Canad. Math. Bull. Vol. **00** (0), 2020 pp. 1–15
http://dx.doi.org/10.4153/xxxx

On meromorphic solutions of certain partial differential equations

Feng Lü

Abstract. **In this paper, we describe meromorphic solutions of certain partial differential equations, which are originated from the algebraic equation $P(f, g) = 0$, where P is a polynomial on $\mathbb{C}^2$. As an application, with the theorem of Coman-Poletsky, we give a proof of the classic theorem: Every meromorphic solution $u(s)$ on $\mathbb{C}$ of $P(u, u') = 0$ belongs to W, which is the class of meromorphic functions on $\mathbb{C}$ that consists of elliptic functions, rational functions and functions of the form $R(e^{as})$, where R is rational and $a \in \mathbb{C}$. In addition, we consider the factorization of meromorphic solutions on $\mathbb{C}^n$ of some well-known PDEs, such as Inviscid Burgers' equation, Riccati equation, Malmquist-Yosida equation, PDEs of Fermat type.**

1 The meromorphic solutions of some PDEs

In this section, we consider meromorphic solution $h(z) = h(z_1, ..., z_n)$ of the following partial differential equation

$$h_{z_1} = R_1[\phi_1(h) + R_2\phi_2(h)], \tag{1.1}$$

where $R_1(\not\equiv 0)$, R_2 are two rational functions on $\mathbb{C}^n$, ϕ_1 and ϕ_2 are two meromorphic functions on $\mathbb{C}$. This PDE is originated from the algebraic equation $P(f, g) = 0$, where P is a polynomial on $\mathbb{C}^2$. Let P be an irreducible polynomial mapping from $\mathbb{C}^2$ to a plane algebraic curve C in $\mathbb{C}^2$. Then a theorem of Picard in [11] says that if the genus of the Riemann surface associated with the closure of C in $\mathbb{P}^2$ is bigger than one, then any such map is constant. One version of the Uniformization Theorem in [10] shows that the curve C of genus g can be parameterized by (i) rational functions if $g = 0$, and (ii) by elliptic functions if $g = 1$. In 2008, Coman-Poletsky [6, Theorem 5.2] obtained the following theorem.

Theorem A. Two meromorphic functions f and g on $\mathbb{C}^n$ satisfy $P(f, g) = 0$, where P is a polynomial on $\mathbb{C}^2$, if and only if one of the following holds:

(i) There exists a meromorphic function h on $\mathbb{C}^n$ and rational functions R_1, R_2 on $\mathbb{C}$ so that $f = R_1(h)$ and $g = R_2(h)$.

(ii) There exists an entire function h on $\mathbb{C}^n$, and elliptic functions ϕ_1, ϕ_2 with the same periods, so that $f = \phi_1(h)$, $g = \phi_2(h)$.

2020 Mathematics Subject Classification: 32A20, 35F20, 32A22, 30D30.
Keywords: Meromorphic solution, Factorization, Partial differential equations, Pseudo-prime.

We assume that $f = \frac{u}{P_1}$ and $g = \frac{u_{z_1}}{P_2}$ in equation $P(f,g) = 0$, where $u = u(z_1, \ldots, z_n)$ is a meromophic function on $\mathbb{C}^n$, $P_1(\not\equiv 0)$ and $P_2(\not\equiv 0)$ are two rational functions on $\mathbb{C}^n$. Then,

$$P\left(\frac{u}{P_1}, \frac{u_{z_1}}{P_2}\right) = 0. \tag{1.2}$$

Due to Theorem A, one gets that

$$\frac{u}{P_1} = f_1(h), \quad \frac{u_{z_1}}{P_2} = f_2(h),$$

where f_1, f_2 and h satisfy the conclusion of Theorem A. Suppose that $f_1 \not\equiv 0$. Then, $u = P_1 f_1(h)$, $u_{z_1} = P_2 f_2(h)$ and

$$h_{z_1} = \frac{P_2 f_2(h) - \frac{\partial P_1}{\partial z_1} f_1(h)}{P_1 f_1'(h)} = R_1[\phi_1(h) + R_2\phi_2(h)], \tag{1.3}$$

where $R_1 = \frac{P_2}{P_1}(\not\equiv 0)$, $R_2 = -\frac{\partial P_1}{\partial z_1}\frac{1}{P_2}$ are two rational functions on $\mathbb{C}^n$, $\phi_1 = \frac{f_2}{f_1'}$ and $\phi_2 = \frac{f_1}{f_1'}$ are meromorphic functions on $\mathbb{C}$. By Theorem A, one gets that both ϕ_1 and ϕ_2 are rational functions or elliptic functions. Therefore, (1.3) is a special case of (1.1).

Now, we consider the equation (1.1). For two functions f, g on $\mathbb{C}$, we define a set $E(f,g)$ as $E(f,g) = \{z \in \mathbb{C} : f(z) = g(z) = 0\}$. And $\sharp E(f,g)$ is the number of elements in $E(f,g)$. More specifically, we obtain that

Theorem 1. Suppose that h is a nonconstant meromorphic solution on $\mathbb{C}^n$ of (1.1).

(1) If $\sharp E(\phi_1 - a, \phi_2 - b) \geq 5$ for two constants $a, b \in \mathbb{C}$, then h_{z_1} is a rational function; In particularly, if $\sharp E(\phi_1 - a, \phi_2 - b) = \infty$, then $h_{z_1} = R_1[a + R_2 b]$.

(2) If ϕ_2 is a rational function on $\mathbb{C}$, then ϕ_1 is also a rational function on $\mathbb{C}$. Furthermore, if h is transcendental, then

$$h_{z_1} = B_0 h^2 + B_1 h + B_2, \tag{1.4}$$

where B_i $(i = 0, 1, 2)$ is a rational function on $\mathbb{C}^n$.

Remark 1. If h is an entire function in (1) of Theorem 1, then the condition $\sharp E(\phi_1 - a, \phi_2 - b) \geq 5$ can be weakened to $\sharp E(\phi_1 - a, \phi_2 - b) \geq 3$. Meanwhile, h_{z_1} reduces to a polynomial on $\mathbb{C}^n$.

Remark 2. If ϕ_2 is a transcendental meromorphic function and ϕ_1 is a rational function on $\mathbb{C}$, then the same argument as in the proof of (2) yields that $R_2 = 0$. Furthermore, if h is transcendental, then (1.4) also holds. Unfortunately, for the case that both ϕ_1 and ϕ_2 are transcendental meromorphic functions on $\mathbb{C}$ the method in the proof of Theorem 1 does not work. We leave this case for further study.

