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Asymptotics and Uniqueness of Travelling Waves for Non-Monotone Delayed Systems on 2D Lattices

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Abstract. We establish asymptotics and uniqueness (up to translation) of travelling waves for delayed 2D lattice equations with non-monotone birth functions. First, with the help of Ikehara's Theorem, the *a priori* asymptotic behavior of travelling wave is exactly derived. Then, based on the obtained asymptotic behavior, the uniqueness of the traveling waves is proved. These results complement earlier results in the literature.

1 Introduction

The discrete growth of mature populations for a single species in a patchy environment or the dynamic distribution of myelinated axons in nerve systems are usually described as time-differential lattice equations. The study of the structure of solutions, particularly the travelling wave solutions and the spreading speeds, has become one of the hot spots of research in this field recently. Chan, Mallet, and Vleck [3] studied the existence of travelling waves for 2D bistable lattice systems

$$\frac{dw_{i,j}(t)}{dt} = D\big[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)\big] + f(w_{i,j}(t))$$

with a bistable birth-rate function f. Later, Cheng, Li, and Wang [8] considered the delayed 2D lattice system

$$(1.1)\quad \frac{dw_{i,j}(t)}{dt} = D\big[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)\big] - dw_{i,j}(t) + \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} \beta(l)\gamma(q)b\big(w_{i+l,j+q}(t-r)\big),$$

and by applying the method developed in [17,19] for 1D lattice differential equations, they showed that the spreading speed coincides with the minimal wave speed of the

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2D lattice system (1.1) in the case where the birth function $b(u)$ is monotone. In [9], they further investigated the stability of travelling waves for the system

$$\frac{dw_{i,j}(t)}{dt} = D\big[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)\big] - dw_{i,j}(t) + b\big(w_{i,j}(t-r)\big)$$

with monotone birth function b. However, the asymptotics and uniqueness of travelling waves for (1.1) with monotone or non-monotone nonlinearity still remain open problems. To study these problems is the main purpose of this paper.

Throughout this paper, we define a travelling wave of (1.1) with speed c in direction θ to be a nonnegative bounded solution in the form $w_{i,j}(t) = \phi(i\cos\theta + j\sin\theta + ct)$ satisfying $\phi(-\infty) = 0$ and $\liminf_{\xi\to+\infty}\phi(\xi) > 0$. Substituting $w_{i,j}(t) = \phi(i\cos\theta + j\sin\theta + ct)$ into (1.1), we have the wave profile equation

(1.2)
$$c\phi'(\xi) = D\big[\phi(\xi+\cos\theta) + \phi(\xi-\cos\theta) + \phi(\xi+\sin\theta) + \phi(\xi-\sin\theta) - 4\phi(\xi)\big] - d\phi(\xi) + \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)b\big(\phi(\xi - l\cos\theta - q\sin\theta - cr)\big).$$

Clearly, when the wave direction is $\theta = 0$ or $\theta = \frac{\pi}{2}$, (1.2) can be reduced to a special wave profile equation of the 1D lattice system

$$\frac{dw_i(t)}{dt} = D[w_{i+1}(t) + w_{i-1}(t) - 2w_i(t)] - dw_i(t) + \sum_{j\in\mathbb{Z}}\beta(i-j)b(w_j(t-r)), \quad i \in \mathbb{Z}, \tag{1.3}$$

which was widely investigated in [12, 13, 15, 17–19].

The uniqueness of monotone travelling waves for various evolution systems has been established; for example, see [2, 5–7, 10, 16, 18] and the references therein. The proof of uniqueness strongly relies on the monotonicity of travelling waves. It seems very difficult to extend the techniques in those papers to the non-monotone evolution systems, because the wave profile may lose the monotonicity (sometimes it is impossible; see, *e.g.,* [14]). Regarding the uniqueness of travelling waves without preconditions, the corresponding study is very limited, see, *e.g.,* [4, 11, 13]. For the noncritical waves case, Diekmann and Kaper [11] and Fang *et al.* [13] studied a nonlinear convolution equation and equation (1.3), respectively. Since the technique developed in [11, 13] seems hard to extend to solving the uniqueness of critical waves for (1.1), we need a new approach to treat the lattice equation (1.1) for the critical waves case. Notice that Carr and Chmaj [4] considered the nonlocal dispersion equation of Fisher-KPP type

$$u_t - J * u + u = f(u),$$

where $J*u = \int_{\mathbb{R}} J(x-y)u(y,t)dy$ is the convolution with a Gaussian-like kernel $J(x)$, and $f(u)$ is monostable. With the help of the established *a priori* asymptotic behavior of the travelling waves and Ikehara's Tauberian theorem for Laplace transforms, they proved that all noncritical and critical travelling waves $\phi(x+ct)$ are unique up to a shift. Inspired by [4], we apply this technique to the 2D discrete system (1.1) and show that all noncritical and critical travelling waves $\phi(i\cos\theta + j\sin\theta + ct)$ for the lattice equations (1.1) are also unique up to translations.

The rest of this paper is organized as follows. In Section 2, the asymptotic behavior of travelling waves of (1.1) for $c \geq c_*(\theta)$ is derived with the help of Ikehara's theorem. In Section 3, the uniqueness of travelling waves is established.

