Appendix G
Relativistic quasielastic scattering

If one scatters an electron from a nucleon at rest to a final state of discrete mass, then, as was shown in chapter 12, the Lorentz invariant response surfaces take the following form[1]

$$W_i(q^2, q \cdot p) = w_i(q^2)\frac{m^2}{p_0'}\delta(p_0 - p_0' - q_0) \qquad ; \; i = 1, 2 \tag{G.1}$$

For a Dirac nucleon

$$\begin{aligned} w_1 &= \frac{q^2}{4m^2}(F_1 + 2mF_2)^2 \\ w_2 &= F_1^2 + \frac{q^2}{4m^2}(2mF_2)^2 \end{aligned} \tag{G.2}$$

For elastic scattering from an isolated nucleon, it was shown in chapter 12 that

$$\begin{aligned} \int d\varepsilon_2\, \delta(m - E' - q_0) &= \frac{E'}{m}r \\ r^{-1} &\equiv \left(1 + \frac{2\varepsilon_1 \sin^2\theta/2}{m}\right) \end{aligned} \tag{G.3}$$

Hence the differential cross section for elastic scattering is given by

$$\frac{d\sigma}{d\Omega} = \sigma_M \left[w_2(q^2) + 2w_1(q^2)\tan^2\frac{\theta}{2}\right] r \tag{G.4}$$

This is the celebrated Rosenbluth cross section.

[1] In this section, momenta denote four-vectors so that $q^2 \equiv q_\mu^2$. We explicitly denote the three-vectors by $\mathbf{q}$, etc.

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An alternative way to proceed is to rewrite the energy-conserving delta function in Eq. (G.1) as

$$\begin{aligned}
\frac{m^2}{p_0'}\delta(p_0 - p_0' - q_0) &= 2m^2\,\delta[(p_0 - q_0)^2 - p_0'^2] \\
&= 2m^2\,\delta[(p - q)^2 - p'^2] \\
&= 2m^2\,\delta(2p\cdot q - q^2) \\
&= m\,\delta\left(\nu - \frac{q^2}{2m}\right) \\
&= \frac{m}{\nu}\,\delta(1 - x)
\end{aligned} \tag{G.5}$$

Here

$$\nu \equiv \frac{p\cdot q}{m} \qquad ; \; x \equiv \frac{q^2}{2m\nu} \tag{G.6}$$

The quantity $\nu = \varepsilon_1 - \varepsilon_2$ is the electron energy loss in the lab, and x is the Bjorken scaling variable. Three-momentum conservation has been used in arriving at the second equality in Eq. (G.5) and the fact that this is elastic scattering so that $p^2 = p'^2 = -m^2$ in the third.

Note also that the combination

$$\begin{aligned}
\frac{q^2}{4m^2}\delta\left(\nu - \frac{q^2}{2m}\right) &= \frac{1}{2m}\frac{q^2}{2m\nu}\delta\left(1 - \frac{q^2}{2m\nu}\right) \\
&= \frac{1}{2m}\delta(1 - x)
\end{aligned} \tag{G.7}$$

Hence for elastic scattering from an isolated nucleon, the response surfaces are given by

$$\begin{aligned}
\frac{\nu}{m}W_2 &= \delta(1 - x)w_2(q^2) \\
\frac{2m}{m}W_1 &= \delta(1 - x)\bar{w}_1(q^2) \\
\bar{w}_1 &\equiv \frac{4m^2}{q^2}w_1(q^2) = (F_1 + 2mF_2)^2 \\
w_2 &= F_1^2 + \frac{q^2}{4m^2}(2mF_2)^2
\end{aligned} \tag{G.8}$$

If one now models the nucleus as a collection of non-interacting nucleons at rest, the nucleon cross sections can just be summed; equivalently, the structure functions take the form

$$\begin{aligned}
\frac{\nu}{m}W_2^{(A)} &= \delta(1 - x)[Z w_2^p(q^2) + N w_2^n(q^2)] \\
2W_1^{(A)} &= \delta(1 - x)[Z \bar{w}_1^p(q^2) + N \bar{w}_1^n(q^2)]
\end{aligned} \tag{G.9}$$