From Theorem 1, we prove the following result.

Corollary 1. Suppose that h is a nonconstant meromorphic solution of (1.1). If ϕ_1 and ϕ_2 are two nonconstant meromorphic periodic functions on $\mathbb{C}$ with the same periods τ. Then, R_2 reduces to a constant A, $\phi_1 + A\phi_2$ also reduces a constant C, and $h_{z_1} = CR_1$ is a rational function on $\mathbb{C}^n$.

Below, we turn attention to meromorphic solutions of (1.2). By combining Theorem 1 and Corollary 1, we immediately get the following result.

Corollary 2. Suppose that u is a nonconstant meromorphic solution of (1.2). Then, u can be written as $u = P_1 f(h)$, where f and h satisfy one of the following assertions.

(i) f is an elliptic function on $\mathbb{C}$, h is an entire function and h_{z_1} is a polynomial on $\mathbb{C}^n$;

(ii) f is a rational function on $\mathbb{C}$, h is a meromorphic function on $\mathbb{C}^n$, and h_{z_1} is a rational function or h_{z_1} satisfies the following equation

$$h_{z_1} = B_0 h^2 + B_1 h + B_2, \tag{1.5}$$

where B_i $(i = 0, 1, 2)$ is a rational function on $\mathbb{C}^n$.

Remark 3. In [30], Saleeby described meromorphic solutions of $P(u, u_{z_1}) = 0$, where P is a polynomial on $\mathbb{C}^2$. Meanwhile, Saleeby gave an outline of an algorithm for finding meromorphic solutions of $P(u, u_{z_1}) = 0$ of genus 0. Some ideas of our theorems are based on Saleeby's results. And some related results can be found in [22, 23, 24].

Remark 4. It should be emphasized that if P_1 and P_2 are constants in (1.2), then the coefficient B_i $(i = 0, 1, 2)$ of (1.5) in (ii) of Corollary 2 reduces to a constant. In [30], Saleeby obtained the forms of meromorphic solutions on $\mathbb{C}^2$ to (1.5) when B_i is a constant.

Below, we give an application of the above results. Let us denote by W the class of meromorphic functions on $\mathbb{C}$ that consists of elliptic functions, rational functions and functions of the form $R(e^{as})$, where R is rational and $a \in \mathbb{C}$. Consider a Briot-Bouquet differential equation

$$P(u, u') = 0, \tag{1.6}$$

where P is a polynomial on $\mathbb{C}^2$. We state the following result, which can be found in [8].

Theorem B. Every meromorphic solution $u = u(s)$ on $\mathbb{C}$ of (1.6) belongs to W.

In [8], it says that Theorem B was known to Abel and Liouville, but probably it was stated for the first time in the work of Briot and Bouquet in [3, 4]. Some further studies which are related with Theorem B can be found in [7, 8, 15, 16, 17]. Below, due to the theorem of Coman-Poletsky(Theorem A), Corollary 2 and Remark 4, we offer a proof of Theorem B.

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

Proof of Theorem B. Suppose that $u(s)$ is transcendental. From Corollary 2, Theorem A and Remark 4, we get that $u = f(h)$, $u' = g(h)$, where f, g, h are three functions on $\mathbb{C}$ which satisfy one of the following assertions.

(1) f, g are two elliptic functions with the same periods and h' is a polynomial;

(2) f, g are two rational functions, h is a transcendental meromorphic function and satisfies the following equation

$$h' = B_0 h^2 + B_1 h + B_2, \tag{1.7}$$

where B_i $(i = 0, 1, 2)$ is a constant.

We firstly consider (1). It is easy to get $h' = \frac{g(h)}{f'(h)} = \alpha(h)$, where $\alpha = \frac{g}{f'}$ is also an elliptic function. Note that h' is a polynomial. Therefore, α is a constant and h is a linear function, which implies that $u = f(h)$ is also an elliptic function, so $u \in W$.

Next, we deal with (2). Suppose $B_0 = 0$. Solve the differential equation $h' = B_1 h + B_0$ yields that $h(s) = Ae^{B_1 s} + B$, where $A(\neq 0)$, B are constants. Then $u = f(h) \in W$, since f is a rational function.

Below, assume that $B_0 \neq 0$. Rewrite (1.7) as

$$h' = B_0(h - a)(h - b) \tag{1.8}$$

where a, b are two constants. It is easy to see that a and b are Picard exceptional values of h. Suppose that $a = b$. Solve the differential equation (1.8) leads to $h(s) = a - \frac{1}{B_0 s + C}$, where C is a constant. So, $u = f(h)$ is a rational function, a contradiction. Thus, $a \neq b$. By the fact that a and b are Picard exceptional values of h, we can set $\frac{h-a}{h-b} = e^{\gamma}$, where γ is an entire function. Rewrite it as

$$h = \frac{be^{\gamma} - a}{e^{\gamma} - 1}. \tag{1.9}$$

Substitute (1.9) into (1.8) yields that $\gamma' = -B_0(b - a)$ is a constant. So, γ is a linear function, which together with (1.9) implies that $u \in W$.

Thus, we finish the proof of Theorem B.

Before the proofs of main results, we assume that the reader is familiarity with the basic notations of Nevanlinna theory, and we utilize four results of a meromorphic function f on $\mathbb{C}^n$ (see e.g., [33, 34]).

(a). The Nevanlinna second fundamental theorem

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f - \omega_j}) + S(r, f)$$

for any q distinct complex numbers $\omega_1, \cdots, \omega_q \in \mathbb{C} \cup \{\infty\}$, where $S(r, f)$ denotes any quantity satisfying that $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside a set of r of finite Lebesgue measure.

(b). Suppose that f and g are two meromorphic functions on $\mathbb{C}^n$. Then, $T(r, \frac{1}{f}) = T(r, f) + O(1), T(r, fg) \leq T(r, f) + T(r, g)$ and $T(r, f+g) \leq T(r, f) + T(r, g) + O(1)$.

(c). Suppose that a is a rational function on $\mathbb{C}^n$, f is a transcendental meromorphic function on $\mathbb{C}^n$. Then, $T(r, a) = S(r, f)$.

(d). The logarithmic derivative lemma $m(r, \frac{f_{z_j}}{f}) = S(r, f), j = 1, \ldots, n$.

