hence $M_n(\mathcal{S}_2^\sharp)$ can be viewed as a subalgebra of $\text{CB}(X^n) \subset M_n(B(\ell_2))$. Moreover, for any $a \in M_n(\mathcal{S}_2^\sharp)$, $a - s_n(a) \otimes I$ is a Hilbert–Schmidt operator on $\ell_2^n(\ell_2)$.

Corollary 6.3. *Suppose that A is an element of $M_n(\mathcal{S}_2^\sharp)$, invertible in $M_n(B(\ell_2))$. Then its inverse belongs to $M_n(\mathcal{S}_2^\sharp)$. Furthermore, the spectrum of A is at most countable, with at most finitely many accumulation points.*

Proof. If A is invertible, then $a = s_n(A)$ is an invertible element of M_n . Thus, multiplying by $a^{-1} \otimes I$ if necessary, we can assume that $A = I + W$, where I is the identity on $\ell_2^n(\ell_2)$, and $W \in M_n(\mathcal{S}_2)$. By Lemma 6.2, $A^{-1} \in M_n(\mathcal{S}_2^\sharp)$.

By [12, Proposition 2.c.10], $A - \lambda \otimes I$ is a Fredholm operator of index 0 for any $\lambda \notin \sigma(s_n(A))$. Then, for any $\lambda \in \sigma(A) \setminus \sigma(s_n(A))$, there exists $\varepsilon > 0$ such that $\mu \notin \sigma(A)$ whenever $0 < |\lambda - \mu| < \varepsilon$. As in the proof of [12, Lemma 2.c.12], it follows that $\sigma(A)$ is at most countable, and $\sigma(s_n(A))$ contains all the accumulation points of $\sigma(A)$. \square

Below, we denote by Q_n , $n \geq 0$, a rank- n idempotent in \mathcal{S}_2 . Clearly, all such idempotents of the same rank are algebraically equivalent.

Lemma 6.4.

- (a) The group $K_1(\mathcal{S}_2)$ is homomorphic to $\{0\}$.
- (b) The group $K_0(\mathcal{S}_2)$ can be identified with \mathbb{Z} , via $n \mapsto [Q_n]_0$ for $n \geq 0$ and $n \mapsto -[Q_{-n}]_0$ for $n < 0$.

Proof. (a) Suppose that A is an invertible element of $M_n(\mathcal{S}_2^\sharp)$. By Corollary 6.3, and by [11, Lemma 3.2], we show that A can be connected to the identity by a path of invertible elements.

(b) For any $p \in \text{IP}_n(\mathcal{S}_2^\sharp)$, $\mathbf{s}_n(p) \in \text{IP}(M_n)$ is similar to a diagonal $n \times n$ matrix. Hence, one can assume that $p = a \otimes I + W$, where $a \in M_n$ is a diagonal projection and W is Hilbert–Schmidt. By the proof of [12, Theorem 2.c.13] (adapted to the operator space setting in [18]), p is algebraically equivalent to $p_1 \oplus \cdots \oplus p_n$, where p_1, \dots, p_n are idempotents in \mathcal{S}_2^\sharp . Moreover, if $P \in \text{IP}(\mathcal{S}_2^\sharp)$, then $P \sim_a \mathbf{s}(P)I + (-1)^{\mathbf{s}(P)}Q_{n(P)}$, with $\mathbf{s}(P) \in \{0, 1\}$, and $n(P) \in \{0\} \cup \mathbb{N}$. It is easy to see that $P_1 \oplus \cdots \oplus P_\ell \sim_a P'_1 \oplus \cdots \oplus P'_k$ if and only if

$$\sum_{j=1}^{\ell} \mathbf{s}(P_j) = \sum_{j=1}^k \mathbf{s}(P'_j) \quad \text{and} \quad \sum_{j=1}^{\ell} (-1)^{\mathbf{s}(P_j)} n(P_j) = \sum_{j=1}^k (-1)^{\mathbf{s}(P'_j)} n(P'_j).$$

Therefore, the Banach algebra \mathcal{S}_2^\sharp has cancellation: if $p_1, p_2 \in \text{IP}_\infty(\mathcal{S}_2^\sharp)$ satisfy $p_1 \sim_0 p_2$ (that is, $p_1 \oplus q \sim_a p_2 \oplus q$ for some $q \in \text{IP}_\infty(\mathcal{S}_2^\sharp)$), then $p_1 \sim_a p_2$. We complete the proof as in [11, Theorem 3.9]. \square

Proof of Theorem 1.1 (iv). We rely on the well-known fact (see, for example, [3, Chapter 9]) that (6.1) gives rise to a sequence of group homomorphisms which is ‘exact in the middle’. That is, $\ker K_\ell(\phi) = \text{ran } K_\ell(i)$, where $\ell = 0, 1$, and the maps i and ϕ are as in (6.1).