2 Asymptotic Behavior of Travelling Waves

In this section, we show the asymptotic behavior of travelling waves of (1.1) for $c \geq c_*(\theta)$ with the help of Ikehara's theorem.

Assume that the function $b(u)$ is differentiable at $u = 0$. Define the characteristic equation

$$\Delta(c,\lambda) = c\lambda - D(e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4) \\ - b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda(l\cos\theta + q\sin\theta + cr)} + d,$$

where c is regarded as a parameter. We make the following assumptions on functions β, γ, and b.

(H1) $\beta(l) = \beta(-l) \geq 0$ and $\gamma(l) = \gamma(-l) \geq 0$ for any $l \in \mathbb{Z}$; $\sum_{l\in\mathbb{Z}}\beta(l) = \sum_{q\in\mathbb{Z}}\gamma(q) = 1$; there exists $\lambda^\sharp > 0$ such that

$$\chi(\lambda) =: \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{\lambda(l+q)}$$

is convergent when $\lambda \in [0, \lambda^\sharp)$ and $\lim_{\lambda\to\lambda^\sharp}\chi(\lambda) = +\infty$, where $\lambda^\sharp$ may be $+\infty$.

(H2) b is continuous from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $b'(0) > d$; there exist $a > 0$, $\delta > 0$, and $\sigma > 1$ such that $b(u) \geq b'(0)u - au^\sigma$ for all $u \in [0,\delta]$.

(H3) For all $u_1, u_2 \geq 0$, $|b(u_1) - b(u_2)| \leq b'(0)|u_1 - u_2|$.

According to (H1), for any $c > 0$, $\Delta(c,\lambda)$ is well defined on $[0,\lambda^\sharp)$. We have the following lemma, whose proof is similar to that of [8, Lemma 4.2].

Lemma 2.1 *Assume that* (H1) *holds and $b'(0) > d$. Then, there exist a unique pair of $c_*(\theta) > 0$ and $\lambda_* > 0$ for any fixed $\theta \in [0, \frac{\pi}{2}]$ such that the following assertions hold.*

(i) $\Delta(c_*(\theta), \lambda_*) = 0$, $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*(\theta),\lambda=\lambda_*} = 0$.

(ii) *For any $c \in (0, c_*(\theta))$ and $\lambda \in [0,\lambda^\sharp)$, $\Delta(c,\lambda) < 0$.*

(iii) *For any* $c \geq c_*(\theta)$, $\Delta(c,\lambda) = 0$ *has two positive roots* $0 < \lambda_1 \leq \lambda_2 < \lambda^\sharp$. *Moreover, if* $c > c_*(\theta)$, $\lambda_1 < \lambda_2$ *and* $\Delta(c,\lambda) > 0$ *for any* $\lambda \in (\lambda_1, \lambda_2)$*; if* $c = c_*(\theta)$*, then* $\lambda_1 = \lambda_2 = \lambda_*$.

We recall a version of Ikehara's Theorem.

Lemma 2.2 ([4]) *Let* $F(\lambda) = \int_0^{+\infty} u(x)e^{-\lambda x}dx$*, with* u *being a positive decreasing function. Assume that* $F(\lambda)$ *has the representation*

$$F(\lambda) = \frac{h(\lambda)}{(\lambda+\alpha)^{k+1}},$$

where $k > -1$ *and* h *is analytic in the strip* $-\alpha \leq \mathrm{Re}\lambda < 0$*. Then*

$$\lim_{x\to+\infty} \frac{u(x)}{x^k e^{-\alpha x}} = \frac{h(-\alpha)}{\Gamma(\alpha+1)} > 0.$$

We now state the asymptotic behavior of the travelling waves for (1.1) as follows.

Theorem 2.3 *Assume that* (H1) *and* (H2) *hold. Let* $\phi(i\cos\theta + j\sin\theta + ct)$ *be a travelling wave of* (1.1) *with* $\phi(-\infty) = 0$*. Then*

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{e^{\lambda_1 \xi}} \text{ exists for } c > c_*(\theta), \qquad \lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi| e^{\lambda_* \xi}} \text{ exists for } c = c_*(\theta).$$

Proof First, claim that ϕ is positive. Suppose on the contrary that there exists $\xi_1 \in \mathbb{R}$ such that $\phi(\xi_1) = 0$. Since ϕ is a nonnegative bounded travelling wave with $\phi(-\infty) = 0$ and $\liminf_{\xi\to+\infty}\phi(\xi) > 0$, $\xi_0 := \sup\{\xi \in \mathbb{R} \mid \phi(\xi) = 0\}$ is well defined and $\phi(\xi_0) = \phi'(\xi_0) = 0$. Thus,

$$\begin{aligned}
0 &= c\phi'(\xi_0)\\
&= D\big[\phi(\xi_0+\cos\theta) + \phi(\xi_0-\cos\theta) + \phi(\xi_0+\sin\theta) + \phi(\xi_0-\sin\theta) - 4\phi(\xi_0)\big]\\
&\quad - d\phi(\xi_0) + \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} \beta(l)\gamma(q) b\big(\phi(\xi_0 - l\cos\theta - q\sin\theta - cr)\big)\\
&\geq D\phi(\xi_0+\cos\theta) + D\phi(\xi_0+\sin\theta) \geq 0,
\end{aligned}$$

which implies that $\phi(\xi_0+\cos\theta) = 0$ and $\phi(\xi_0+\sin\theta) = 0$. This contradicts the definition of ξ_0 for any $\theta \in [0, \frac{\pi}{2}]$.