2 Proofs of the Theorem 1 and Corollary 1

Proof of Theorem 1. We firstly consider the case (1). Assume that $z_0 \in E(\phi_1 - a, \phi_2 - b)$. Then, $\phi_1(z_0) = a$ and $\phi_2(z_0) = b$. If $h(\alpha) = z_0$, then, substituting α into (1.1) yields that

$$\begin{aligned} h_{z_1}(\alpha) &= R_1(\alpha)[\phi_1(h(\alpha)) + R_2(\alpha)\phi_2(h(\alpha))] = R_1(\alpha)[\phi_1(z_0) + R_2(\alpha)\phi_2(z_0)] \\ &= R_1(\alpha)[a + R_2(\alpha)b], \end{aligned} \tag{2.1}$$

which implies that α is a zero of $h_{z_1} - R_1[a + R_2 b]$. For each $z_0 \in E(\phi_1 - a, \phi_2 - b)$, the above discussion shows that all the zeros of $h - z_0$ are zeros of $h_{z_1} - R_1[a + R_2 b]$. Suppose that h_{z_1} is transcendental. Then, $h_{z_1} - R_1[a + R_2 b] \not\equiv 0$. Obviously, h is transcendental and $T(r, R_1[a + R_2 b]) = S(r, h)$. Suppose that $q = \sharp E(\phi_1 - a, \phi_2 - b) \geq 5$. Observe that all the poles of h_{z_1} are poles of h and h_{z_1} just has multiple poles. Suppose that β is a pole of h_{z_1} with multiplicity m. Then $m \geq 2$ and β is a pole of h with multiplicity at least $m - 1$. So, $N(r, h_{z_1}) \leq 2N(r, h)$. Together with the second fundamental theorem, one obtains that

$$\begin{aligned} (q-2)T(r, h) &\leq \sum_{z_0 \in E(\phi_1 - a, \phi_2 - b)} \overline{N}(r, \frac{1}{h - z_0}) + S(r, h) \\ &\leq \overline{N}(r, \frac{1}{h_{z_1} - R_1[a + R_2 b]}) + S(r, h) \leq T(r, h_{z_1} - R_1[a + R_2 b]) + S(r, h) \\ &\leq T(r, h_{z_1}) + S(r, h) = m(r, h_{z_1}) + N(r, h_{z_1}) + S(r, h) \\ &\leq m(r, \frac{h_{z_1}}{h}) + m(r, h) + 2N(r, h) + S(r, h) \leq 2T(r, h) + S(r, h), \end{aligned} \tag{2.2}$$

which is a contradiction since $q \geq 5$. Therefore, h_{z_1} must be a rational function.

In particularly, suppose $q = \sharp E(\phi_1 - a, \phi_2 - b) = \infty$. Observe that $h_{z_1} - R_1[a + R_2 b]$ is a rational function. Without loss of generality, we assume that

$$T(r, h_{z_1} - R_1[a + R_2 b]) \leq A \log r, \;\; as \;\; r \to \infty, \tag{2.3}$$

where A is a positive constant. For each $z_0 \in E(\phi_1 - a, \phi_2 - b)$, the above discussion shows that all the zeros of $h - z_0$ are the zeros of $h_{z_1} - R_1[a + R_2 b]$. If $h_{z_1} - R_1[a + R_2 b] \not\equiv$

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

0, then,

$$
\begin{aligned}
T(r, h_{z_1} - R_1[a + R_2 b]) &\geq N(r, \frac{1}{h_{z_1} - R_1[a + R_2 b]}) \\
&\geq \sum_{z_0 \in E(\phi_1 - a, \phi_2 - b)} \overline{N}(r, \frac{1}{h - z_0}) \geq (A+1)\log r, \;\; as \;\; r \to \infty,
\end{aligned}
$$

since $q = \sharp E(\phi_1 - a, \phi_2 - b) = \infty$. This contradicts (2.3). Therefore, one gets that

$$h_{z_1} - R_1[a + R_2 b] = 0. \tag{2.4}$$

Below, we consider case (2). Rewrite (1.1) as

$$\frac{1}{R_1} h_{z_1} - R_2 \phi_2(h) = \phi_1(h). \tag{2.5}$$

On the contrary, suppose that ϕ_1 is a transcendental function. Note that ϕ_2 is a rational function. It follows from (2.5) that h is transcendental. By [5, Theorem 4.1], we have

$$\lim_{r\to\infty} \frac{T(r, \phi_1(h))}{T(r, h)} = \infty. \tag{2.6}$$

Suppose that $\deg \phi_2 = d$. Then, (2.5) yields that

$$
\begin{aligned}
T(r, \phi_1(h)) &= T(r, \frac{1}{R_1} h_{z_1} - R_2 \phi_2(h)) \\
&\leq T(r, R_1) + T(r, R_2) + T(r, h_{z_1}) + T(r, \phi_2(h)) + O(1) \\
&\leq T(r, h_{z_1}) + T(r, \phi_2(h)) + O(\log r) \\
&= m(r, h_{z_1}) + N(r, h_{z_1}) + dT(r, h) + S(r, h) \\
&\leq m(r, \frac{h_{z_1}}{h}) + m(r, h) + 2N(r, h) + dT(r, h) + S(r, h) \\
&\leq (d+2)T(r, h) + S(r, h) = O(T(r, h)),
\end{aligned}
$$

which contradicts (2.6). Thus, ϕ_1 is also a rational function.

Note that ϕ_1 and ϕ_2 are rational functions. We rewrite (1.1) as

$$h_{z_1} = R_1 \phi_1(h) + R_1 R_2 \phi_2(h) = R(h), \tag{2.7}$$

where $R(h)$ is a rational function in h, whose coefficients are rational functions. If h is transcendental, then, either [30, Proposition 1] or [32, Corollary 3] yields that (2.7) becomes the following equation

$$h_{z_1} = B_0 h^2 + B_1 h + B_2,$$

where B_i $(i = 0, 1, 2)$ is a rational function on $\mathbb{C}^n$.

This finishes the proof of Theorem 1.