To prove the theorem for K_1 , note that, if T is an invertible element in $M_n(\text{CB}(X)) = \text{CB}(X^n)$, then $\det \phi_n(T)$ does not vanish on \mathbb{A} (here $\phi_n = I_{M_n} \otimes \phi$). Write $T = \tilde{T}_a + W$, where $a \in M_n(\ell_1(C^{|j|}, \mathbb{Z}))$ corresponds to $\phi_n(T)$, and W is Hilbert–Schmidt. Since T is invertible, \tilde{T}_a must have Fredholm index 0. Thus, by [5, Chapter 1] or [15], $\text{wind} \det(\phi_n(T)) = 0$, and therefore, by Lemma 6.1, $K_1(q)([T]_1) = 0$. In other words, $K_1(\text{CB}(X)) = \ker K_1(q)$. However, $\ker K_1(q) = \text{ran } K_1(i)$, by ‘exactness in the middle’, and $\text{ran } K_1(i) = \{0\}$, by Lemma 6.4. Thus, $K_1(\text{CB}(X)) = \{0\}$.

Next we deal with K_0 . By Lemma 6.1, $K_0(q)([p]_0 - [q]_0) = \text{rank } \phi_n(p) - \text{rank } \phi_k(q)$ for $p \in \text{IP}_n(\text{CB}(X))$ and $q \in \text{IP}_k(\text{CB}(X))$. Thus, $\text{ran } K_0(q)$ can be identified with \mathbb{Z} , and $\text{ran } K_0(q) = K_0(\mathcal{Y})$. Moreover, define a group homomorphism $\mu : \mathbb{Z} = K_0(\mathcal{Y}) \rightarrow K_0(\text{CB}(X))$ by setting $\mu(n) = [I^{(n)}]_0$ for $n \geq 0$ and $\mu(n) = -[I^{(n)}]_0$ for $n < 0$. Clearly, $K_0(q) \circ \mu$ is the identity on \mathbb{Z} . ‘Exactness in the middle’ then implies that $K_0(\text{CB}(X)) = \mathbb{Z} \oplus \text{ran } K_0(i)$.

It remains to show that $\text{ran } K_0(i) = \mathbb{Z}/m\mathbb{Z}$. By Lemma 6.4, we have a bijective group homomorphism $\nu : \mathbb{Z} \rightarrow K_0(\mathcal{S}_2)$, taking n to $[Q_n]_0$ for $n \geq 0$ and to $-[Q_{-n}]_0$ for $n < 0$.

Thus, we need to prove that $K_0(i)(\nu(n_1)) = K_0(i)(\nu(n_2))$ if and only if m divides $n_1 - n_2$. We consider the case of $n_1 \leq 0$ and $n_2 \geq 0$; all other combinations of signs are handled in a similar manner.

By the above results, $K_0(i)(\nu(n_1)) = K_0(i)(\nu(n_2))$ if and only if $-[Q_{-n_1}]_0 = [Q_{n_2}]_0$. By the definition of \sim_0 , the latter occurs if and only if $Q_{-n_1} \oplus Q_{n_2} \oplus I^{(k)} \sim_a I^{(k)}$ for some $k \in \mathbb{N}$. But the left-hand side is algebraically equivalent to $Q_s \oplus I^{(k)}$, where $s = n_2 - n_1$. Equivalently, there exists an operator space Z , containing X^k as a subspace of codimension s and completely isomorphic to it. Let $n = \lceil s/m \rceil$, and $\ell = mn - s$. X^k is completely isomorphic to all of its subspaces of codimension mn , and hence so is Z . Therefore, X^k is completely isomorphic to its subspace of codimension ℓ . By Lemma 3.13, m divides ℓ , and hence m also divides s . □

7. Further properties of $X(C, m)$

7.1. Subspaces with few completely bounded maps

We show that the shift ‘degenerates’ on some subspaces of $X(C, m)$.

Proposition 7.1. *For any $n \in \mathbb{N}$ there exists a subspace Z of X such that Z is a direct sum of mutually orthogonal infinite-dimensional subspaces Z_1, \dots, Z_n , and $T : Z \rightarrow X$ is completely bounded if and only if $T = \sum_{k=1}^n c_k J_k + W$, where $W \in \mathcal{S}_2$, and J_k is the composition of the orthogonal projection from Z onto Z_k and the embedding of Z_k into X . For such $(c_k)_{k=1}^n$ and W ,*

$$\max \left\{ |c_1|, \dots, |c_n|, \frac{\|W\|_2}{21\sqrt{2}\sqrt{n}} \right\} \leq \|T\|_{cb} \leq C^n(|c_1| + \dots + |c_n|) + \|W\|_2.$$

Moreover, Z is not completely isomorphic to a complete direct sum of more than n infinite-dimensional operator spaces.

The following lemma establishes Proposition 7.1 for $n = 1$.

Lemma 7.2. *Suppose that $|\lambda| = 1$, and a shift lacunary orthonormal sequence (ξ_i) exists such that $\|S^m \xi_i - \lambda \xi_i\| < \varepsilon_i = \alpha^{2i}(1 + i/\alpha)^{-5} 2^{-(2i+4)}$ for every $i \in \mathbb{N}$. Then $T : Z \rightarrow X$ is completely bounded if and only if $T = cJ + W$, where J is the embedding of $Z = \text{span}\{\xi_i \mid i \in \mathbb{N}\}$ into X . Moreover,*

$$\max \left\{ |c|, \frac{\|W\|_2}{21\sqrt{2}} \right\} \leq \|T\|_{cb} \leq |c| + \|W\|_2.$$

Proof. By a simple induction argument,

$$\|S^{mj} \xi_i - \lambda^j \xi_i\| \leq \|\lambda(S^{m(j-1)} \xi_i - \lambda^{j-1}) \xi_i\| + \|S^{m(j-1)}(S^m \xi_i - \lambda \xi_i)\| < j\varepsilon_i \tag{7.1}$$

for any $j \in \mathbb{N}$, and therefore, the sequence (ξ_i) is regular, with $\alpha_j = \lambda^j$.