Second, claim that there exists $\rho > 0$ such that $\phi(\xi) = O(e^{-\rho\xi})$ as $\xi \to -\infty$. Define $f(\phi)(x) := \phi(\xi+\cos\theta) + \phi(\xi-\cos\theta) + \phi(\xi+\sin\theta) + \phi(\xi-\sin\theta) - 4\phi(\xi)$. Since $b'(0) > d$ and $\sum_{l\in\mathbb{Z}}\beta(l) = \sum_{q\in\mathbb{Z}}\gamma(q) = 1$, there exist ϵ_0, N_1, and N_2 such that

$$A := (1-\epsilon_0)b'(0)\sum_{|l|\leq N_1}\sum_{|q|\leq N_2}\beta(l)\gamma(q) - d > 0.$$

For such $\epsilon_0 > 0$, there exists $\delta_0 > 0$ such that $b(u) \geq (1-\epsilon_0)b'(0)u$ for any $u \in [0,\delta_0]$ according to (H2). Since $\phi(-\infty) = 0$, there exists $M > 0$ such that $\phi(\xi) < \delta_0$ for

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any $\xi \le -M$. Integrating (1.2) from η to ξ with $\xi \le -M - N_1 - N_2 + cr$, it follows that

$$
\begin{aligned}
(2.1)\quad & c[\phi(\xi) - \phi(\eta)] \\
&= D\int_\eta^\xi f(\phi)(y)dy - d\int_\eta^\xi \phi(y)dy \\
&\quad + \int_\eta^\xi \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} \beta(l)\gamma(q)b(\phi(y - l\cos\theta - q\sin\theta - cr))dy \\
&\ge D\int_\eta^\xi f(\phi)(y)dy - d\int_\eta^\xi \phi(y)dy \\
&\quad + \int_\eta^\xi \sum_{|l|\le N_1}\sum_{|q|\le N_2} \beta(l)\gamma(q)b(\phi(y - l\cos\theta - q\sin\theta - cr))dy \\
&\ge D\int_\eta^\xi f(\phi)(y)dy - d\int_\eta^\xi \phi(y)dy \\
&\quad + b'(0)(1-\epsilon_0)\int_\eta^\xi \sum_{|l|\le N_1}\sum_{|q|\le N_2} \beta(l)\gamma(q)\phi(y - l\cos\theta - q\sin\theta - cr)dy \\
&= D\int_\eta^\xi f(\phi)(y)dy + A\int_\eta^\xi \phi(y)dy \\
&\quad + b'(0)(1-\epsilon_0)\int_\eta^\xi \sum_{|l|\le N_1}\sum_{|q|\le N_2} \beta(l)\gamma(q) \\
&\quad \times [\phi(y - l\cos\theta - q\sin\theta - cr) - \phi(y)]dy.
\end{aligned}
$$

Since $\phi(\xi)$ is differentiable, we have

$$
\begin{aligned}
&\int_\eta^\xi f(\phi)(y)dy \\
&\quad = \int_\eta^\xi \int_0^{\cos\theta} \phi'(y+\tau)d\tau dy + \int_\eta^\xi \int_0^{-\cos\theta} \phi'(y+\tau)d\tau dy \\
&\qquad + \int_\eta^\xi \int_0^{\sin\theta} \phi'(y+\tau)d\tau dy + \int_\eta^\xi \int_0^{-\sin\theta} \phi'(y+\tau)d\tau dy \\
&\quad = \int_0^{\cos\theta} [\phi(\xi+\tau) - \phi(\eta+\tau)]d\tau + \int_0^{-\cos\theta} [\phi(\xi+\tau) - \phi(\eta+\tau)]d\tau \\
&\qquad + \int_0^{\sin\theta} [\phi(\xi+\tau) - \phi(\eta+\tau)]d\tau + \int_0^{-\sin\theta} [\phi(\xi+\tau) - \phi(\eta+\tau)]d\tau.
\end{aligned}
$$

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Similarly, we know that

$$\begin{aligned}\int_\eta^\xi \sum_{|l|\le N_1}\sum_{|q|\le N_2}&\beta(l)\gamma(q)\big[\phi(y-l\cos\theta-q\sin\theta-cr)-\phi(y)\big]\,dy\\&=-\sum_{|l|\le N_1}\sum_{|q|\le N_2}\beta(l)\gamma(q)(l\cos\theta+q\sin\theta+cr)\\&\quad\times\int_0^1\Big[\phi\big(\xi-\tau(l\cos\theta+q\sin\theta+cr)\big)-\phi\big(\eta-\tau(l\cos\theta+q\sin\theta+cr)\big)\Big]\,d\tau.\end{aligned}$$