Proof of Corollary 1. We know that ϕ_1 and ϕ_2 are two nonconstant periodic functions on $\mathbb{C}$ with the same period. Suppose that $z_0 \in \mathbb{C}$ is not a pole of ϕ_1 and ϕ_2. And assume

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

that $\phi_1(z_0) = a$ and $\phi_2(z_0) = b$. Then $\sharp E(\phi_1 - a, \phi_2 - b) = \infty$. It follows from (1) of Theorem 1 that

$$h_{z_1} - R_1[a + R_2 b] = 0. \tag{2.8}$$

Because ϕ_2 is nonconstant, there must exist a point t_0 such that $\phi_2(t_0) = d \neq b$ and $\phi_1(t_0) = c \neq \infty$. Then $\sharp E(\phi_1 - c, \phi_2 - d) = \infty$. The same argument as above yields that

$$h_{z_1} - R_1[c + R_2 d] = 0. \tag{2.9}$$

Combining (2.8) and (2.9) leads to that $R_2 = \frac{c-a}{b-d} = A$, a constant. Rewrite (1.1) as

$$h_{z_1} = R_1[\phi_1(h) + A\phi_2(h)] = R_1\phi_3(h), \tag{2.10}$$

where $\phi_3 = \phi_1 + A\phi_2$ is also a periodic function. Furthermore, ϕ_3 is a constant. Otherwise, the left side of (2.10) is a rational function and the right side of (2.10) is a transcendental meromorphic function, which is impossible. Let $\phi_3 = C$. Then, $h_{z_1} = CR_1$.

This finishes the proof of Corollary 1.

3 The factorization of meromorphic functions on $\mathbb{C}^n$

In this section, we turn attention to the factorization of meromorphic solutions of some certain PDEs. If $h = f(g)$ is a meromorphic function on $\mathbb{C}^n$, where $g : \mathbb{C}^n \to \mathbb{C}$ is an entire function and $f : \mathbb{C} \to \mathbb{P} = \mathbb{C} \cup \{\infty\}$ is a meromorphic function, then h is said to have a factorization with right factor g and left factor f (g may be a meromorphic function on $\mathbb{C}^n$ when f is a rational function from $\mathbb{C}$ to $\mathbb{P}$). A meromorphic function h is said to be prime if every such factorization implies that either f or g is linear. Furthermore, h is said to be pseudo-prime if every factorization of the above form implies that either f is a rational function or g is a polynomial.

The first prime function $e^s + s$ on $\mathbb{C}$ was introduced by Rosenbloom in [27]. Some further results on the factorization of meromorphic functions on $\mathbb{C}$ can be found in [12, 13, 26]. Observe that $e^s + s$ is a solution of the following equation

$$f^{(n)}(s) + a_{n-1}(s)f^{(n-1)}(s) + \cdots + a_0(s)f(s) = a(s), \tag{3.1}$$

where $a(s), a_0(s), \cdots, a_{n-1}(s)$ are rational functions on $\mathbb{C}$. Therefore, it is natural to consider the primeness or pseudo-primeness of meromorphic solutions of (3.1). It is pointed out that the meromorphic solutions of (3.1) maybe not prime. Steinmetz in [33] proved that any meromorphic solution on $\mathbb{C}$ of (3.1) is pseudo-prime. Later on, Chang-Li-Yang generalized this result from one variable to several variables as follows.

Theorem C. Let m be a positive integer and

$$D^{(m)}F(z) + A_m(z)D^{(m-1)}F(z) + \cdots + A_1(z)F(z) + A_0(z) = 0$$

be a differential equation, where the coefficients $A_j(z)$, $0 \leq j \leq m$, $z = (z_1, \ldots, z_n)$, are rational functions in $\mathbb{C}^n$ and the operator $D^{(l)} = \frac{\partial^{l_1+\cdots+l_n}}{\partial z_1^{l_1}\cdots\partial z_n^{l_n}}$, $l = l_1 + \cdots l_n$, $(1 \leq$

$l \leq m$). Then any meromorphic solution $F(z)$ of finite order must be pseudo-prime provided that $F(z)$ is not a constant along each z_j-axis ($1 \leq j \leq n$).

Here, for a meromorphic function f on $\mathbb{C}^n$, the order of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If $\rho(f)$ is finite (respectively infinite), then f is of finite order (respectively infinite order).

In [5], Chang-Li-Yang offered an example to show that in contrast to the case of one complex variable, the condition "finite order" in Theorem C can not be dropped.

Below, we continue to study the pseudo-primeness of entire solutions to certain PDEs. Consider $P(u, u_{z_1}) = 0$, where u is an entire function on $\mathbb{C}^n$ and P is a polynomial on $\mathbb{C}^2$. From Theorem A, we see that u has a factorization $u = f(g)$, where f, g satisfy some conditions. So, it's natural to ask whether u is pseudo-prime or not? In this section, we consider this question and obtain the following result.

Theorem 2. Suppose that an entire function $u = u(z_1, ..., z_n)$ on $\mathbb{C}^n$ is a solution of the following PDE

$$P(z, u, u_{z_1}) = \sum_{\lambda \in I} a_\lambda(z) u^{\lambda_0} u_{z_1}^{\lambda_1} = 0, \tag{3.2}$$

where I is a finite set of multi-indices (λ_0, λ_1) with nonnegative integers λ_0, λ_1, and $a_\lambda(z)(\not\equiv 0)$ is a rational function on $\mathbb{C}^n$. If u is of finite order and $u_{z_1} \not\equiv 0$, then u is pseudo-prime.

Remark 5. For the case $n = 1$, Liao-Yang in [21, Theorem 2] proved that any entire solution of (3.2) must be pseudo-prime. The main theorems in this section are inspired by Liao-Yang's result. Below, we offer two examples to show that the conditions "finite order" and $u_{z_1} \not\equiv 0$ are necessary.

Example 1. Consider

$$u(z) = e^{z_1 + \phi(z_2, \ldots, z_n)},$$

where $\phi(z_2, \ldots, z_n)$ is a transcendental entire function on $\mathbb{C}^n$. Obviously, u is an entire solution of infinite order to $P(z, u, u_{z_1}) = u_{z_1} - u = 0$. But u is not pseudo-prime, since $u = f(g)$, where $f = e^s$ on $\mathbb{C}$ and $g = z_1 + \phi(z_2, \ldots, z_n)$ on $\mathbb{C}^n$.

Example 2. Suppose that $f(s)$ is an entire function of finite order on $\mathbb{C}$ which is not pseudo-prime. Let $u(z) = f(z_2)$. Obviously, $P(z, u, u_{z_1}) = u_{z_1} = 0$, and u is of finite order but not pseudo-prime.