Consider $T \in \text{CB}(Z, X)$ satisfying $\|T\|_{cb} < \frac{1}{4}$. Then $T = \tilde{T}_a|_Z + W$, for some $a = (a_j)_{j \in \mathbb{Z}} \in \ell_1(C^{|j|}, \mathbb{Z})$, such that $\|a\|_w < 1$ and $\|W\|_2 < 5\sqrt{2}$. Let $c = \sum_j a_j \lambda^j$. Clearly,

$|c| \leq \|a\|_w$. Next we show that

$$\|\tilde{T}_a|_Z - cJ\|_2^2 = \sum_i \left\| \left(\sum_j a_j S^{mj} - c \right) \xi_i \right\|^2 < \frac{1}{8}. \tag{7.2}$$

By (7.1),

$$\begin{aligned} \left\| \left(\sum_j a_j S^{mj} - c \right) \xi_i \right\| &= \left\| \sum_j a_j (S^{mj} - \lambda^j) \xi_i \right\| \\ &\leq \sum_{|j| \geq K_i+1} |a_j| \|(S^{mj} - \lambda^j) \xi_i\| + \sum_{|j| \leq K_i} |a_j| \|(S^{mj} - \lambda^j) \xi_i\| \\ &< 2 \sum_{|j| \geq K_i+1} |a_j| + \|a\|_w \sum_{|j| \leq K_i} |j| \varepsilon_i \\ &< C^{-(K_i+1)} + \frac{\alpha^{2i}}{(1+i/\alpha)^{4 \cdot 2^{2i+4}}} < 3e^{-2i} \end{aligned}$$

(recall that $C^{-(K_i+1)} \leq C^{-2i \ln C} = e^{-2i}$). Therefore,

$$\|\tilde{T}_a|_Z - cJ\|_2^2 = \sum_{i=1}^{\infty} \|(\tilde{T}_a - c) \xi_i\|^2 < 9 \sum_{i=1}^{\infty} e^{-4i},$$

which establishes (7.2). Thus,

$$\|T - cJ\|_2 \leq \|T - \tilde{T}_a\|_2 + \|\tilde{T}_a - cJ\|_2 \leq 5\sqrt{2} + \frac{1}{2\sqrt{2}}.$$

To finish the proof, observe that $|c| \leq \|T\|$. □

Proof of Proposition 7.1. For $1 \leq k \leq n$ let $\lambda_k = e^{2\pi\sqrt{-1}(k-1)/n}$. Select a shift lacunary sequence (ξ_i) such that $\|\lambda_k \xi_i - S^m \xi_i\| < \alpha^{2i} (1 + i/\alpha)^{-5} 2^{-(2i+4)}$ whenever $i = n(\ell - 1) + k$, for some $\ell \in \mathbb{N}$. Let $Z_k = \text{span}[\xi_{n(\ell-1)+k} \mid \ell \in \mathbb{N}]$, and $Z = \text{span}[\xi_i \mid i \in \mathbb{N}] = \text{span}[Z_1, \dots, Z_n]$.

Suppose $T \in \text{CB}(Z, X)$. By Lemma 7.2, for every k there exist $c_k \in \mathbb{C}$ and $W_k \in \mathcal{S}_2$ such that $|c_k| \leq \|T|_{Z_k}\| \leq \|T\|$, $\|W_k\|_2 \leq 476\|T\|_{\text{cb}}$ and $T|_{Z_k} = c_k J_k + W_k$. The spaces (Z_k) are mutually orthogonal, and hence

$$\left\| T - \sum_{k=1}^n c_k J_k \right\|_2 = \left(\sum_{k=1}^n \|W_k\|_2^2 \right)^{1/2} \leq 476\sqrt{n}\|T\|_{\text{cb}}.$$

On the other hand, by the proof of Lemma 7.2,

$$\left\| S^{mj}|_Z - \sum_{k=1}^n \lambda_k^j J_k \right\|_2 = \sum_{k=1}^n \sum_{\ell=0}^{\infty} \|(S^{mj} - \lambda_k^j) \xi_{n\ell+k}\|^2 < 9 \sum_{i=1}^{\infty} e^{-4i} < 1$$

for $0 \leq j \leq n - 1$. Therefore, $\|\sum_{k=1}^n \lambda_k^j J_k\|_{cb} \leq 1 + \|S^{m,j}\|_{cb} < C^n$. However,

$$J_i = \frac{1}{n} \sum_{j=0}^{n-1} \bar{\lambda}_i^j \sum_{k=1}^n \lambda_k^j J_k,$$

and hence $\|J_i\|_{cb} \leq C^n$ for $1 \leq i \leq n$.