Letting $\eta\to-\infty$ in (2.1), we obtain

$$\begin{aligned}A\int_{-\infty}^\xi \phi(y)dy&\le c\phi(\xi)-D\Big[\int_0^{\cos\theta}\phi(\xi+\tau)d\tau+\int_0^{-\cos\theta}\phi(\xi+\tau)d\tau\\&\qquad+\int_0^{\sin\theta}\phi(\xi+\tau)d\tau+\int_0^{-\sin\theta}\phi(\xi+\tau)d\tau\Big]\\&\quad+b'(0)(1-\epsilon_0)\sum_{|l|\le N_1}\sum_{|q|\le N_2}\beta(l)\gamma(q)|l\cos\theta+q\sin\theta+cr|g_{l,q}(\phi)(\xi),\end{aligned}\tag{2.2}$$

where

$$g_{l,q}(\phi)(\xi)=\int_0^1\phi\big(\xi-\tau(l\cos\theta+q\sin\theta+cr)\big)\,d\tau.$$

It then follows from (H1) that $\int_{-\infty}^\xi\phi(y)dy<+\infty$. Letting $\Phi(\xi)=\int_{-\infty}^\xi\phi(y)dy$ and integrating (2.2) from $-\infty$ to ξ, we have

$$\begin{aligned}A\int_{-\infty}^\xi \Phi(y)dy&\le c\Phi(\xi)-D\Big[\int_0^{\cos\theta}\Phi(\xi+\tau)d\tau+\int_0^{-\cos\theta}\Phi(\xi+\tau)d\tau\\&\qquad+\int_0^{\sin\theta}\Phi(\xi+\tau)d\tau+\int_0^{-\sin\theta}\Phi(\xi+\tau)d\tau\Big]\\&\quad+b'(0)(1-\epsilon_0)\sum_{|l|\le N_1}\sum_{|q|\le N_2}\beta(l)\gamma(q)|l\cos\theta+q\sin\theta+cr|g_{l,q}(\Phi)(\xi)\\&\le\varrho\Phi(\xi+\kappa)\end{aligned}$$

for some $\kappa>0$ and $\varrho>0$ according to the monotonicity of $\Phi(\xi)$. Letting $\varpi>0$ such that $\varrho<A\varpi$, and letting $\xi\le -M-N_1-N_2+cr$, it then follows that

$$\Phi(\xi-\varpi)\le\frac{1}{\varpi}\int_{\xi-\varpi}^\xi\Phi(y)dy\le\frac{1}{\varpi}\int_{-\infty}^\xi\Phi(y)dy\le\frac{\varrho}{A\varpi}\Phi(\xi+\kappa).$$

Define $h(\xi) = \Phi(\xi)e^{-\rho\xi}$, where $\rho = \frac{1}{\kappa+\varpi} \ln \frac{A\varpi}{\varrho} > 0$. Hence,

$$h(\xi - \varpi) = \Phi(\xi - \varpi)e^{-\rho(\xi-\varpi)} \leq \frac{\varrho}{A\varpi} e^{\rho(\kappa+\varpi)} h(\xi + \kappa) = h(\xi + \kappa),$$

which implies that h is bounded. Therefore, $\Phi(\xi) = O(e^{\rho\xi})$ when $\xi \to -\infty$. Integrating (1.2) from $-\infty$ to ξ, it follows from (H3) that

$$\begin{aligned} c\phi(\xi) &= D\big[\Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi)\big] \\ &\quad - d\Phi(\xi) + \int_{-\infty}^{\xi} \sum_{l\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \beta(l)\gamma(q) b\big(\phi(\xi - l\cos\theta - q\sin\theta - cr)\big) \\ &\leq f(\Phi)(\xi) - d\Phi(\xi) + b'(0) \sum_{l\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \beta(l)\gamma(q)\Phi(\xi - l\cos\theta - q\sin\theta - cr). \end{aligned}$$

Thus, we prove $\phi(\xi) = O(e^{\rho\xi})$ when $\xi \to -\infty$.

By the above argument, for any $0 < \mathrm{Re}\lambda < \rho$ we can now define the two-sided Laplace transform of ϕ:

$$L(\lambda) \equiv \int_{\mathbb{R}} \phi(y)e^{-\lambda y}\,dy.$$

We claim that $L(\lambda)$ is analytic for any $\mathrm{Re}\lambda \in (0, \lambda_1)$ and has a singularity at $\lambda = \lambda_1$. Indeed, since

$$\int_{\mathbb{R}} e^{-\lambda y} f(\phi)(y)\,dy = L(\lambda)(e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4)$$

and

$$\begin{aligned} &c\phi'(\xi) - Df(\phi)(\xi) + d\phi(\xi) - b'(0) \sum_{l\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \beta(l)\gamma(q)\phi(\xi - l\cos\theta - q\sin\theta - cr) \\ &\quad = \sum_{l\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \beta(l)\gamma(q)\big[-b'(0)\phi(\xi - l\cos\theta - q\sin\theta - cr) \\ &\qquad\qquad + b(\phi(\xi - l\cos\theta - q\sin\theta - cr))\big] \\ &\quad =: R(\phi)(\xi), \end{aligned}$$

we obtain

$$\Delta(c,\lambda)L(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda y} R(\phi)(y)\,dy. \tag{2.3}$$

It is easily seen that the left-hand side of (2.3) is analytic for $\lambda \in (0, \nu)$, where $\nu = \min\{\rho, \lambda^\sharp\}$. According to (H3), for any $\bar{u} > 0$, there exists $\bar{a} > 0$ such that $b(u) \geq b'(0)u - \bar{a}u^\sigma, \forall\, u \in [0, \bar{u}]$, where $\bar{a} := \max\{a, \delta^{-\sigma} \max_{u\in[\delta,\bar{u}]}\{b'(0)u - b(u)\}\}$. Thus,