It is pointed out that Examples 1-2 can be found in [5], given by Chang-Li-Yang.

Let I and J be finite sets of multi-indices as in Theorem 2. Next, we consider more general PDEs

$$P(z,u,u_{z_1},u_{z_2}) = \sum_{\lambda\in I} a_\lambda(z)u^{\lambda_0}u_{z_1}^{\lambda_1} + \sum_{\mu\in J} b_\mu(z)u^{\mu_0}u_{z_2}^{\mu_1} = 0, \tag{3.3}$$

where $a_\lambda(z)(\not\equiv 0)$, $b_\mu(z)(\not\equiv 0)$ are rational functions on $\mathbb{C}^2$, and $z = (z_1, z_2)$. Then, we obtain the following result.

Theorem 3. Suppose that $u = u(z_1, z_2)$ on $\mathbb{C}^2$ is a nonconstant entire solution of (3.3). If u is of finite order and $u_{z_1} \not\equiv 0$, $u_{z_2} \not\equiv 0$, then one of the following assertions holds.

(1) u is pseudo-prime;

(2) $I = J$, and u has the form $u = f(g)$, where f is transcendental meromorphic function on $\mathbb{C}$ with at most one pole, g is a transcendental entire function on $\mathbb{C}^2$ satisfying that for any $\lambda = (\lambda_0, \lambda_1) \in I$ there exist co-prime polynomials c_λ, d_λ such that $c_\lambda g_{z_1} + d_\lambda g_{z_2} = 0$ and $[\frac{c_\lambda(z)}{d_\lambda(z)}]^{\lambda_1} = -\frac{a_\lambda(z)}{b_\lambda(z)}$.

Remark 6. Suppose that $a_\lambda(z) = \alpha(z_2)$ and $b_\lambda(z) = \beta(z_1)$ are two polynomials on $\mathbb{C}$ for some $\lambda \in I$. Then, $c_\lambda(z) = P_1(z_2)$ and $d_\lambda(z) = Q_1(z_1)$ reduce to a function of one variable. And $c_\lambda g_{z_1} + d_\lambda g_{z_2} = 0$ becomes

$$P_1(z_2)g_{z_1} + Q_1(z_1)g_{z_2} = 0.$$

Solving the above PDE yields that $g = \gamma(Q(z_1) - P(z_2))$, where $P(z_2) = \int P_1(z_2)dz_2$, $Q(z_1) = \int Q_1(z_1)dz_1$ are two polynomials on $\mathbb{C}$, and γ is an arbitrary entire function on $\mathbb{C}$. Furthermore, $u(z_1, z_2) = f(\gamma(Q(z_1) - P(z_2)))$.

It is pointed out that the case (2) indeed occurs in Theorem 3, which can be shown by the following example.

Example 3. Consider that $u(z_1, z_2) = e^{\sin(\frac{{z_1}^2 - {z_2}^2}{2})}$. Then, u is an entire solution of $z_2 u_{z_1} + z_1 u_{z_2} = 0$. It is easy to see that $u = f(g)$, where $f(s) = e^s$ and $g(z_1, z_2) = \sin(\frac{{z_1}^2 - {z_2}^2}{2})$ are two transcendental entire functions and $z_2 g_{z_1} + z_1 g_{z_2} = 0$. So, the assertion (2) occurs.

In Theorem 3, if we add some conditions to guarantee that the assertion (2) cannot occur, then entire solutions of (3.3) must be pseudo-prime. Inspired by this idea, we get a corollary by Theorem 3.

Corollary 3. Suppose that entire function u of finite order on $\mathbb{C}^2$ satisfies (3.3) and $u_{z_1} \not\equiv 0$, $u_{z_2} \not\equiv 0$. Then, u is pseudo-prime if u satisfies one of the following conditions.

(1) $I \neq J$;

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

(2) There exists an index $\lambda = (\lambda_0, \lambda_1) \in I \cap J$ such that $\lambda_1 \geq 2$ and $\frac{a_\lambda(z)}{b_\lambda(z)}$ is a nonconstant irreducible polynomial.

We pay attention to entire solutions of some well-known PDEs. From Theorems 2-3 and Corollary 3, we immediately get the following results. It is noted that some of them may have been recorded in the literature, the author is not able to find such a reference.

Corollary 4. Suppose that $u(z) = u(z_1, z_2)$ is an entire function of finite order on $\mathbb{C}^2$ and $u_{z_1} \not\equiv 0$, $u_{z_2} \not\equiv 0$. Then u is pseudo-prime, if u satisfies one of the following PDEs.

(1) Inviscid Burgers' equation $u_{z_1} + uu_{z_2} = 0$;

(2) Riccati equation $u_{z_1} = au^2 + bu + c$, where $a(\not\equiv 0)$, b, c are rational functions on $\mathbb{C}^2$;

(3) Malmquist-Yosida equation $u_{z_1}^m = \sum_{i=0}^{2m} a_i u^i$, where m is a positive integer, $a_{2m}(\not\equiv 0)$,..., a_0 are rational functions on $\mathbb{C}^2$;

(4) PDEs of Fermat type of $au^m + bu_{z_1}^n = 1$, where m, n are two positive integers, $a(\not\equiv 0)$ and $b(\not\equiv 0)$ are two rational functions on $\mathbb{C}^2$.

In the above theorems and corollaries, the condition "finite order" is necessary. Below, we omit this condition by considering a system of Malmquist-Yosida type of partial differential equations, which is inspired by a theorem of Hu-Yang in [19, Theorem 6.2]. In fact, we obtain the following result.

Theorem 4. If a transcendental meromorphic function $u(z) = u(z_1, ..., z_n)$ on $\mathbb{C}^n$ satisfies the following system of partial differential equations

$$(u_{z_i})^{m_i} + \sum_{j=1}^{m_i - 1} a_{i,j} (u_{z_i})^j = \sum_{l=0}^{2m_i} b_{i,l} u^l, \quad i = 1, 2, \ldots, n$$

where m_i ($i = 1, ..., n$) is a positive integer and $a_{i,j}, b_{i,l}$ are rational functions on $\mathbb{C}^n$, then u is of finite order. In particularly, if u is an entire function on $\mathbb{C}^n$, then u is pseudo-prime.

Remark 7. In [19, Theorem 6.2], Hu-Yang considered the pseudo-primeness of meromorphic solutions of a certain system of partial differential equations. In Theorem 4, if u is meromorphic function on $\mathbb{C}^n$, we have not deduced that u is pseudo-prime and leave it for further study.