Now suppose that R_1, \dots, R_{n+1} are completely bounded projections in Z and that $R_i R_j = 0$ if $i \neq j$. We must show that at least one of the projections has finite-dimensional range. To this end, define a map $\pi : CB(Z) \rightarrow \ell_\infty^n$ by setting

$$\pi\left(\sum_{k=1}^n c_k J_k + W\right) = (c_k)_{k=1}^n$$

(here $(c_k) \subset \mathbb{C}$, and $W \in \mathcal{S}_2$). By the above, π is well defined and is a bounded algebraic homomorphism. Therefore, $\pi(R_i) = \chi_{E_i}$ for any i (here E_i is a subset of $\{1, \dots, n\}$). Moreover,

$$\chi_{E_i} \chi_{E_j} = \pi(R_i) \pi(R_j) = \pi(R_i R_j) = \pi(0) = 0,$$

and therefore, by the pigeon-hole principle, E_i is empty for some i . For such i , R_i is Hilbert–Schmidt, and hence of finite rank. □

7.2. Compact operators on subspaces of $X(C, m)$

We conclude the paper by pointing out that, for a generic subspace $Y \hookrightarrow X = X(C, m)$, compactness of an operator $T \in CB(Y, X)$ does not imply any further ‘interesting’ properties of T (cf. Lemma 4.3). First show that subspaces of X allow a large variety of diagonal operators.

Proposition 7.3. *Suppose the function $f : \mathbb{A} \rightarrow \mathbb{C}$ belongs to \mathcal{Y} , and $\{\lambda_1, \lambda_2, \dots\} \subset f(\mathbb{T})$. There then exists a subspace Y of X , spanned by a shift lacunary basis $(\xi_i)_{i \in \mathbb{N}}$, and $T \in CB(X)$, so that $T\xi_i = \lambda_i \xi_i$ for any $i \in \mathbb{N}$.*

Proof. Find the points $z_1, z_2, \dots \in \mathbb{T}$ such that $f(z_n) = \lambda_n$ for any $n \in \mathbb{N}$. By approximating ‘very spiky’ functions concentrated around the points z_n by trigonometric polynomials, we find that, for each n ,

$$g_n(z) = \sum_{j=-K_n}^{K_n} \alpha_{jn} z^j$$

such that $\|g_n\|_2 = 1$, and $\|(f - \lambda_n)g_n\|_2 < 2^{-n}$. Find an increasing sequence (M_n) of positive integers such that $M_n > K_n$ for every n , and moreover, the sequence (ξ_n) is shift lacunary, where

$$\xi_n = \sum_{j=-K_n}^{K_n} \alpha_{jn} \delta_{m(M_n+j)}.$$

Write $f(z) = \sum_{j \in \mathbb{Z}} a_j z^j$. By the above, $\|\tilde{T}_a \xi_n - \lambda_n \xi_n\| < 2^{-n}$. Define an operator $W : X \rightarrow X$ by setting

$$W \xi_n = \tilde{T}_a \xi_n - \lambda_n \xi_n \quad \text{and} \quad W|_{\text{span}\{\xi_n | n \in \mathbb{N}\}^\perp} = 0.$$

Clearly, W is Hilbert–Schmidt, and hence $\tilde{T}_a - W \in \text{CB}(X)$. However, $(\tilde{T}_a - W)\xi_n = \lambda_n \xi_n$. □

It easily follows from the above proposition that, for any separable symmetric sequence space \mathcal{E} not isomorphic to c_0 , there exists a subspace Y of X and a compact $T \in \text{CB}(X)$ such that the sequence of singular numbers $(\sigma_i(T))_{i \in \mathbb{N}}$ does not belong to \mathcal{E} (in other words, such a T does not belong to the ideal $\mathcal{S}_{\mathcal{E}}$; see, for example, [7] for the definition). Indeed, we find a sequence $(s_n)_{n \in \mathbb{N}} \in c_0 \setminus \mathcal{E}$ such that $s_n \in (0, \frac{1}{2})$ for any n , and consider $f(z) = 1 - z$. We then find a sequence (z_n) such that $|f(z_n)| = s_n$. By Proposition 7.3, there exists $T \in \text{CB}(\text{span}\{\xi_i | i \in \mathbb{N}\})$ such that $T \xi_i = f(z_i) \xi_i$. Then (s_n) is the sequence of singular numbers of the operator T .

Several versions of ‘non-commutative compactness’ have been put forward over the years (see, for example, [17, 18]). If $u \in \text{CB}(Y, Z)$ (Y and Z are operator spaces), one can introduce the sequence of *complete Kolmogorov numbers* by setting, for $n \in \mathbb{N}$,

$$d_n^{\text{cb}}(u) = \inf\{\|q_F u\|_{\text{cb}} \mid F \hookrightarrow Z, \dim F \leq n\}$$

(here $q_F : Z \rightarrow Z/F$ is a quotient map). Similarly, the *complete Gelfand numbers* are defined by setting

$$c_n^{\text{cb}}(u) = \inf\{\|u j_G\|_{\text{cb}} \mid G \hookrightarrow Y, \dim Y/G \leq n\},$$

where j_G is the canonical embedding. A map $u \in \text{CB}(Y, Z)$ is called *operator compact* (*Gelfand compact*) if $\lim_n d_n^{\text{cb}}(u) = 0$ (respectively, $\lim_n c_n^{\text{cb}}(u) = 0$).

Below we show that a compact $T \in \text{CB}(Y, X)$ need not be either operator or Gelfand compact.