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$-\bar{a}\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)\phi^{\sigma}(\xi - l\cos\theta - q\sin\theta - cr) \le R(\phi)(\xi) \le 0$. Choose $\nu_1 > 0$ such that $\frac{\nu_1}{\sigma-1} < \rho$ and $\nu + \nu_1 < \lambda^{\sharp}$. Then for any $\lambda \in (0, \nu+\nu_1)$, we have

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty} e^{-\lambda y} R(\phi)(y)dy\right| \\
&\quad \le \bar{a}\int_{-\infty}^{\infty} e^{-\lambda y}\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)\phi^{\sigma}(y - l\cos\theta - q\sin\theta - cr)dy \\
&\quad = \bar{a}\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda(l\cos\theta+q\sin\theta+cr)}\int_{-\infty}^{\infty} e^{-\lambda y}\phi^{\sigma}(y)dy \\
&\quad \le \bar{a}\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda(l\cos\theta+q\sin\theta+cr)}L(\lambda-\nu_1)\Big(\sup_{\xi\in\mathbb{R}}\phi(\xi)e^{-\frac{\nu_1\xi}{\sigma-1}}\Big)^{\sigma-1} \\
&\quad < +\infty.
\end{aligned}
$$

We now use a property of the Laplace transform ([20, p. 58]). Since $\phi > 0$, there exists a real B such that $L(\lambda)$ is analytic for $0 < \mathrm{Re}\lambda < B$, and $L(\lambda)$ has a singularity at $\lambda = B$. Hence for $c \ge c_*(\theta)$, $L(\lambda)$ is analytic for $\lambda \in (0,\lambda_1)$ and $L(\lambda)$ has a singularity at $\lambda = \lambda_1$.

We rewrite (2.3) as

$$\int_{-\infty}^{0}\phi(\theta)e^{-\lambda\theta}d\theta = \frac{\int_{\mathbb{R}} e^{-\lambda\theta}R(\phi)(\theta)d\theta}{\Delta(c,\lambda)} - \int_{0}^{\infty}\phi(\theta)e^{-\lambda\theta}d\theta.$$

Note that $\int_0^{\infty}\phi(\theta)e^{-\lambda\theta}d\theta$ is analytic for $\mathrm{Re}\lambda > 0$. Also, $\Delta(c,\lambda) = 0$ does not have any solution with $\mathrm{Re}\lambda = \lambda_1$ other than $\lambda = \lambda_1$. Indeed, let $\lambda = \lambda_1 + i\tilde{\lambda}$, then

$$
\begin{aligned}
0 = c\lambda_1 - D\big[&e^{\lambda_1\cos\theta}\cos(\tilde{\lambda}\cos\theta) + e^{-\lambda_1\cos\theta}\cos(\tilde{\lambda}\cos\theta) \\
&+ e^{\lambda_1\sin\theta}\cos(\tilde{\lambda}\sin\theta) + e^{-\lambda_1\sin\theta}\cos(\tilde{\lambda}\sin\theta) - 4\big] \\
&- b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda_1(l\cos\theta+q\sin\theta+cr)}\cos(\tilde{\lambda}(l\cos\theta+q\sin\theta+cr)) + d
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
0 = c\tilde{\lambda} - D\big[&e^{\lambda_1\cos\theta}\sin(\tilde{\lambda}\cos\theta) - e^{-\lambda_1\cos\theta}\sin(\tilde{\lambda}\cos\theta) \\
&+ e^{\lambda_1\sin\theta}\sin(\tilde{\lambda}\sin\theta) - e^{-\lambda_1\sin\theta}\sin(\tilde{\lambda}\sin\theta)\big] \\
&+ b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda_1(l\cos\theta+q\sin\theta+cr)}\sin\big(\tilde{\lambda}(l\cos\theta+q\sin\theta+cr)\big).
\end{aligned}
\tag{2.5}
$$

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Thus, according to (2.4) and $\Delta(c,\lambda_1)=0$, we have

$$\begin{aligned}0 = D\big[e^{\lambda_1\cos\theta}(1-\cos(\tilde\lambda\cos\theta)) + e^{-\lambda_1\cos\theta}(1-\cos(\tilde\lambda\cos\theta)) \\ + e^{\lambda_1\sin\theta}(1-\cos(\tilde\lambda\sin\theta)) + e^{-\lambda_1\sin\theta}(1-\cos(\tilde\lambda\sin\theta))\big] \\ + b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda_1(l\cos\theta+q\sin\theta+cr)} \\ \times\big(1-\cos(\tilde\lambda(l\cos\theta+q\sin\theta+cr))\big),\end{aligned}$$

which implies that

$$\cos(\tilde\lambda\cos\theta)=\cos(\tilde\lambda\sin\theta))=\cos(\tilde\lambda(l\cos\theta+q\sin\theta+cr))=1. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain $\tilde\lambda=0$.