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This is the world's most naive model of the nucleus; however, it does have the following features to recommend it:

- It is completely covariant, assuming only that the nucleons are at rest in the lab frame and remain nucleons after the scattering. The nuclear response tensor has the correct Lorentz covariant structure;
- The nuclear current is conserved, and the structure of the nuclear response tensor reflects this fact;
- The nucleons can have *arbitrarily large* final four-momentum $p' = p - q$; the calculation still holds;
- When divided by the appropriate single-nucleon response functions, the nuclear response tensors exhibit *Bjorken scaling*, depending only on the variable ν through the Bjorken scaling variable x appearing in the factor $\delta(1 - x)$.

It is a simple matter to generalize the above to the situation where the target nucleon is moving with momentum $\mathbf{p}$. There are two changes that one has to consider:

1) From the definition of the initial flux as the number of particles crossing unit area transverse to the beam per unit time, one has

$$\begin{aligned} I_{\text{inc}} &= \frac{1}{\Omega}\mathbf{v}_{\text{rel}} \cdot \left(\frac{\mathbf{k}_1}{k_1}\right) \\ &= \frac{1}{\Omega}\left(\frac{\mathbf{k}_1}{k_1} - \frac{\mathbf{p}}{E}\right) \cdot \left(\frac{\mathbf{k}_1}{k_1}\right) \\ &= \frac{1}{\Omega}\frac{\sqrt{(p \cdot k_1)^2}}{Ek_1} \end{aligned} \tag{G.10}$$

This is exactly the same expression used previously in obtaining the invariant form of the cross section in Eq. (11.20). Hence it is appropriate to start from there.

2) Since the electron tensor is conserved, the terms in the nucleon tensor proportional to q_μ and q_ν can be discarded in the contraction of the two. The required replacements are therefore:

$$\begin{aligned} \eta_{\mu\nu}\delta_{\mu\nu} &\rightarrow \quad \eta_{\mu\nu}\delta_{\mu\nu} \\ \eta_{\mu\nu}\frac{p_\mu p_\nu}{m^2} &\rightarrow \quad \frac{1}{m^2}[2(p \cdot k_1)(p \cdot k_2) + (k_1 \cdot k_2)m^2] \\ &\quad = \frac{1}{m^2}\left[2(p \cdot k_1)^2 + q^2(p \cdot k_1) - \frac{1}{2}q^2 m^2\right] \end{aligned} \tag{G.11}$$

The first contraction, given entirely in electron variables, is unchanged. For a nucleon at rest in the lab frame with $p_\mu = (\mathbf{0}, im)$, the second contraction

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takes the previous form

$$\frac{1}{m^2}[2(p \cdot k_1)(p \cdot k_2) + (k_1 \cdot k_2)m^2] = 2\varepsilon_1\varepsilon_2 \cos^2\frac{\theta}{2} \tag{G.12}$$

For a moving nucleon, one simply evaluates Eqs. (G.11) for

$$p_\mu = (\mathbf{p}, iE) = (\mathbf{p}, i\sqrt{\mathbf{p}^2 + m^2}) \tag{G.13}$$

The cross section for scattering a massless Dirac electron from a Dirac nucleon moving with initial momentum $\mathbf{p}$ in the lab is thus given by

$$\left(\frac{d^2\sigma}{d\varepsilon_2 d\Omega_2}\right)_{\text{mov nucl}} = \sigma_M \frac{m\varepsilon_1}{\sqrt{(k_1 \cdot p)^2}}\, 2m\,\delta(2p \cdot q - q^2)\left\{2w_1(q^2)\tan^2\frac{\theta}{2}\right.$$
$$\left. + w_2(q^2)\frac{1}{2m^2\varepsilon_1\varepsilon_2\cos^2\theta/2}\left[2(p \cdot k_1)^2 + q^2(p \cdot k_1) - \frac{1}{2}q^2m^2\right]\right\} \tag{G.14}$$

Now suppose the nucleus is modeled as a collection of non-interacting nucleons where there are $n(\mathbf{p}^2)\,d^3p$ nucleons moving with momentum between $\mathbf{p}$ and $\mathbf{p}+d\mathbf{p}$. This could, for example, be the momentum distribution for nucleons in an independent-particle shell model[2]

$$n^{(\alpha)}(\mathbf{p}^2) = \sum_i |\phi_i^{(\alpha)}(\mathbf{p})|^2 \qquad ;\ \alpha = p, n \tag{G.15}$$

One can again just add the individual cross sections.