Proposition 7.4. *There exist a subspace Y of the space $X = X(C, m)$ and a compact operator $T \in \text{CB}(Y)$ such that*

$$\inf\{\|q_F T j_G\|_{\text{cb}} \mid F \hookrightarrow X, G \hookrightarrow Y, \dim F < \infty, \dim Y/G < \infty\} > 0.$$

Remark 7.5. *A fortiori, T cannot be approximated by finite-rank operators in the c.b. norm. This contrasts sharply with operators on X : among the compact operators on X itself, the norms $\|\cdot\|_{\text{cb}}$ and $\|\cdot\|_2$ are equivalent. Hence, for any compact $T \in \text{CB}(X)$ there exists a sequence of finite-rank maps (T_n) such that*

$$\lim_n \|T - T_n\|_{\text{cb}} = \lim_n \|T - T_n\|_2 = 0.$$

The following technical lemma seems to be part of the folklore. We state and prove it for the sake of completeness.

Lemma 7.6. *Suppose that Y and Z are operator spaces, $u \in \text{CB}(Y, Z)$, and there exists a net (F_i) of finite-dimensional subspaces of Z , such that $Z = \bigcup_i F_i$. Then*

$$\inf\{\|q_F u\|_{\text{cb}} \mid F \hookrightarrow X, \dim F < \infty\} = \inf_i \|q_{F_i} u\|_{\text{cb}}.$$

Proof. Assume, without loss of generality, that $\|u\|_{\text{cb}} \leq \frac{1}{3}$. Let

$$\lambda = \inf\{\|q_F u\|_{\text{cb}} \mid F \hookrightarrow X, \dim F < \infty\}.$$

For any $\varepsilon \in (0, \frac{1}{3})$, there exists a finite-dimensional subspace $F \hookrightarrow Z$ such that $\|q_F u\|_{\text{cb}} < \lambda + \varepsilon$. Then for every $x \in X \otimes \mathcal{K}$ with $\|x\| < 1$ there exists $f \in F \otimes \mathcal{K}$ satisfying

$$\|f + (u \otimes I_{\mathcal{K}})x\| < \lambda + \varepsilon.$$

By the triangle inequality, $\|f\| < 1$. A simple perturbation argument (see [23, §2.13]) shows that there exists i such that for any $f \in F \otimes \mathcal{K}$ there exists $f' \in F_i \otimes \mathcal{K}$, satisfying $\|f - f'\| < \varepsilon$. By the triangle inequality, $\|f' + (u \otimes I_{\mathcal{K}})x\| < \lambda + 2\varepsilon$. Hence, $\|q_{F_i} u\|_{\text{cb}} \leq \lambda + 2\varepsilon$. However, ε can be taken to be arbitrarily small. \square

Proof of Proposition 7.4. Find a sequence (ℓ_i) of positive integers such that $m \leq \ell_1 \leq \ell_2 \leq \dots$, $\lim_i \ell_i = \infty$ and $\sum_i 1/\ell_i = \infty$. Consider a shift lacunary sequence $(\xi_i)_{i \in \mathbb{N}}$, consisting of vectors

$$\xi_i = \frac{1}{\sqrt{\ell_i}} \sum_{k=N_i+1}^{N_i+\ell_i} \delta_k,$$

where $N_k + \ell_k + 2m(1 + \lfloor k/\alpha \rfloor) < N_{k+1}$ for any k (as before, $\alpha = \min\{1, \ln C\}/2$). We shall show that $Y = \text{span}\{\xi_i \mid i \in \mathbb{N}\} \hookrightarrow X$ and $T = (I - S^m)|_Y$ satisfy the conditions of Theorem 7.4. Clearly, $\|T\|_{\text{cb}} \leq 1 + C$. Moreover, $T\xi_i = \sqrt{2m/\ell_i}\eta_i$, where

$$\eta_i = \frac{1}{\sqrt{2m}} \sum_{j=0}^{m-1} (\delta_{N_i+j} - \delta_{N_i+\ell_i+j}).$$

Note that $\lim_i \|T\xi_i\| = 0$ and the vectors η_i are orthonormal; hence, T is compact.

Applying Lemma 7.6 to T and then to T^* , we see that it suffices to show that, for any $K \in \mathbb{N}$,

$$\|q_K T|_{Y_K}\|_{\text{cb}} \geq \frac{1}{2}, \tag{7.3}$$

where $Y_K = \text{span}\{\xi_i \mid i > K\}$, $F_K = \text{span}\{\delta_1, \dots, \delta_K\}$ and $q_K : X \rightarrow X/F_K$ is the quotient map. To this end, fix $M \in \mathbb{N}$ so that

$$c = \left(\sum_{i=K+1}^{K+M} 1/\ell_i \right)^{-1/2} < (2m)^{-1}.$$

Define $v \in B(\ell_2^M, Y_K)$ and $u \in B(X, \ell_2^M)$ by setting $ve_i = \xi_{K+i}$, and $u^*e_i = c_i\eta_{K+i}$, where $(e_i)_{i=1}^M$ is the canonical basis for ℓ_2^M , $c_i = c/\sqrt{\ell_{K+i}}$. Note that $\|v\| = 1 = \|u\|_2$. Pick $\varepsilon \in (0, 1)$ and find (as in the proof of Lemma 2.2) $s > \varepsilon^{-1}$ such that $n_s = M$,

and $\|u_s - u\|_1 < \varepsilon$. We identify the domain of v and the range of u with E_s^* and E_s , respectively. Note that $F_K = \ker q_K \subset \ker u$, and hence there exists a linear operator $\tilde{u} \in B(X/F_K, E_s)$ such that $\tilde{u}q_K = u$. We obviously have

$$\|q_K T|_{Y_K}\|_{cb} \geq \frac{\|\tilde{u}q_K T v\|_{cb}}{\|\tilde{u}\|_{cb}\|v\|_{cb}}. \tag{7.4}$$