Assume that $\phi(\xi)$ is increasing for large $-\xi>0$. Then we can choose a translation of ϕ such that it is increasing for $\xi<0$. Letting $u(\xi)=\phi(-\xi)$ and

$$\begin{aligned}T(u)(\xi) := \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)\big[-b'(0)u(\xi+l\cos\theta+q\sin\theta+cr) \\ + b(u(\xi+l\cos\theta+q\sin\theta+cr))\big],\end{aligned}$$

it is clear that $u(\xi)$ is decreasing $\xi>0$ and

$$\int_0^\infty u(\theta)e^{\lambda\theta}d\theta=\frac{\int_{\mathbb{R}}e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c,\lambda)}-\int_{-\infty}^0u(\theta)e^{\lambda\theta}d\theta:=\frac{h(\lambda)}{(\lambda-\lambda_1)^{k+1}},$$

where $k=0$ for $c>c_*(\theta)$, and $k=1$ for $c=c_*(\theta)$, and

$$h(\lambda)=\frac{(\lambda-\lambda_1)^{k+1}\int_{\mathbb{R}}e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c,\lambda)}-(\lambda-\lambda_1)^{k+1}\int_{-\infty}^0u(\theta)e^{\lambda\theta}d\theta.$$

By Lemma 2.1, $\lim_{\lambda\to\lambda_1}h(\lambda)$ exists. Therefore, $h(\lambda)$ is analytic for all $0<\mathrm{Re}\lambda\le\lambda_1$. Then Lemma 2.2 implies that

$$\lim_{\xi\to+\infty}\frac{u(\xi)}{\xi^ke^{-\lambda_1\xi}}\ \text{exists},\quad i.e.,\quad \lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|^ke^{\lambda_1\xi}}\quad\text{exists},$$

that is,

$$\lim_{\xi\to-\infty}\frac{\phi(\xi)}{e^{\lambda_1\xi}}\quad\text{exists for}\quad c>c_*(\theta),\qquad \lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|e^{\lambda_*\xi}}\quad\text{exists for}\quad c=c_*(\theta).$$

Now we assume that $\phi(\xi)$ is not monotone for large $-\xi>0$. Letting $p=\frac{4D+d}{c}$ and $\widehat\phi(\xi)=\phi(\xi)e^{p\xi}>0$, it follows that

$$\begin{aligned}c\widehat\phi'(\xi)=D\big[\widehat\phi(\xi+\cos\theta)e^{-p\cos\theta}+\widehat\phi(\xi-\cos\theta)e^{p\cos\theta} \\ +\widehat\phi(\xi+\sin\theta)e^{-p\sin\theta}+\widehat\phi(\xi-\sin\theta)e^{p\sin\theta}\big] \\ +(cp-4D-d)\widehat\phi(\xi) \\ +\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)b(\phi(\xi-l\cos\theta-q\sin\theta-cr))e^{p\xi},\end{aligned}$$

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which implies that $\widehat{\phi}'(\xi) > 0$ for any $\xi \in \mathbb{R}$. Letting $\widehat{u}(\xi) = \widehat{\phi}(-\xi)$, it is obvious that $\widehat{u}(\xi)$ is decreasing on $\xi > 0$. Let $\widehat{L}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\widehat{\phi}(\xi)d\xi$. Noting that $\widehat{L}(\lambda) = L(\lambda - p)$ and repeating the above argument, we have

$$\lim_{\xi\to+\infty} \frac{\widehat{u}(\xi)}{\xi^k e^{-(p+\lambda_1)\xi}} = \lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|^k e^{\lambda_1\xi}} \quad \text{exists.}$$

Thus, it follows that

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{e^{\lambda_1\xi}} \quad \text{exists for} \quad c > c_*(\theta), \qquad \lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|e^{\lambda_*\xi}} \quad \text{exists for} \quad c = c_*(\theta).$$

This completes the proof. ■

3 Uniqueness of Travelling Waves

In this section, we investigate the uniqueness of travelling waves for (1.1).

Theorem 3.1 *Assume that* (H1)–(H3) *hold and the functions β and γ are compact supported. For any given $\theta \in [0, \frac{\pi}{2}]$, let φ, ψ be two travelling waves of* (1.1) *with direction θ and speed $c \geq c_*(\theta)$. Then ϕ is a translation of ψ; more precisely, there exists $\bar{\xi} \in \mathbb{R}$ such that $\phi(\xi) = \psi(\xi + \bar{\xi})$.*

Proof Let φ, ψ be two travelling waves for $c \geq c_*(\theta)$. According to Theorem 2.3, there exist some positive numbers θ_1 and θ_2 such that

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \omega_1^k \quad \text{and} \quad \lim_{\xi\to-\infty} \frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \omega_2^k,$$

where $k = 0$ for $c > c_*(\theta)$, and $k = 1$ for $c = c_*(\theta)$. For $\epsilon > 0$, define

$$w(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{e^{\lambda_1\xi}} \qquad \text{for } c > c_*(\theta),$$
$$w_\epsilon(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{(\epsilon|\xi| + 1)e^{\lambda_*\xi}} \qquad \text{for} \quad c = c_*(\theta),$$

where $\bar{\xi} = \frac{1}{\lambda_1} \ln \frac{\omega_1^k}{\omega_2^k}$. Then $w(\pm\infty) = 0$ and $w_\varepsilon(\pm\infty) = 0$.