Remark 8. In [35], Zimogljad proved that every entire transcendental solution of a second-order algebraic differential equation on $\mathbb{C}$ with rational coefficients has a positive order. Therefore, as Theorem 2-4, one can consider the pseudo-primeness of entire solutions of some second-order PDEs, such as Burgers' equation $u_{z_1} + uu_{z_2} = \nu u_{z_1 z_1}$, One-dimensional diffusion equation $u_{z_1} = \nu u_{z_2 z_2}$ and some generalizations of them,

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

where v is a constant.

4 Proofs of the Theorems 2-4

In order to prove the above theorems, we employ the following lemmas. The first one is a theorem of Brownawell in [2], which is a generalization of the remarkable Steinmetz's Reduction Theorem in [33].

Lemma 1. Let

$$F_1, \dots, F_k : \mathbb{C} \to \mathbb{P}^1, \ \ h_1, \dots, h_k : \mathbb{C}^n \to \mathbb{P}^1$$

be meromorphic functions, none of which is identically zero. Also let $g : \mathbb{C}^n \to \mathbb{C}$ be a nonconstant entire function. For some $C > 0$, suppose that the characteristic functions satisfy

$$\sum_{j=1}^{k} T(r, h_j) \leq CT(r, g) + S(r, g).$$

If

$$F_1(g)h_1 + F_2(g)h_2 + \cdots + F_k(g)h_k = 0,$$

then there exist polynomials $Q_1(z), \dots, Q_k(z)$ not all identically zero in $\mathbb{C}[z]$ such that

$$Q_1F_1 + Q_2F_2 + \cdots + Q_kF_k = 0.$$

The second one is due to Strelitz in [31].

Lemma 2. Every entire transcendental solution of a first-order algebraic differential equation with rational coefficients has an order no less than $1/2$.

The last one is given by Chang-Li-Yang, which is contained in the proof of [5, Theorem 3.1]. It also appears in [19]. For $n = 1$, see Edrei-Fuchs in [9].

Lemma 3. Let $g : \mathbb{C}^n \to \mathbb{C}$ be a transcendental entire function, and let $f : \mathbb{C} \to \mathbb{P}^1$ be a meromorphic function of positive order. Then, $F = f(g)$ is of infinite order.

Proof of Theorem 2. The following proof is relied heavily on the proof of Liao-Yang in [21, Theorem 2]. Assume, to the contrary, that $u = f(g)$ where $f : \mathbb{C} \to \mathbb{P}^1$ is a transcendental meromorphic function and $g : \mathbb{C}^n \to \mathbb{C}$ is a transcendental entire function. Substituting $u = f(g)$ into (3.2) yields that

$$\sum_{\lambda \in I} a_\lambda(z)[f(g(z))]^{\lambda_0} [f'(g(z))]^{\lambda_1} [g_{z_1}(z)]^{\lambda_1} = 0. \tag{4.1}$$

Obviously, $g_{z_1} \not\equiv 0$, since $u_{z_1} \not\equiv 0$. Since g is transcendental, one gets that

$$T(r, a_\lambda(z)[g_{z_1}(z)]^{\lambda_1}) \leq CT(r, g) + S(r, g),$$

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

where C is a fixed positive constant. By Lemma 1, there exist polynomials Q_λ which are not all identically zero such that

$$\sum_{\lambda\in I} Q_\lambda(s)[f(s)]^{\lambda_0}\,[f'(s)]^{\lambda_1}=0. \tag{4.2}$$

In [21, Theorem 2], Liao-Yang proved that f has positive order. For the sake of completeness, we will state their proof in all details. We know that $u=f(g)$ is an entire function. If $\alpha\in\mathbb{C}$ is a pole of f, then α must be a finite Picard exceptional value of g. Observe that g is a transcendental entire function. Picard's little theorem tells us that g has at most one finite Picard exceptional value. So, f has at most one pole on $\mathbb{C}$. If f is an entire function, then, Lemma 2 yields the order $\rho(f)>0$. If f has one pole at a with multiplicity m, then we can write $f(s)=\frac{h(s)}{(s-a)^m}$, where h is an entire function on $\mathbb{C}$. Taking derivative of f yields that $f'(s)=\frac{h'(s)(s-a)-mh(s)}{(s-a)^{m+1}}$. Substitute the forms of f and f' into (4.2) yields that

$$\sum_{J\in\Lambda} P_J(s)[h(s)]^{J_0}\,[h'(s)]^{J_1}=0,$$

where $P_J(\not\equiv 0)$ is a polynomial. Then again by Lemma 2 we get $\rho(h)>0$. So, $\rho(f)=\rho(h)>0$. The above discussion yields that f is of positive order. By Lemma 3, one can get a contradiction.

This finishes the proof of Theorem 2.

Proof of Theorem 3. Suppose that the conclusion (1) is invalid. Then, we can set $u=f(g)$, where $f:\mathbb{C}\to\mathbb{P}^1$ is a transcendental meromorphic function and $g:\mathbb{C}^n\to\mathbb{C}$ is a transcendental entire function. Substituting $u=f(g)$ into (3.3) yields that

$$\begin{aligned}&\sum_{\lambda\in I} a_\lambda(z)[f(g(z))]^{\lambda_0}\,[f'(g(z))]^{\lambda_1}\,[g_{z_1}(z)]^{\lambda_1}\\&+\sum_{\mu\in J} b_\mu(z)[f(g(z))]^{\mu_0}\,[f'(g(z))]^{\mu_1}\,[g_{z_2}(z)]^{\mu_1}=0.\end{aligned} \tag{4.3}$$

Suppose that $\Lambda=I\cap J$. Then, we rewrite (4.3) as

$$\begin{aligned}&\sum_{\lambda\in I\setminus\Lambda} a_\lambda(z)[f(g(z))]^{\lambda_0}\,[f'(g(z))]^{\lambda_1}\,[g_{z_1}(z)]^{\lambda_1}\\&+\sum_{\mu\in J\setminus\Lambda} b_\mu(z)[f(g(z))]^{\mu_0}\,[f'(g(z))]^{\mu_1}\,[g_{z_2}(z)]^{\mu_1}\\&+\sum_{\lambda\in\Lambda}[f(g(z))]^{\lambda_0}\,[f'(g(z))]^{\lambda_1}\,[a_\lambda(z)g_{z_1}(z)^{\lambda_1}+b_\lambda(z)g_{z_2}(z)^{\lambda_1}]=0.\end{aligned} \tag{4.4}$$