To establish (7.3), we find suitable estimates for the numerator and the denominator of (7.4). Look at the numerator first. Recall that $\tilde{u}q_K = u$. By the properties of the space E_s (see § 2),

$$\begin{aligned} (4 + \varepsilon)\|\tilde{u}q_K T v\|_{cb} &\geq \|u T v\|_1 = \sum_{i=1}^M |\langle T v e_i, u^* e_i \rangle| = \sum_{i=1}^M \left| \left\langle \sqrt{\frac{2m}{\ell_{K+i}}} \eta_{K+i}, c_i \eta_{K+i} \right\rangle \right| \\ &= \sum_{i=1}^M \frac{\sqrt{2m}}{\ell_{K+i}} c = \sum_{i=1}^M \frac{\sqrt{2m}}{\ell_{K+i}} \left(\sum_{i=1}^M \frac{1}{\ell_{K+i}} \right)^{-1/2} \\ &= \frac{\sqrt{2m}}{c}. \end{aligned}$$

Next we show that $\|v\|_{CB(E_s^*, X)} < \sqrt{m}/(\sqrt{2}c) + \varepsilon$. As in the proof of Lemma 2.2,

$$\|v\|_{cb} \leq \max \left\{ 1, \sup_{p, q \geq 0} C^{-(p+q)} \|u_s S^{mp} S^{*mq} v\|_1 \right\}.$$

However,

$$\begin{aligned} \|u S^{mp} S^{*mq} v\|_1 &\leq \sum_{i, j=1}^M |\langle u^* e_i, S^{mp} S^{*mq} v e_j \rangle| \\ &= \sum_{i=1}^M |\langle u^* e_i, S^{mp} S^{*mq} v e_i \rangle| + \sum_{i \neq j} |\langle u^* e_i, S^{mp} S^{*mq} v e_j \rangle|. \end{aligned}$$

The first term on the right can be estimated as follows:

$$\sum_{i=1}^M |\langle u^* e_i, S^{mp} S^{*mq} v e_i \rangle| = \sum_{i=1}^M |\langle c_i \eta_{i+K}, S^{mp} S^{*mq} \xi_{i+K} \rangle| \leq \sum_{i=1}^M \frac{c_i m}{\sqrt{2m} \sqrt{\ell_{K+i}}} = \frac{\sqrt{m}}{\sqrt{2}c}.$$

As for the second term, the conditions on the sequence (N_k) imply that

$$\langle u^* e_i, S^{mp} S^{*mq} v e_j \rangle = \langle c_i \eta_{i+K}, S^{mp} S^{*mq} \xi_{j+K} \rangle = 0$$

whenever $i \neq j$, and $\max\{i, j\} \leq \alpha(p + q)$. Therefore,

$$\sum_{i \neq j} |\langle u e_i, S^{mp} S^{*mq} v e_j \rangle| \leq |\{(i, j) \in \mathbb{N}^2 \mid i \neq j, \max\{i, j\} \leq \alpha(p + q)\}| \leq (\alpha(p + q))^2.$$

Thus,

$$\begin{aligned} \|u_s S^{mp} S^{*mq} v\|_1 &\leq \|u_s - u\|_1 + \|u S^{mp} S^{*mq} v\|_1 \\ &< \varepsilon + \frac{\sqrt{m}}{\sqrt{2c}} + (\alpha(p + q))^2 \\ &< \varepsilon + \frac{\sqrt{m}}{\sqrt{2c}} (1 + \alpha(p + q))^2 \\ &\leq \varepsilon + \frac{\sqrt{m}}{\sqrt{2c}} C^{p+q}, \end{aligned}$$

and therefore $\|v\|_{cb} \leq \varepsilon + \sqrt{m}/\sqrt{2c}$.

Finally, by definition of the quotient of operator spaces,

$$\|\tilde{u}\|_{cb} = \|u\|_{cb} \leq \|u_s\|_{cb} + \|u - u_s\|_{cb} < 1 + \varepsilon$$

(here we use (2.1)). Thus, (7.4) yields

$$\|T\|_{cb} \geq \frac{\sqrt{2m}/c}{(4 + \varepsilon)(\varepsilon + \sqrt{m}/\sqrt{2c})(1 + \varepsilon)}.$$