First, we consider $c > c_*(\theta)$. Since $w(\xi)$ is continuous and $w(\pm\infty) = 0$, $\sup_{\xi\in\mathbb{R}}\{w(\xi)\}$ and $\inf_{\xi\in\mathbb{R}}\{w(\xi)\}$ are finite. Without loss of generality, we assume

$\sup_{\xi\in\mathbb{R}}\{w(\xi)\} \geq |\inf_{\xi\in\mathbb{R}}\{w(\xi)\}|$ (otherwise, we may take $w(\xi) := \frac{\psi(\xi+\bar{\xi})-\phi(\xi)}{e^{\lambda_1\xi}}$). If $w(\xi) \not\equiv 0$, there exists ξ_0 such that

$$w(\xi_0) = \max_{\xi\in\mathbb{R}}\{w(\xi)\} = \sup_{\xi\in\mathbb{R}}\{w(\xi)\} > 0 \quad \text{and} \quad w'(\xi_0) = 0.$$

We claim that $w(\xi_0 \pm \cos\theta) = w(\xi)$ for any $\theta \in [0, \frac{\pi}{2}]$. If this does not hold, either $w(\xi_0 + \cos\theta_0) < w(\xi_0)$ or $w(\xi_0 - \cos\theta_0) < w(\xi_0)$ for some $\theta_0 \in [0, \frac{\pi}{2}]$. According to (1.2) and (H3), we have

$$\begin{aligned} 0 = cw'(\xi_0) &= -c\lambda_1 w(\xi_0) + D\Big[w(\xi_0+\cos\theta_0)e^{\lambda_1\cos\theta_0} + w(\xi_0-\cos\theta_0)e^{-\lambda_1\cos\theta_0} \\ &\qquad + w(\xi_0+\sin\theta_0)e^{\lambda_1\sin\theta_0} + w(\xi_0-\sin\theta_0)e^{-\lambda_1\sin\theta_0} - 4w(\xi_0)\Big] \\ &\quad - dw(\xi_0) + \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)\Big[b\big(\phi(\xi_0 - l\cos\theta_0 - q\sin\theta_0 - cr)\big) \\ &\qquad - b\big(\psi(\xi_0+\bar{\xi} - l\cos\theta_0 - q\sin\theta_0 - cr)\big)\Big]e^{-\lambda_1\xi_0} \\ &< w(\xi_0)\big[-c\lambda_1 + D(e^{\lambda_1\cos\theta_0} + e^{-\lambda_1\cos\theta_0} + e^{\lambda_1\sin\theta_0} + e^{-\lambda_1\sin\theta_0} - 4) \\ &\qquad - d + b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda_1(l\cos\theta_0 + q\sin\theta_0 + cr)}\big] \\ &= -w(\xi_0)\Delta(c,\lambda_1) = 0, \end{aligned}$$

which is a contradiction. Again by bootstrapping, $w(\xi_0 \pm j\cos\theta) = w(\xi_0)$ for all $j \in \mathbb{Z}$ and $\theta \in [0, \frac{\pi}{2}]$, and since $w(+\infty) = 0$, we get $\phi(\xi) \equiv \psi(\xi+\bar{\xi})$ for $\xi \in \mathbb{R}$, which contradicts $w(\xi) \not\equiv 0$.

Next, we consider $c = c_*(\theta)$. Similarly to the above argument, we assume that

$$\sup_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} \geq |\inf_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}|.$$

If $w_\epsilon(\xi) \not\equiv 0$, there exists ξ_0^ϵ such that

$$w_\epsilon(\xi_0^\epsilon) = \max_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} = \sup_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} > 0 \quad \text{and} \quad w'_\epsilon(\xi_0) = 0.$$

We first suppose that $\xi_0^\epsilon \to \infty$ as $\epsilon \to 0$. Choose $\epsilon > 0$ sufficiently small such that

$$\xi_0^\epsilon > \max\big\{\sup\{j : j \in \operatorname{supp}\beta(j)\}, \sup\{j : j \in \operatorname{supp}\gamma(j)\}\big\} + \max\{1, c_*(\theta)r\}.$$

Note that

$$\phi'(\xi_0^\epsilon) - \psi'(\xi_0^\epsilon + \bar{\xi}) = w'_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon} + w_\epsilon(\xi_0^\epsilon)\epsilon e^{\lambda_*\xi_0^\epsilon} + w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\lambda_* e^{\lambda_*\xi_0^\epsilon}.$$

Thus, we get

$$
\begin{aligned}
(3.1)\qquad & c_*(\theta)w_\epsilon(\xi_0^\epsilon)\epsilon + c_*(\theta)\lambda_* w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1) \\
&\le D\{w_\epsilon(\xi_0^\epsilon + \cos\theta)[\epsilon(\xi_0^\epsilon + \cos\theta) + 1]e^{\lambda_* \cos\theta} \\
&\quad + w_\epsilon(\xi_0^\epsilon - \cos\theta)[\epsilon(\xi_0^\epsilon - \cos\theta) + 1]e^{-\lambda_* \cos\theta} \\
&\quad + w_\epsilon(\xi_0^\epsilon + \sin\theta)[\epsilon(\xi_0^\epsilon + \sin\theta) + 1]e^{\lambda_* \sin\theta} \\
&\quad + w_\epsilon(\xi_0^\epsilon - \sin\theta)[\epsilon(\xi_0^\epsilon - \sin\theta) + 1]e^{-\lambda_* \sin\theta} \\
&\quad - 4w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\} - dw_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1) \\
&\quad + b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)|w_\epsilon(\xi_0^\epsilon - l\cos\theta - q\sin\theta - c_*(\theta)r)|\times \\
&\quad [\epsilon(\xi_0^\epsilon - l\cos\theta - q\sin\theta - c_*(\theta)r) + 1]e^{-\lambda_*(l\cos\theta + q\sin\theta + c_*(\theta)r)}.
\end{aligned}
$$