Obviously,

$$T(r,l(z))\le CT(r,g)+S(r,g),$$

where $l(z)\in\{a_\lambda(z)[g_{z_1}(z)]^{\lambda_1},\ b_\mu(z)[g_{z_2}(z)]^{\mu_1},\ a_\lambda(z)g_{z_1}(z)^{\lambda_1}+b_\lambda(z)g_{z_2}(z)^{\lambda_1}\}$. If $I\setminus\Lambda\neq\emptyset$ or $J\setminus\Lambda\neq\emptyset$, then the same argument as the proof of Theorem 2 yields a

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

contradiction. Below, we assume that $I \setminus \Lambda = \emptyset$ and $J \setminus \Lambda = \emptyset$. That is $I = J = \Lambda$. Then, (4.4) becomes

$$\sum_{\lambda \in \Lambda} [f(g(z))]^{\lambda_0} [f'(g(z))]^{\lambda_1} [a_\lambda(z) g_{z_1}(z)^{\lambda_1} + b_\lambda(z) g_{z_2}(z)^{\lambda_1}] = 0. \tag{4.5}$$

If one of $a_\lambda(z) g_{z_1}(z)^{\lambda_1} + b_\lambda(z) g_{z_2}(z)^{\lambda_1}$ $(\lambda \in \Lambda)$ is not identically zero, then the same argument as the proof in Theorem 2 yields a contradiction. Therefore,

$$a_\lambda(z) g_{z_1}(z)^{\lambda_1} + b_\lambda(z) g_{z_2}(z)^{\lambda_1} = 0, \ \ \lambda \in \Lambda. \tag{4.6}$$

Rewrite (4.6) as $[\frac{g_{z_2}}{g_{z_1}}]^{\lambda_1} = -\frac{a_\lambda}{b_\lambda}$. Furthermore, we can derive that $\frac{g_{z_2}}{g_{z_1}} = \frac{c_\lambda}{d_\lambda}$, where $c_\lambda(z)$ and $d_\lambda(z)$ are two polynomials on $\mathbb{C}^n$ and $[\frac{c_\lambda}{d_\lambda}]^{\lambda_1} = -\frac{a_\lambda}{b_\lambda}$.

This finishes the proof of Theorem 3.

Proof of Theorem 4. The following proof is based on the idea of Hu-Yang in [19, Theorem 6.2]. We firstly prove that there exist two positive constants σ and r_0 such that for $|z| \geq r_0$,

$$\frac{|u_{z_i}|^2}{(1+|u|^2)^2} \leq |z|^\sigma, \ \ i = 1, \dots, n. \tag{4.7}$$

On the contrary, for any $N > 0$, there exist an index $i \in \{1, 2, \dots, n\}$ and a sequence $\{v_t\} \subset \mathbb{C}^n$ such that

$$|v_t| = r_t \to \infty, \ \ as \ \ t \to \infty, \ \ \frac{|u_{z_i}(v_t)|^2}{(1+|u(v_t)|^2)^2} \geq r_t^N. \tag{4.8}$$

Without loss of generality, we assume that

$$N = 2(1 + \max_{i,j,l}\{\deg a_{i,j}, \deg b_{i,l}\}). \tag{4.9}$$

From (4.8), one gets

$$|u_{z_i}(v_t)| \geq (1+|u(v_t)|^2) r_t^{N/2} \geq r_t^{N/2} \to \infty, \ \ as \ \ t \to \infty.$$

Further, for t large enough, we get that

$$\begin{aligned} \left|(u_{z_i})^{m_i} + \sum_{j=1}^{m_i-1} a_{i,j}(u_{z_i})^j\right|_{z=v_t} &\geq |u_{z_i}(v_t)|^{m_i}[1 - \sum_{j=1}^{m_i-1} \frac{|a_{i,j}(v_t)|}{|(u_{z_i})^{m_i-j}(v_t)|}] \\ &\geq \frac{1}{2}(u_{z_i})^{m_i}(v_t), \end{aligned} \tag{4.10}$$

since $\frac{|a_{i,j}(v_t)|}{|(u_{z_i})^{m_i-j}(v_t)|} \to 0$ as $t \to \infty$ for any $j = 1, \dots, m_i - 1$. Meanwhile, we have

$$\left|(u_{z_i})^{m_i} + \sum_{j=1}^{m_i-1} a_{i,j}(u_{z_i})^j\right|_{z=v_t} = \left|\sum_{l=0}^{2m_i} b_{i,l} u^l\right|_{z=v_t} \leq \sum_{l=0}^{2m_i} |b_{i,l}(v_t)||u^l(v_t)|. \tag{4.11}$$

https://doi.org/10.4153/S0008439525000347 Published online by Cambridge University Press

For t large enough, combining (4.10) and (4.11) yields that

$$\frac{|u_{z_i}(\nu_t)|^{m_i}}{(1+|u(\nu_t)|^2)^{m_i}} \leq \frac{2\sum_{l=0}^{2m_i}|b_{i,l}(\nu_t)||u^l(\nu_t)|}{(1+|u(\nu_t)|^2)^{m_i}} \leq 2\sum_{l=1}^{2m_i}|b_{i,l}(\nu_t)| < r_t^{\frac{N}{2}}, \tag{4.12}$$

which implies that

$$\frac{|u_{z_i}(\nu_t)|^2}{(1+|u(\nu_t)|^2)^2} < r_t^{\frac{N}{m_i}} \leq r_t^N. \tag{4.13}$$

By (4.8) and (4.13), we derive a contradiction. Thus, (4.7) is valid. Without loss of generality, we assume that (4.7) holds for any $z \in \mathbb{C}^n$. The same argument of Hu-Yang in [19, Theorem 6.2] yields that u is of finite order. For the sake of completeness, we give it in all detail. So,

$$\begin{aligned} A_u(r) &= \frac{i}{2\pi}r^{2-2n}\int_{C^n|r|}\left(1+|u|^2\right)^{-2} du \wedge \overline{du} \wedge \nu^{n-1} \\ &= n^{-1}r^{2-2n}\int_{C^n|r|}\left\{\left(1+|u|^2\right)^{-2}\sum_{i=1}^{n}\left|u_{z_i}\right|^2\right\}\nu^n \\ &\leq r^{2-2n+\sigma}\int_{C^n|r|}\nu^n = Kr^{\sigma+2}, \end{aligned} \tag{4.14}$$

where K is a positive constant. By the definition of $T(r,u)$ (see e.g.,[20, 33]), one gets that

$$T(r,u) = \int_{r_0}^{r} A_u(t)\frac{dt}{t} + O(1) \leq \frac{K}{(\sigma+2)}r^{\sigma+2} + O(1).$$

Therefore, u is of finite order. Furthermore, if u is a transcendental entire function, then by Theorem 2, u is pseudo-prime.