However, $\varepsilon > 0$ can be arbitrarily small. Hence, we have (7.3). □

8. Representations of $\ell_1(C^{|j|}, \mathbb{Z})$ and $\ell_1(C^j, \mathbb{Z}^+)$

In this section, we modify the construction presented above to exhibit operator spaces arising from faithful unital representations of weighted convolution algebras $\ell_1(C^{|j|}, \mathbb{Z})$ and $\ell_1(C^j, \mathbb{Z}^+)$. As before, $C > 1$ and $m \in \mathbb{N}$ are fixed. We shall use S_b to denote the bilateral shift on $\ell_2(\mathbb{Z})$. For the semigroup $\Gamma = \mathbb{Z}$ or $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, consider the following contractive unital representations:

$$\begin{aligned} \pi_{\mathbb{Z},b} : \ell_1(C^{|j|}, \mathbb{Z}) &\rightarrow B(\ell_2(\mathbb{Z})) : \delta_j \mapsto C^{-|j|} S_b^{mj}; \\ \pi_{\mathbb{Z}^+,b} : \ell_1(C^j, \mathbb{Z}^+) &\rightarrow B(\ell_2(\mathbb{Z})) : \delta_j \mapsto C^{-|j|} S_b^{mj}; \\ \pi_{\mathbb{Z}^+,r} : \ell_1(C^j, \mathbb{Z}^+) &\rightarrow B(\ell_2) : \delta_j \mapsto C^{-|j|} S^{mj}; \\ \pi_{\mathbb{Z}^+,\ell} : \ell_1(C^j, \mathbb{Z}^+) &\rightarrow B(\ell_2) : \delta_j \mapsto C^{-|j|} S^{*mj}. \end{aligned}$$

Here, the letters b, ℓ and r stand for ‘bilateral’, ‘left’ and ‘right’, respectively. It turns out that these representations give rise to operator spaces.

Theorem 8.1. *Suppose that $C > 1$, $m \in \mathbb{N}$, $(\Gamma, d) \in \{(\mathbb{Z}, b), (\mathbb{Z}^+, b), (\mathbb{Z}^+, r), (\mathbb{Z}^+, \ell)\}$, and $\pi_{\Gamma,d}$ is one of the representations described above. Let $\Gamma' = \mathbb{Z}$ if $\Gamma = \mathbb{Z}$ and let $\Gamma' = \mathbb{N}$ if $\Gamma = \mathbb{Z}^+$. There then exists an operator space $X_{\Gamma,d} = X_{\Gamma,d}(C, m)$, isometric to $\ell_2(\Gamma')$, such that $\pi_{\Gamma,d}$ induces an isomorphism from $\ell_1(C^{|j|}, \Gamma)$ into $\text{CB}(X_{\Gamma,d})$. Moreover, the following conditions hold.*

- (i) For any $a \in \ell_1(C^{|j|}, \Gamma)$, $\|\pi_{\Gamma,d}(a)\|_{cb} \geq \frac{1}{4}\|a\|_w$.
- (ii) $T \in CB(X_{\Gamma,d})$ if and only if $T = \pi_{\Gamma,d}(a) + W$ for some $a \in \ell_1(C^{|j|}, \Gamma)$ and $W \in \mathcal{S}_2$. In this case,

$$\|a\|_w + \|W\|_2 \geq \|T\|_{cb} \geq \max\{\|a\|_w/4, \gamma\|W\|_2\},$$

where γ depends only on C and m .

- (iii) If Y is an infinite-dimensional subspace of $X_{\Gamma,d}$, then for any $T \in CB(Y, X_{\Gamma,d})$ there exists $a \in \ell_1(C^{|j|}, \Gamma)$ such that $T - \pi_{\Gamma,d}(a)|_Y$ is compact, and $\|a\|_w \leq 4\|T\|_{cb}$. Moreover, Y contains an infinite-dimensional subspace Z such that any $T \in CB(Z, X_{\Gamma,d})$ can be written as $T = \pi_{\Gamma,d}(a)|_Z + W$, with $a \in \ell_1(C^{|j|}, \Gamma)$, $W \in \mathcal{S}_2$, $\|W\|_2 \leq 20\sqrt{2}\|T\|_{cb}$ and $\|a\|_w \leq 4\|T\|_{cb}$.
- (iv) Any infinite-dimensional subspace of $X_{\Gamma,d}$ fails the OAP.
- (v) $X_{\Gamma,d}$ is not completely isomorphic to any of its proper subspaces.
- (vi) $K_0(CB(X_{\Gamma,d})) = \mathbb{Z}^2$, $K_1(CB(X_{\mathbb{Z}^+,d})) = \{0\}$ and $K_1(CB(X_{\mathbb{Z},b})) = \mathbb{Z}$.

The proof of this theorem is similar to those presented above, so we outline it only briefly. We use the spaces E_i described in §2. The maps $u_i : \ell_2(\Gamma') \rightarrow E_i$ are such that, for all $n, N \in \mathbb{N}$ and $u : \ell_2(\Gamma') \rightarrow \ell_2^N$ with $\|u\|_2 = 1$, there exists $i > N$ such that $\dim E_i = n$ and $\|u - u_i\|_1 < 2^{-N}$. We define the operator space structure, for $x \in \ell_2(\Gamma') \otimes \mathcal{K}$, by setting

$$\|x\|_{X_{\Gamma,d} \otimes \mathcal{K}} = \sup\{C^{-|j|}\|(u_i S_d^{mj} \otimes I_{\mathcal{K}})x\|_{E_i \otimes \mathcal{K}} \mid i \in \mathbb{N}, j \in \Gamma\},$$

where S_d stands for S_b ($d = b$), S ($d = r$), or S^* ($d = \ell$). Then $X_{\Gamma,d}$ is an operator space, isometric to $\ell_2(\Gamma')$, and $\|\pi_{\Gamma,d}(a) + W\|_{cb} \leq \|a\|_w + \|W\|_2$ for any $a \in \ell_1(C^{|j|}, \Gamma)$ and $W \in \mathcal{S}_2$.