The third modification required for this case, in addition to the previous two, is as follows:

3) The expression for the energy-conserving delta function now takes the form

$$\begin{aligned}
2m\int n(\mathbf{p}^2)\,d^2p_\perp\,dp_\parallel\,\delta(2p \cdot q - q^2) &= \frac{2m}{2q}\int n(\mathbf{p}^2)\,d^2p_\perp\,dW\left(\frac{\partial p_\parallel}{\partial W}\right)\delta(W) \\
&= \frac{m}{q}\int n(\mathbf{p}_\perp^2 + p_\parallel^2)\,d^2p_\perp\left(\frac{\partial p_\parallel}{\partial W}\right) \\
W \equiv p_\parallel + \frac{\omega}{q}(\mathbf{p}_\perp^2 + p_\parallel^2 + m^2)^{1/2} - \frac{q_\mu^2}{2q} &= 0 \\
\frac{\partial W}{\partial p_\parallel} &= \frac{Eq + \omega p_\parallel}{Eq}
\end{aligned} \tag{G.16}$$

The equation $W = 0$ determines $p_\parallel(\mathbf{p}_\perp^2, q, \omega)$ where now $q \equiv |\mathbf{q}|$ and $\omega = -q_0 = \varepsilon_1 - \varepsilon_2$.

[2] Closed shells are assumed and hence the distribution is a function of $\mathbf{p}^2$.

The resulting nuclear cross section is given by an incoherent sum

$$\left(\frac{d^2\sigma}{d\varepsilon_2 d\Omega_2}\right)^{(A)} = \sigma_M \frac{m}{q} \sum_{\alpha=n,p} \int n^{(\alpha)}(\mathbf{p}_\perp^2 + p_\parallel^2)\, d^2 p_\perp$$

$$\times \left(\frac{Eq}{Eq + \omega p_\parallel}\right) \frac{m\varepsilon_1}{\sqrt{(k_1 \cdot p)^2}} \left\{ 2w_1^{(\alpha)}(q^2) \tan^2 \frac{\theta}{2} \right.$$

$$\left. + w_2^{(\alpha)}(q^2) \frac{1}{2m^2 \varepsilon_1 \varepsilon_2 \cos^2 \theta/2} \left[2(p \cdot k_1)^2 + q^2 (p \cdot k_1) - \frac{1}{2} q^2 m^2 \right] \right\} \quad \text{(G.17)}$$

Here $p_\parallel$ is again determined from $W = 0$.

These are exact results within this model. The nuclear current is again conserved, and the nucleon can be scattered through arbitrarily large (q, ω). While achieving these goals, it is important to note that the kinematics for electron scattering on a free nucleon have been employed, as well as the dispersion relation for a free initial nucleon in Eq. (G.13). Final-state interactions and modification of the initial nucleon spinors have been neglected.

To obtain some insight into this answer, specialize to the case where $|\mathbf{p}/E| = |(\mathbf{v}/c)_{\text{initial}}| \ll 1$. To leading order, the coefficients in the cross section reduce to those in Eq. (G.8), and the only change is to introduce a new quantity into the previous y-scaling analysis in Eq. (23.36)

$$\tilde{y} \equiv \frac{m\omega}{q} - \frac{q_\mu^2}{2q} \quad \text{(G.18)}$$

This is energy–momentum conservation to order $(v/c)^2_{\text{initial}}$. Note again, (q, ω) can be arbitrarily large as long as the nucleon remains a nucleon. y-scaling is discussed in much more detail in the review article [Da90], and also in [Do99].

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