It follows from (3.1) and Lemma 2.1(i) that for any $\theta \in [0, \frac{\pi}{2}]$,

$$w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon \pm \cos\theta).$$

Indeed, assume $w_\epsilon(\xi_0^\epsilon) > w_\epsilon(\xi_0^\epsilon + \cos\theta_0)$ or $w_\epsilon(\xi_0^\epsilon) > w_\epsilon(\xi_0^\epsilon - \cos\theta_0)$ for some $\theta_0 \in [0, \frac{\pi}{2}]$, then

$$
\begin{aligned}
& c_*(\theta_0)\epsilon + c_*(\theta_0)\lambda_*(\epsilon\xi_0^\epsilon + 1) \\
&\quad < D\Big\{\Big[\epsilon(\xi_0^\epsilon + \cos\theta_0) + 1\Big]e^{\lambda_*\cos\theta_0} + \Big[\epsilon(\xi_0^\epsilon - \cos\theta_0) + 1\Big]e^{-\lambda_*\cos\theta_0} \\
&\qquad + \Big[\epsilon(\xi_0^\epsilon + \sin\theta_0) + 1\Big]e^{\lambda_*\sin\theta_0} + \Big[\epsilon(\xi_0^\epsilon - \sin\theta_0) + 1\Big]e^{-\lambda_*\sin\theta_0} - 4(\epsilon\xi_0^\epsilon + 1)\Big\} \\
&\quad - d(\epsilon\xi_0^\epsilon + 1) + b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)\Big[\epsilon\Big(\xi_0^\epsilon - l\cos\theta_0 - q\sin\theta_0 - c_*(\theta_0)r\Big) + 1\Big] \\
&\quad \times e^{-\lambda_*(l\cos\theta_0 + q\sin\theta_0 + c_*(\theta_0)r)},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
c_*(\theta_0) &< D(e^{\lambda_*\cos\theta_0}\cos\theta_0 - e^{-\lambda_*\cos\theta_0}\cos\theta_0 + e^{\lambda_*\sin\theta_0}\sin\theta_0 - e^{-\lambda_*\sin\theta_0}\sin\theta_0) \\
&\quad - b'(0)\sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\beta(l)\gamma(q)e^{-\lambda_*(l\cos\theta_0 + q\sin\theta_0 + c_*(\theta_0)r)}(l\cos\theta_0 + q\sin\theta_0 + c_*(\theta_0)r).
\end{aligned}
$$

This contradicts $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*(\theta),\lambda=\lambda_*} = 0$. Repeating the above arguments, we have $w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon \pm j\cos\theta)$ for $j \in \mathbb{Z}$ and $\theta \in [0, \frac{\pi}{2}]$, which implies that w_ϵ is a constant. Since $w_\epsilon(+\infty) = 0$, we get $\phi(\xi) \equiv \psi(\xi + \bar{\xi})$ for $\xi \in \mathbb{R}$. This contradicts $w_\epsilon(\xi) \not\equiv 0$.

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Next we assume $\xi_0^\epsilon \to -\infty$ as $\epsilon \to 0$, then $w_\epsilon(\xi_0^\epsilon) \to 0$, as $\epsilon \to 0$. Since

$$\lim_{\epsilon \to 0} w_\epsilon(\xi) = w_0(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{e^{\lambda_* \xi}} \quad \text{for all} \quad \xi \in \mathbb{R}, \tag{3.2}$$

and $w_\epsilon(x) \le w_\epsilon(\xi_0^\epsilon)$, we have $w_0(\xi) \le 0$ for all $\xi \in \mathbb{R}$. Note that $w_\epsilon(\xi_0^\epsilon) > 0$ implies $\phi(\xi_0^\epsilon) - \psi(\xi_0^\epsilon + \bar{\xi}) > 0$ and hence $w_0(\xi_0^\epsilon) > 0$, which gives a contradiction.

Last we assume that $\{\xi_0^\epsilon\}$ is bounded, then we can take a subsequence $\xi_0^\epsilon \to \xi_1$ as $\epsilon \to 0$ for some finite ξ_1. From uniform convergence of w_ϵ to w on compact sets, $w_\epsilon(\xi_0^\epsilon) \to w(\xi_1)$ as $\epsilon \to 0$, where $w_0(\xi)$ is defined by (3.2). Thus, $w_0(\xi) = \lim_{\epsilon\to 0} w_\epsilon(\xi) \le \lim_{\epsilon \to 0} w_\epsilon(\xi_0^\epsilon) = w_0(\xi_1)$ for all $\xi \in \mathbb{R}$ and $w_0(\xi_1) \ge 0$. This is similar to the argument in the case $c > c_*(\theta)$, and we can also get $w_0(\xi) \equiv 0$; that is, we have $\phi(\xi) \equiv \psi(\xi + \bar{\xi})$ for $\xi \in \mathbb{R}$. This completes the proof. ■

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