For notational simplicity, we henceforth use X and π for $X_{\Gamma,d}$ and $\pi_{\Gamma,d}$, respectively.

Our next goal is to split an operator $T \in CB(Y, X)$, $Y \hookrightarrow X$, into its compact and Toeplitz parts, thus proving (iii). When dealing with right or left shifts ($d = r$ or ℓ), we use shift lacunary sequences as before. In the bilateral case we employ bilateral shift lacunary sequences. We say that a sequence $(\xi_i)_{i \in \mathbb{N}}$ in $\ell_2(\mathbb{Z})$ is *bilaterally shift lacunary* if it is normalized, and there exists a sequences of mutually orthogonal subspaces $(F_i)_{i \in \mathbb{N}}$ of $\ell_2(\mathbb{Z})$, such that $\dim F_i = 2K_i + 1$, $S_b^{mj}\xi_i \in F_i$ whenever $|j| \leq K_i$, if $i \neq k$. As in Proposition 3.4, we prove: suppose (Y_n) is a sequence of infinite-dimensional subspaces of $\ell^2(\mathbb{Z})$. There then exists a bilaterally shift lacunary sequence $(\xi_n)_{n \in \mathbb{N}}$, such that $\xi_n \in Y_n$ for any n . As in §3, we show that, for any $Y \hookrightarrow X$ and $T \in CB(Y, X)$, there exists $a \in \ell_1(C^{|j|}, \Gamma)$ for which $T - \pi(a)|_Y$ is compact.

As in §§4 and 5, we define c -nice and regular sequences, and establish parts (i), (ii) and (iv) of Theorem 8.1.

To deal with parts (v) and (vi), it is convenient to represent $\ell_1(C^{|j|}, \Gamma)$ as a function algebra \mathcal{Y}_Γ . More precisely, by [4, Theorems 4.6.9 and 4.6.12], the spectrum of $\ell_1(C^{|j|}, \Gamma)$ is

$$\mathbb{A}_\Gamma = \begin{cases} \{z \in \mathbb{C} \mid C^{-1} \leq |z| \leq C\}, & \Gamma = \mathbb{Z}, \\ \{z \in \mathbb{C} \mid |z| \leq C\}, & \Gamma = \mathbb{Z}^+. \end{cases}$$

\mathcal{Y}_Γ is the algebra of functions f on \mathbb{A}_Γ with the power series decomposition

$$f(z) = \sum_{j \in \Gamma} a_j z^j$$

and the norm

$$\|f\| = \sum_{j \in \Gamma} |a_j| C^{|j|}.$$

As in § 3, define $\phi : \text{CB}(X) \rightarrow \mathcal{Y}_\Gamma$ by setting $\phi(T)(z) = \sum_{j \in \Gamma} a_j z^j$, where $a = (a_j)_{j \in \Gamma}$ is such that $T - \pi(a)$ is compact (there are no non-trivial compact Toeplitz operators, and hence such an a is unique).

We can now show that $X^n = X \oplus \cdots \oplus X$ (n times) is not completely isomorphic to any of its proper subspaces (this is a counterpart of Lemma 3.13, and yields part (v) of our theorem). To minimize the clutter, we use ϕ_n and π_n for $I_{M_n} \otimes \phi$ and $I_{M_n} \otimes \pi$, respectively. Suppose that $T \in \text{CB}(X^n)$ is a complete isomorphism onto its range. As in the proof of Lemma 3.11, T is Fredholm. Thus, the range of T has finite codimension, and there exists $T_1 \in \text{CB}(X^n)$ such that $T_1 T = I_{X^n}$. Then $f_1(z) f(z) = I_{M_n}$ for any $z \in \mathbb{A}_\Gamma$, where $f_1 = \phi_n(T_1)$ and $f = \phi_n(T)$. π_n is a unital representation. Hence, $\pi_n(f)$ is invertible. But $T - \pi_n(f)$ is compact, and hence T has Fredholm index 0.

Part (vi) of our theorem can be deduced from (v) by imitating § 6. Alternatively, one can observe that $\phi \circ \pi = I_{\mathcal{Y}_\Gamma}$. Hence, the short exact sequence

$$0 \rightarrow \mathcal{S}_2 \xrightarrow{i} \text{CB}(X) \xrightarrow{\phi} \mathcal{Y}_\Gamma \rightarrow 0,$$

splits, and therefore, $K_s(\text{CB}(X)) = K_s(\mathcal{S}_2) \oplus K_s(\mathcal{Y}_\Gamma)$ for $s = 0, 1$ [3, Proposition 8.3.6]. The K -groups of \mathcal{S}_2 have already been described in Lemma 6.4, while $K_s(\mathcal{Y}_\Gamma) = K_s(C(\mathbb{A}_\Gamma))$, by [27]. But $K_0(C(\mathbb{A}_\Gamma)) = \mathbb{Z}$ (this follows from [3, Example 5.3.2] or [10, Proposition 8.3.1]). As noted in § 6, $K_1(C(\mathbb{A}_\mathbb{Z})) = \mathbb{Z}$, while it is easy to show that $K_1(C(\mathbb{A}_{\mathbb{Z}^+})) = \{0\}$ (see [13, Example 9.6.1]).

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