This finishes the proof of Theorem 4.

Acknowledgements. The author is grateful to the reviewers for valuable suggestions and comments. This work was supported by Natural Science Foundation of Shandong Province (ZR2022MA014).

References

[1] W.F. Ames, Nonlinear partial differential equations in engineering. *Academic Press*, New York, (1965).
[2] W.D. Brownawell, On the factorization of partial differential equations. *Can. J. Math.* 39(1987), 825-834.
[3] C.H. Briot and J. Bouquet, Théorie des fonctions doublement périodiques et, en particulier, desfonctions elliptiques. *Mallet-Bachelier*, Paris, (1859).
[4] C.H. Briot and J. Bouquet, Intégration des équations différentielles au moyen de fonctions elliptiques. *J. Ecole Polytechnique.* 21(1856), 199-254.
[5] D.C. Chang, B.Q. Li and C.C. Yang, On composition of meromorphic functions in several complex variables. *Forum Math.* 7(1995), 77-94.
[6] D. Coman and E.A. Poletsky, Stable algebras of entire functions. *Proc. Amer. Math. Soc.* 136(2008), 3993-4002.
[7] A. Eremenko, Meromorphic solutions of equations of Briot-Bouquet type. *Teor. Funktsii, Funk. Anal. i Prilozh.* 38(1982), 48-56. English translation: *Amer. Math. Soc. Trans.*1 133(1986), 15-23.

[8] A. Eremenko, L.W. Liao and T.W. Ng, Meromorphic solutions of higher order Briot-Bouquet differential equations. *Math. Proc. Cambridge Philos. Soc.* 146(2009), 197-206.

[9] A. Edrei and W.H.J. Fuchs, On thc zeros of $f(g(z))$, where f and g are entire functions. *J. Analyse Math.* 12(1964), 243-255.

[10] L.R. Ford, Automorphic functions. 1st ed, *McGraw-Hill*, New York, (1929).

[11] M. Green, Some Picard theorems for holomorphic maps to algebraic varieties. *Am. J. Math.* 97(1975), 43-75.

[12] F. Gross, On factorization of meromorphic functions. *Trans. Amer. Math. Soc.* 131(1968), 215-222.

[13] F. Gross, C.C. Yang and C. Osgood, Primeable entire functions. *Nagoya Math J.* 51(1973), 123-130.

[14] A.A. Gol'dberg, On single-valued solutions of first-order differential equations. *Ukrainian Math. J.* 8(1956), 254-261.

[15] E. Hille, Higher order Briot-Bouquet differential equations. *Ark. Mat.* 16(1978), 271-286.

[16] E. Hille, Remarks on Briot-Bouquet differential equations I. *Comment. Math.* 1(1978), 119-132.

[17] E. Hille, Some remarks on Briot-Bouquet differential equations II. *J. Math. Anal. Appl.* 65(1978), 572-585.

[18] W.K. Hayman, Meromorphic functions. *Clarendon Press*, Oxford, (1964).

[19] P.C. Hu and C.C. Yang, Malmquist type theorem and factorization of meromorphic solutions of partial differential equations. *Complex Variables. Theory Appl.* 27(1995), 269-285.

[20] R.O. Kujala, Functions of finite-type in several complex variables. *Trans. Amer. Math. Soc.* 161(1971), 327-358.

[21] L.W. Liao and C.C. Yang, On the growth and factorization of entire solutions of algebraic differential equations. *Ann. Acad. Sci. Fenn. Math.* 25(2000), 73-84.

[22] B.Q. Li, Entire solutions of certain partial differential equations and factorization of partial derivatives. *Trans. Am. Math. Soc.* 357(2004), 3169-3177.

[23] B.Q. Li, Entire solutions of certain partial differential equations in $\mathbb{C}^n$. *Isr. J. Math.* 143(2004), 131-140.

[24] B.Q. Li and E.G. Saleeby, Entire solutions of first-order partial differential equations. *Complex Variables. Theory Appl.* 48(2003), 657-661.

[25] R. Nevanlinna, Analytic functions. *Springer*, Berlin, (1970).

[26] Y. Noda, On factorization of entire functions. *Kodai. Math J.* 4(1981), 480-494.

[27] P.C. Rosenbloom, The fix-points of entire functions. *Medd. Lunds Univ. Mat. Sem. Suppl. Bd. M. Riesz.* 186(1952), 186-192.

[28] E.G. Saleeby, On complex analytic solutions of certain trinomial functional and partial differential equations. *Aequationes Math.* 85(2013), 553-562.

[29] E.G. Saleeby, Meromorphic solutions of generalized inviscid Burgers' equations and a family of quadratic PDEs. *J. Math. Anal. Appl.* 425(2015), 508-519.

[30] E.G. Saleeby, On meromorphic solutions of first-order Briot-Bouquet type PDEs. *J. Math. Anal. Appl.* 482(2020), 123517.

[31] Sh. Strelitz, Three theorems on the growth of entire transcendental solutions of algebraic differential equations. *Canad. J. Math.* 35(1983), 1110-1128.

[32] Z.H. Tu, On meromorphic solutions of some algebraic partial differential equations on $\mathbb{C}^n$. *J. Math. Anal. Appl.* 214(1997), 1-10.

[33] N. Steinmetz, Über die Faktorisierbaren Losungen gewöhnlicher differential Gleichungen. *Math. Z.* 170(1980), 169-180.

[34] A. Vitter, The lemma of the logarithmic derivative in seveal complex variables. *Duke Math. J.* 44(1977), 89-104.

[35] V.V. Zimogljad, On the growth of entire transcendental solutions of second-order algebraic differential equations. *Math. USSR Sb.* 14(1971), 281-296.

[36]

College of Science, China University of Petroleum, Qingdao, Shandong, P.R. China., 266580
e-mail: lvfeng18@gmail.com.